Security Analysis & Portfolio Management
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Lecture 54
Arbitrage Pricing Model - I

Welcome back, so let us continue our discussion of the arbitrage pricing theory. We now move to the arbitrage pricing theory in detail, in a lot of detail rather. As far as the diversification aspect is concerned, as far as the diversification of unsystematic risks are concerned, both the arbitrage pricing theory and the CAPM are having similar presumptions, or similar assumptions.

Both the CPM and APT agree on the diversibility, or diversification of idiosyncratic risks, that is the unsystematic risk component both these models agree, that this part of the risk can be diversified away, adequate by adequate manipulation, or adequate incorporation of securities into our portfolio. This cancellation is called the principle of diversification.

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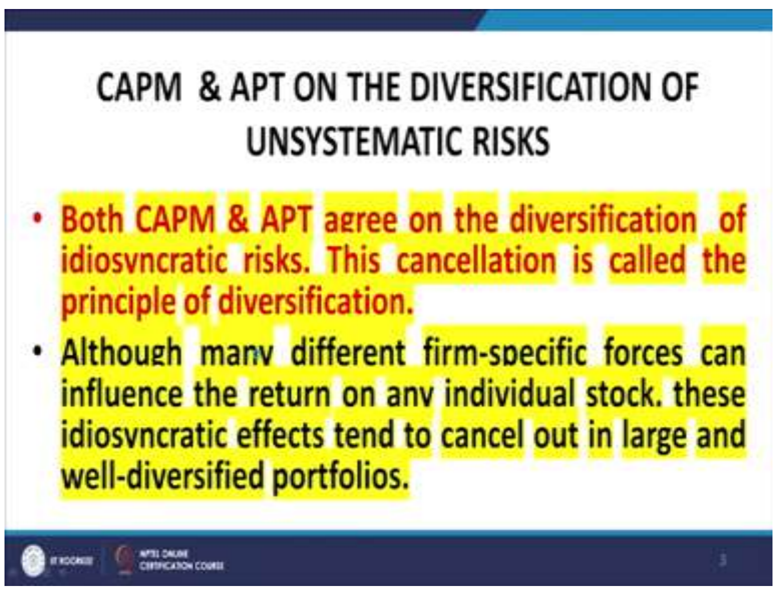

Although, many different firms specific forces can influence the return on in any individual stock, these idiosyncratic effects tend to cancel out in large and well diversified portfolios. Let me repeat, although many different firm specific forces can influence, the return on any individual stock, these idiosyncratic effects tend to cancel out in large and well diversified portfolios.

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But the systematic ratio, therefore the important issue, that needs attention in this kind of models in this models of risk return tradeoff is the issue of systematic risk. Large and well diversified portfolios are not risk free. Although, we can diversify a way, we can eliminate a component of the total risk, which is the unsystematic risk, but the systematic risk component remains. We have seen that, through mathematical explanation exposition as well.

So, nevertheless large well-diversified portfolios are not risk-free, because common economic forces may pervasively influence, all stock returns and are not eliminated by diversification. So, although some component, that is the random, random component of the total risk, that arises from firm specific factors, or industry factors can be diversified away.

And by choosing an appropriate and appropriate mix of assets, the other component, that cannot be diversified away, that is the systematic risk is what is important. And as a result of which even well diversified portfolios are not completely risk free. In the CAPM and the APT, this common forces are called systematic, or pervasive risks.

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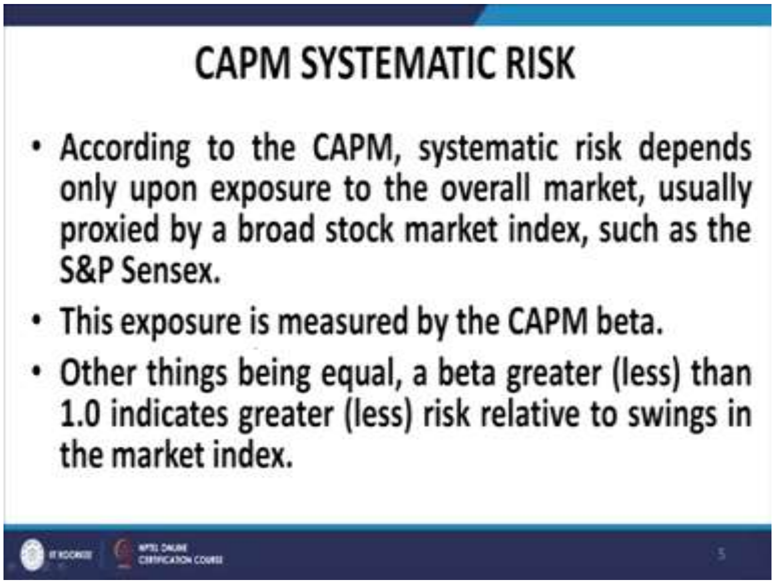

The CAPM systematic risk, we have already discussed it. But it is worth recapitulating according to the CAPM model, systematic risk depends only upon exposure to the overall market, usually proxied by a broad stock market index, such as the S and P Sensex.

This exposure is measured by the CAPM beta, this exposure is measured by the CAPM beta. We discussed it at the beginning of the previous class. Other things being equal a beta greater or less than 1.0 indicates greater, or less risk relative to swings in the market index. So, all this we have already discussed.

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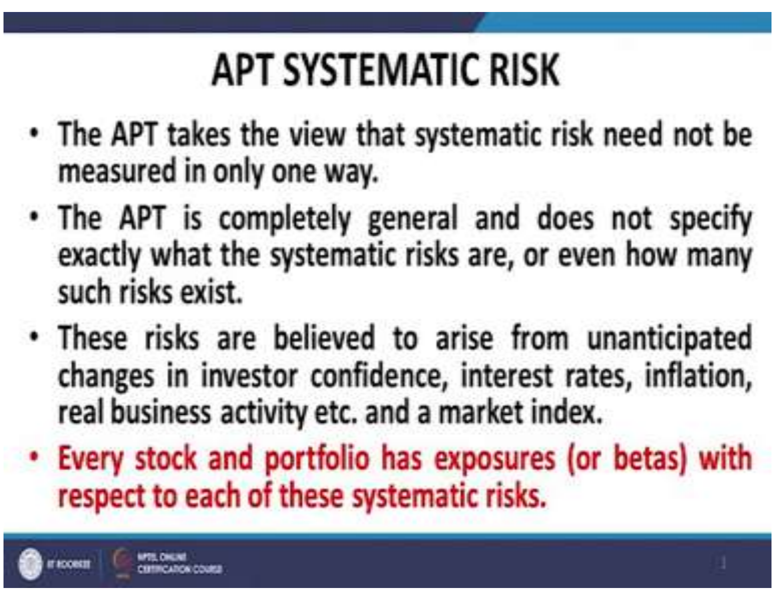

What about the APT systematic risk now? This is the fundamental part, the APT takes the view that systematic risk need not be measured in only one way. The APT is completely general and does not specify exactly, what the systematic risks are, or even how many of such risks exist, that is why I said in some sense, the APT is a generalization of the capital asset rising model.

The capital asset pricing model, relates to only one source of risk, that is the market, which it assumes is able to capture all the components of systematic risk. However, the APT is more general, it does not specify exactly what the systematic risks are, or even how many such risks exist. These risks are believed to arise from unanticipated changes in investor confidence, interest rates, inflation, real business activity and a market index. This is an illustrated list of factors, which are believed to contribute to the total systematic risk of a portfolio.

Now, APT exposures and APT betas, every stock and every portfolio has exposures, or betas with respect to each of the systematic risk. Whatever is are the number of sources of systematic risk that are identified by the analyst, the relationship between the expected return of a portfolio and each of those resources is captured by the beta in relation to that resource. So, these are this together from the risk profile, risk exposure profile, in the terminology of the APT model.

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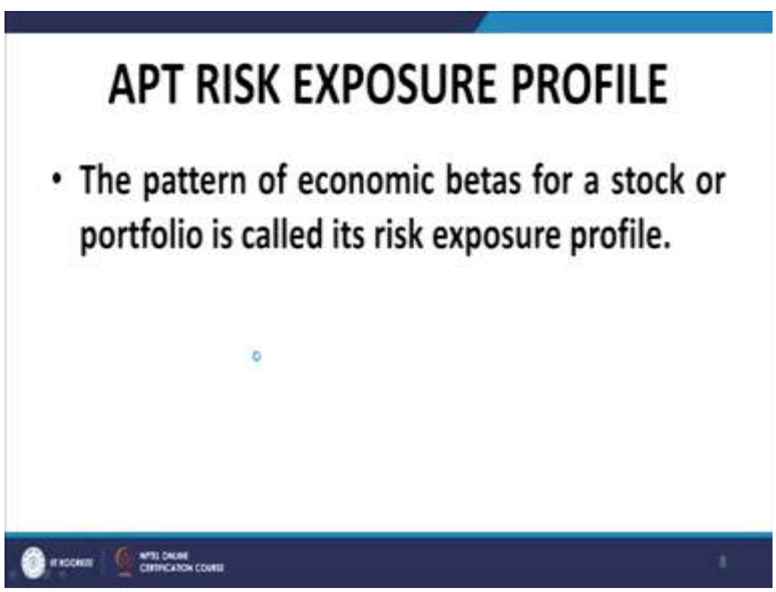

So, the pattern of economic betas of a security, or portfolio is called its risk exposure profile. The pattern of economic betas of a stock, or a portfolio is called its risk exposure profile.

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Now, the risk exposure profile and the expected return, as I mentioned it is this set of betas in relation to each of these risks, which are identified by the analyst as contributing to the total systematic risk of the portfolio, that will contribute to the expected return on the portfolio. So, risk exposures are rewarded in the market with additional expected return, the greater is your exposure with reference to a particular risk source, the greater would be the expected return corresponding to that resource, or the cumulative expected return.

The profile also indicates how a stock, or portfolio will perform under different economic conditions. Thus, the risk exposure profile determines, the volatility and performance of a well diversified portfolio.

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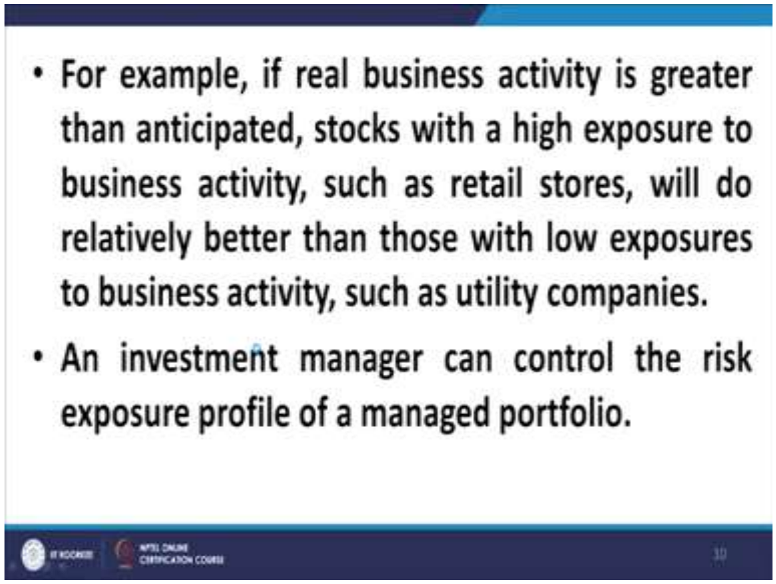

For example if real business activity is greater than anticipated stocks with a high exposure to business activity, such as retail stores, you will do relatively better than those with low exposure to business activities, such as utility companies. An investment manager can control the risk exposure profile of a managed portfolio by manipulating the composition of the portfolio.

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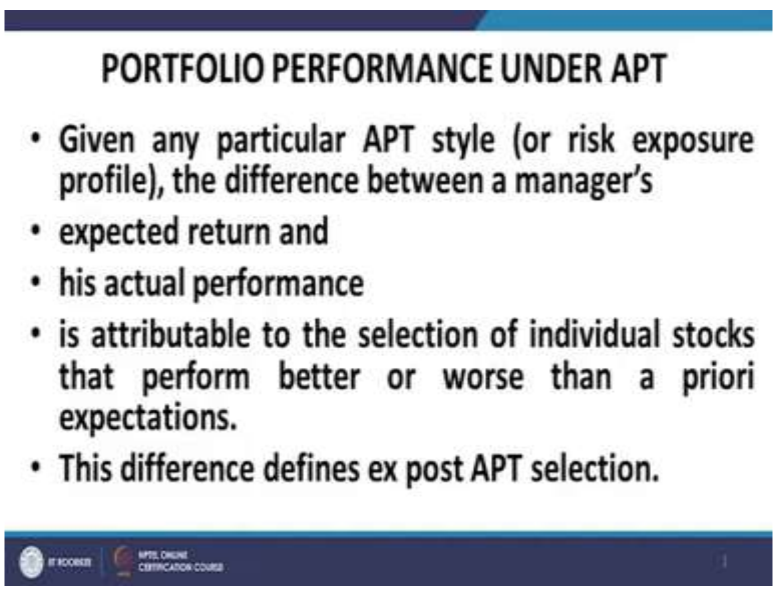

Now, the portfolio performance under APT how do we measure portfolio performance under APT given any particular APT style as captured by the exposure profile, I repeat given any particular APT style as captured by the risk exposure profile the difference between the managers expected return and his actual performance is attributable to the selection of individual stocks that perform better, or worse than a priori expectations. This difference defines, the ex post APT selection.

So, it is basically the difference between the expected return and the actual performance, which defines, or which captures the ex post APT selection, or the performance of the portfolio manager.

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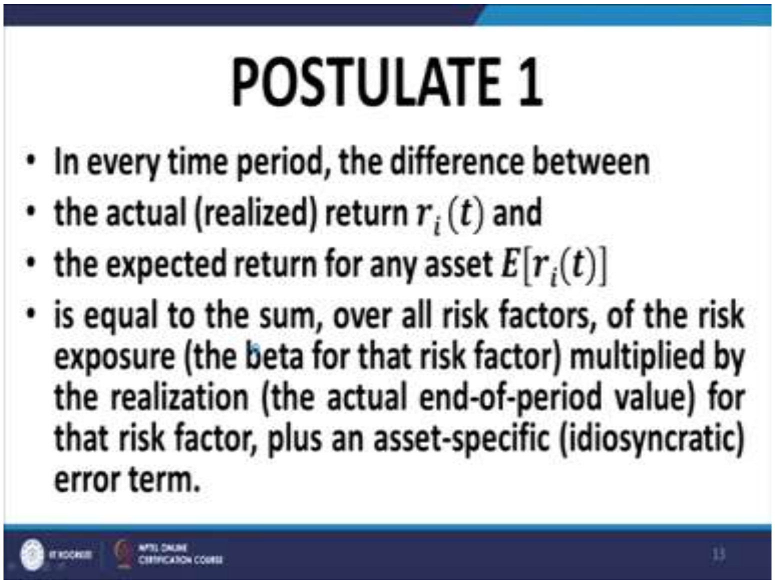

Now, we come to the APT postulates. Postulate number 1, in every time period, the difference between the actual realized return, small ri t and the expected return for any asset that is E of ri t, please note the expected return is the return is the expectation of the return calculated at say t equal to 0 and the for a period of time t. And then we measure the actual return, that has been achieved over that pi and that time t, with reference to a particular security i.

Then the difference between the actual realized return ri t and the expected return, expected return is obviously pre estimated, that is the estimate of the return, or the expectation of the return at t equal to 0 in respect of the time period 0 to small t.

So, in every time period, the difference between the actual realized return that is r i t and the expected return for any asset that is E of ri t is equal to the sum is equal to the sum overall risk factors of the risk exposure, that is represented by beta, of course the risk exposure in the APT model, the risk exposure is captured by the family of betas, or the sequence of betas, or the set of betas and multiplied by the realization, that is the actual end of period value for that risk factor plus a factor, that is usually termed as epsilon i, which is the asset specific it ideosyncratic error term.

So, the difference between the actual return and the expected return is represented by the sum of the product of the various risk factors multiplied by the realizations of this various risk factors, at the end of the period plus a random error term, that we kept that very believe is to represent the unsystematic risk, or the asset specific risk. The ideosyncretic risk. So, this is represented by equation number 1, the statement that I read out just now is represented by equation number 1

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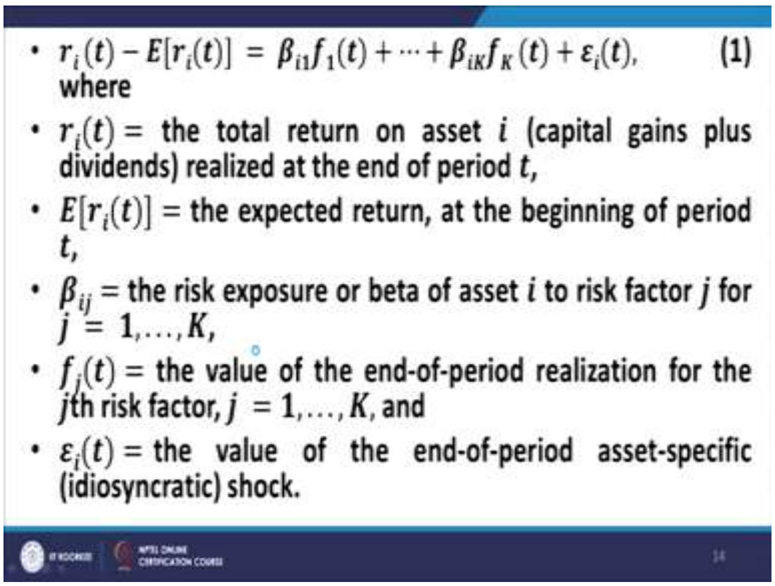

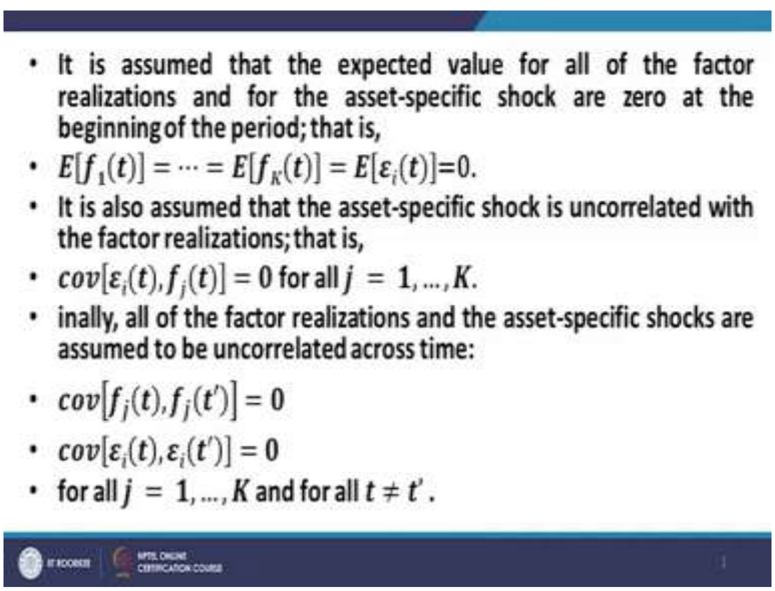

ri t as I mentioned is the total return on asset i, that includes capital gains, dividends, realized at the end of period t. E of ri t is the expected return, at the beginning of the period, that is the estimate, or the expectation of the return over the period t at the beginning of t, that is let us say t equal to 0, for the immediately following time interval 0 to small t.

Beta ij is the exposure, or beta of asset i, in relation to risk factor j, beta ij is the most important quantity perhaps, it represents the risk exposure of the asset i, to risk factor j, j can vary from 1 to k, we are assuming, that there are k risk factors, which contribute to the generation of expected return on a given security i. fj t is the value of the end of period realization for the jth risk factor, I repeat fj t is the value of the end of period realization for the jth factor, please note these are random variables fj 1, fj 2, fj 3, up to fj k, are random variables they represent resources.

So, just as we have the market return in the case of CAPM and the market return is random variable and it contributes to the overall return on the security i. Similarly, this factors f1 t, f2 t, fk t are all random variables. And what it says is fj t is the actual realization of this random variables, at the point t equal to t. And epsilon i t as I mentioned is the value of the end of period asset specific idiosyncratic shock, or risk.

It is assumed that the expected value of all the factor realizations and for the asset specific shock a 0 at the beginning of the period. We make this assumption, that the expected value for all the factor realizations and for the asset specific shock as 0 at the beginning of the period, please note this does not in any way disturb the generality of the problem. We can always rescale this factors f i the values of f i at realization to account for this particular assumption.

So, this is essentially a simplifying assumption that helps us in maintaining, or reducing the complexity of the problem. I repeat this assumption, it is assumed that the expected value of all the factor realizations and for the asset specific shock are 0, at the beginning of the periods. It is also assumed, that the asset specific shock is uncorrelated with the factor realization.

Now, now please note the as I as we have discussed again in the CAPM model, the idiosyncratic risk, or the epsilon term is also a random variable. So, we assume that r m and epsilon i are independent of each other, they are not correlated with each other. And this assumption is carried forward in the APT model, which also assumes that, the idiosyncratic risk, or the idiosyncratic shock is not correlated with any of the factors. So, any of the systematic risk factors are not correlated with the idiosyncratic risk of the security.

Finally, all of the factor realizations and the assets asset specific shocks are assumed to be uncorrelated across time I repeat finally all of the factor realizations and asset specific shocks are assumed to be uncorrelated across time, in other words they represent a truly random process. So, the value of the factor, or the realization of the factor, at any point t and there are realization of the factor at any later, or earlier point t dash are not correlated in any way.

And so this is another important assumption, that the values, or the realizations possible realizations of the of the factors at different points of any factor, at different points in time are uncorrelated completely. And similarly, the idiosyncratic term is also uncorrelated across time, that means the value of this random term at any point in time t and any point in point prime t dash are mutually uncorrelated.

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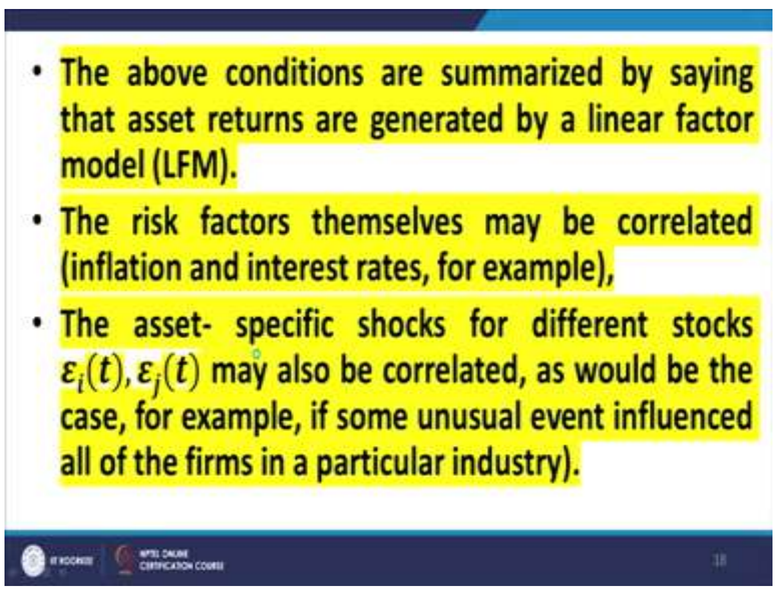

So, the above conditions are summarized by saying, that the asset returns are generated by a linear factor model L f m, I repeat the above conditions are summarized by saying, that the asset returns generated by a linear factor model. This risk factor themselves may be correlated, please note we have not put in any condition, that fi and fj should be uncorrelated, I repeat this is important we have not put in any restriction, we have not put in any condition, that fi and fj need to be uncorrelated, they may be correlated. For example, inflation and interest rates.

The asset specific shocks for different stocks may also be correlated, please note this is different to the single index model. And this is something, which we had, we had assumed as valid, as necessary in the single index model, but that does not hold in this APT model. In the APT model, we have generalized this to accommodate, or to include the fact, that asset specific shocks for different stocks i and j may be correlated, as would be the case for example, if some unusual event influenced all of these firms in a particular industry.

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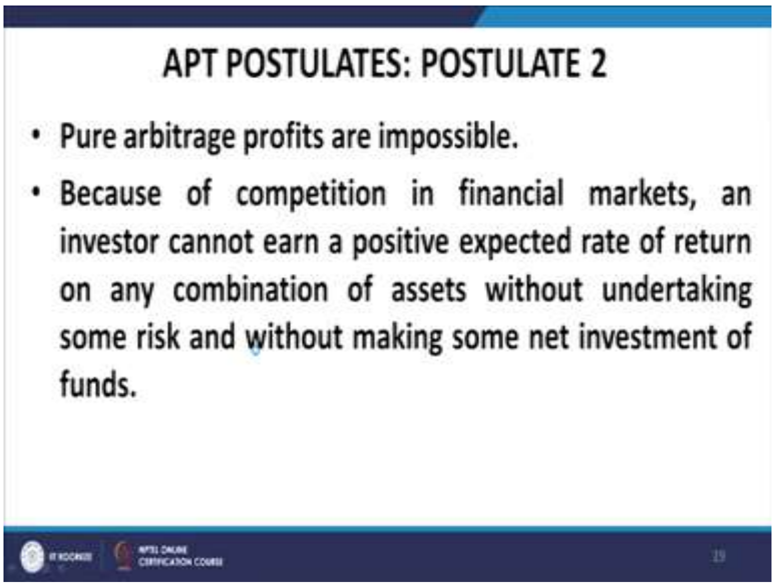

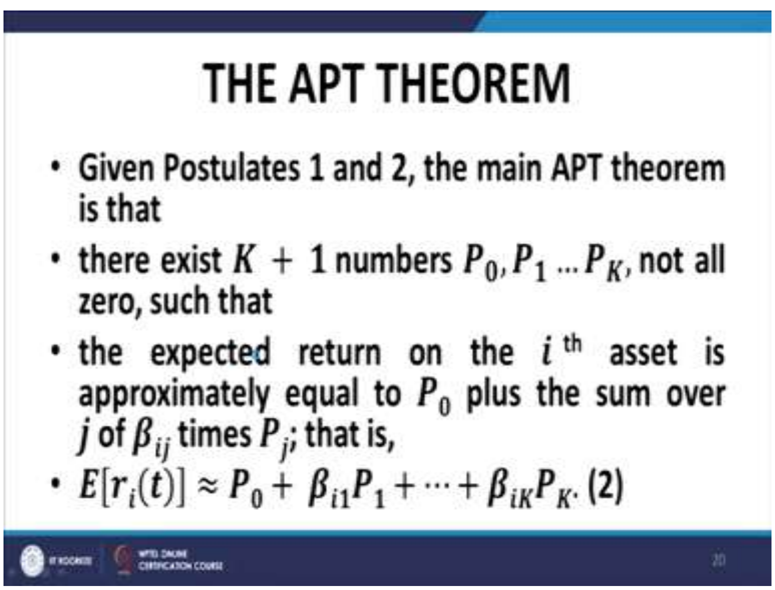

Now, postulate number 2. Pure arbitrage profits are impossible, pure arbitrage profits are impossible, that means what, that is why is it so rather, why is it so? Because competition in financial markets is so much, is so extensive and the markets as a result of which become so efficient, that an investor cannot earn a positive expected rate of return on any combination of assets without undertaking some incremental risk and without making some net investment of funds. So, please note this important point, it arises on account of the competition in the market and the consequential market efficiency.

So, higher the market efficiency in other words, higher the market efficiency, lower is the chance of making any arbitrage profits. So, we make this assumption that in the ideal case pure arbitrage profits are impossible and no investor can earn a positive expected rate of return on any combination of assets without undertaking some risk and without making some net investment of funds.

Now, what is the APT theorem, these were the postulates. The APT theorem, is that given the postulates 1 and 2, the APT theorem says that, there exists k plus 1 numbers p0, p1, p2, p3, up to pk not all 0, such that the expected return on the ith asset is approximately equal to p 0 plus the sum over j of beta j times pj, that is equation number 2. Expected return on a security i, can be represented as a sum of p 0 and a sum of the product of the beta profile, of the risk exposure profile, of the of this asset, each term being multiplied by the by a particular number p1, p2, p3, and pk.

What are these ps? Now, first of all this approximation is has been proved to hold substantially and as a result of which it may be replaced by the equality sign. So, now we come to the issue, what are these pjs? These pjs, are the price of risk, just as we had the equity risk premium in the case of the CAPM model, whatever the beta was we multiplied by the expression, which represented the equity risk premium. But there was only one term there.

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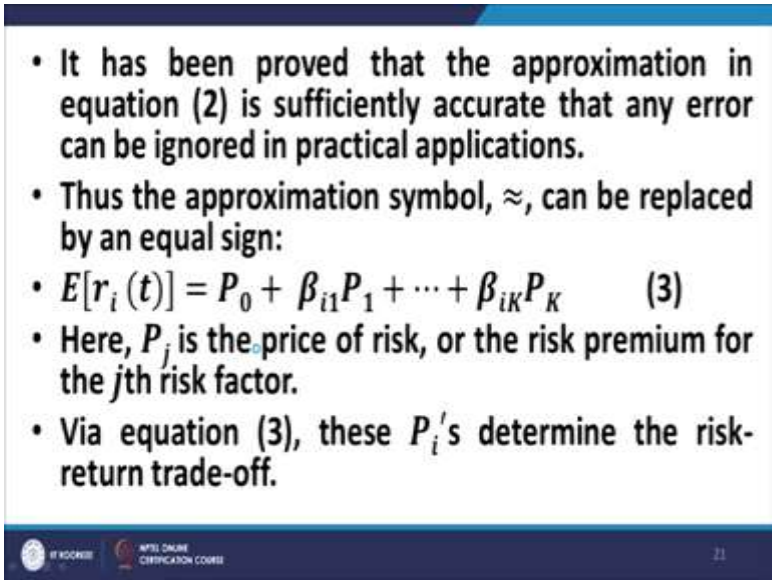

Here what we are saying is that, the entire set of betas, which represents the risk exposure profile of the asset is multiplied by the corresponding values of the risk price of risk to arrive at and then the cumulative effect represents together with p 0, the expected return on a security i.

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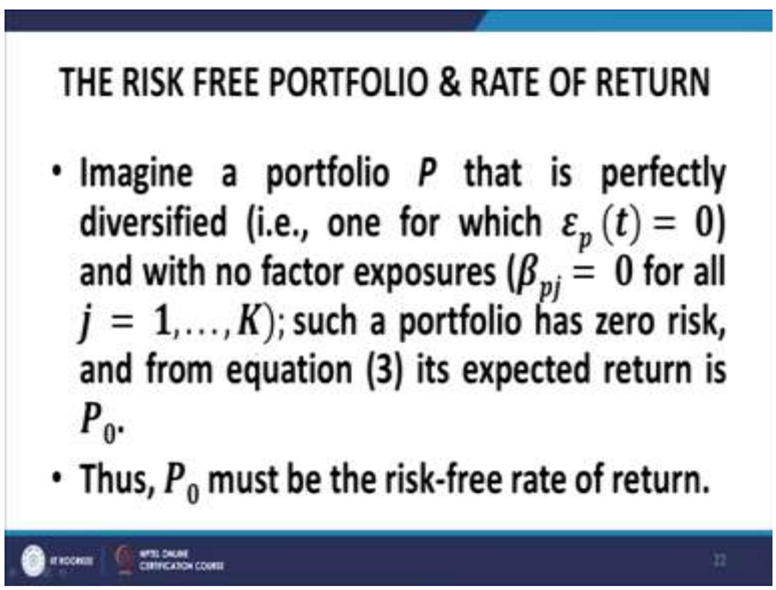

Now, what about the risk free rate, it is easy to see, that the risk free rate is equal to the to the term p0. How do we see it? We imagine a portfolio P, that is perfectly diversified, that is for with the idiosyncratic risk, or the unsystematic risk is 0. And with no factor exposures to any of this risk factors, it has all beta is equal to 0, that means it has no factor exposure to any of these risk contributors, or resources, which contribute to the total expected return on the portfolio.

Then beta pj is equal to 0 for all j equal to 1 to k and that means what? That means the portfolio has 0 risk and from equation number 3, we find that its expected return is equal to p0, this p0, must be the risk free rate of return, this is quite easy to see.

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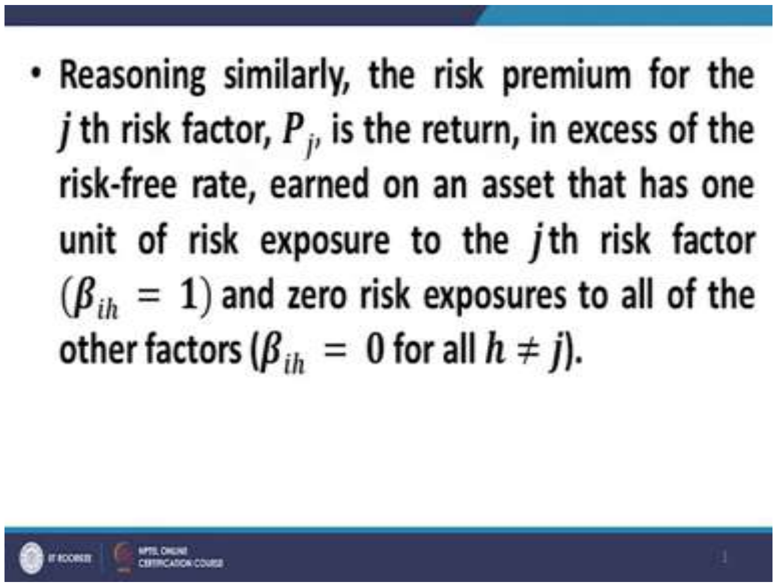

Reasoning similarly, the risk premium, now if you we can extend this logic, this rational to define the various items of risk premium, or risk price. Reasoning similarly, the risk premium, or the risk price for the j th risk factor pj is the return in excess of the risk free rate, earned on an asset, that is that is one unit of risk exposure to the j th risk factor, that is the beta with reference to that particular risk factor is equal to 1 and zero risk factor, zero risk exposures with reference to the other factors.

So, let me repeat, if we have a portfolio, that has one unit of one unit of sensitivity you may call it, or the exposure to a particular risk factor, let us say pj, or ph and zero risk exposure reference to all the other risk factors, then the excess return, excess expected return over the risk free rate, that is generated on that portfolio will be called the risk premium, with reference to that risk factor.

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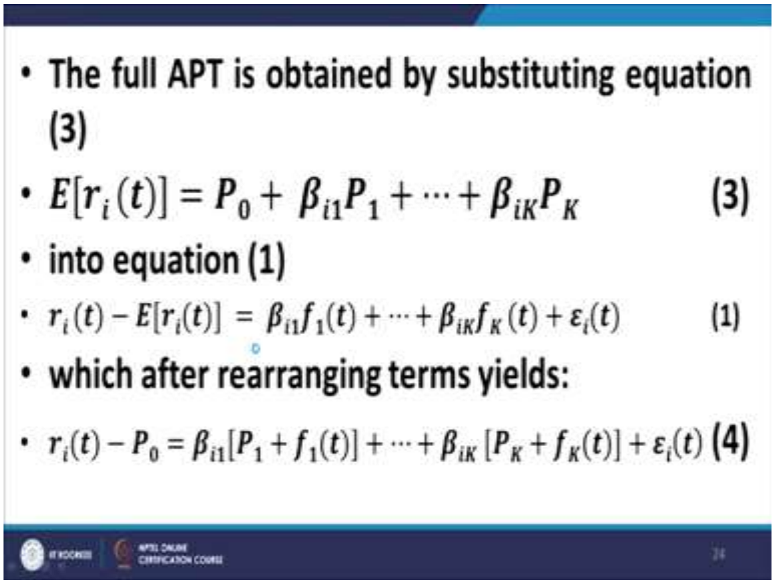

The full APT is obtained by substituting. Now, we see the derivation of the full APT theorem. The full APT is obtained by substituting, the expected return, that is equation number 3, into equation number 1. The expected return is given by p0 plus beta i1, p1 plus beta up to ik Pk, we are considering a model with k plus 1, or rather k risk factors, of course P0 is the risk free rate, we substitute this into equation 1, and we simplify a bit. When we simplify this expression, what we get is equation number 4.

So, equation number 4 is easily obtained simply by substituting equation number 3, in equation number 1. When you substitute the value of E of ri t into from equation number 3, in equation number 1 and rearrange the terms, what we get is equation number 4.

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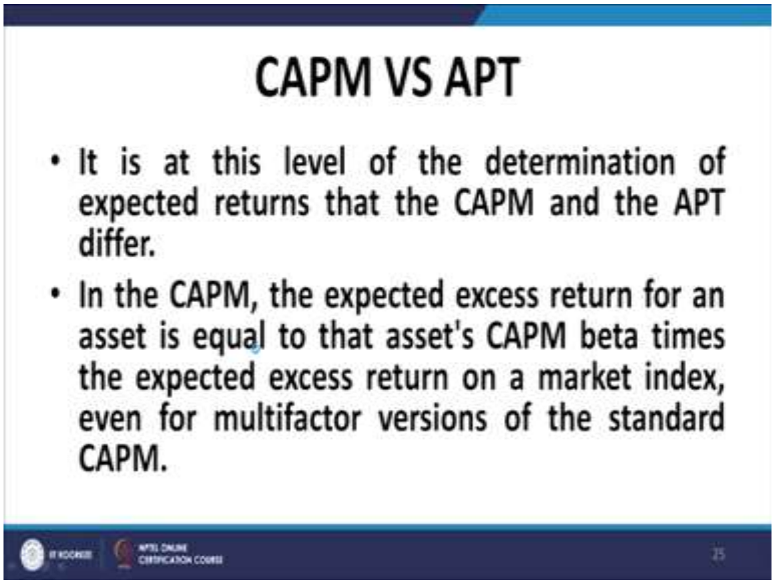

Now, again we come back to the CAPM versus APT relationship. It is at this level of the determination of expected returns that the CAPM model and the APT model differ. In the CAPM model, the expected excess return for an asset is equal to that assets CAPM beta times. The expected excess return on a market index, even for multi-factor versions of the standard CAPM.

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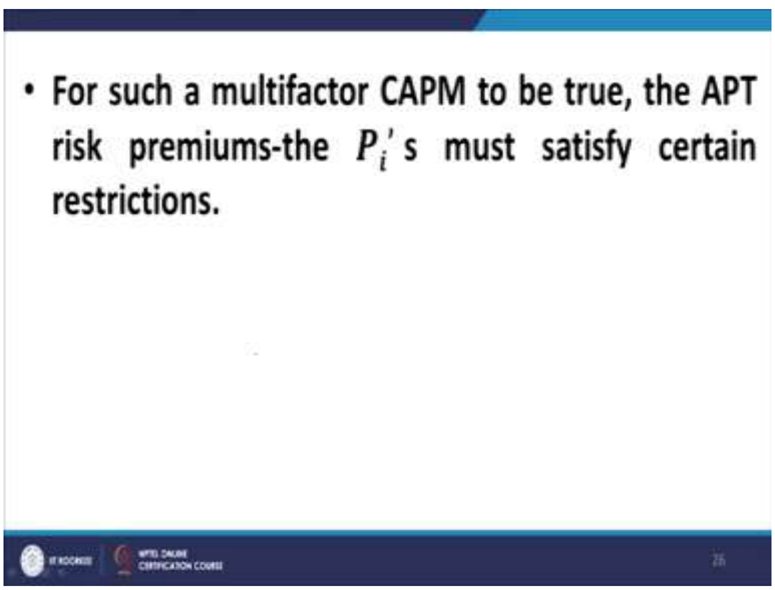

For such a multifactor CAPM model to be true, the APT risk premium must satisfy a certain relationship. What is the relationship between the Pi's of the APT model and the betas of the CAPM model. Let us now investigate that.

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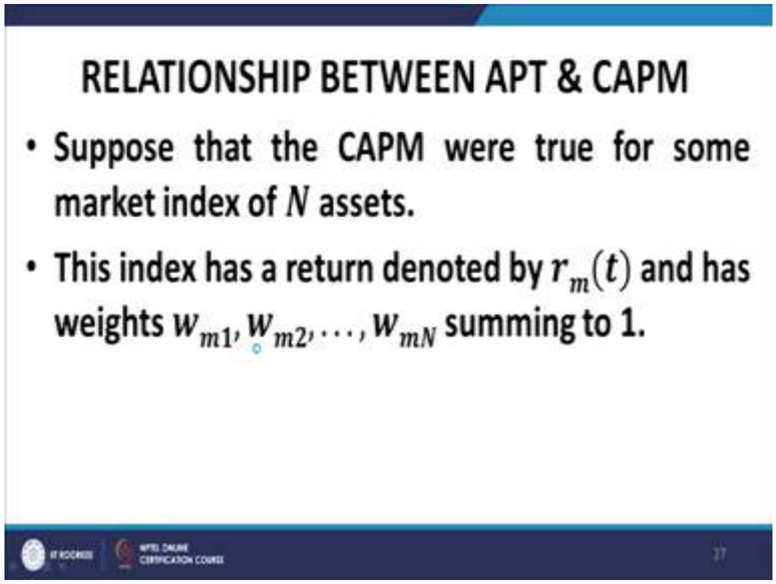

So, relationship between the APT and the CAPM, that is in essence what we are trying to work out is the relationship between the CAPM beta and the APT prices of risk, or risk premia. Suppose, that the CAPM were true for some market index of N assets.

Let me repeat, suppose that the CAPM was true for some market index of N assets. This asset has a return, which we denote by rm t and has weights w m1, w m2, w mN, of each of the N securities and the total sum of the weights is equal to 1. We assume this, this particular part of the exposition.

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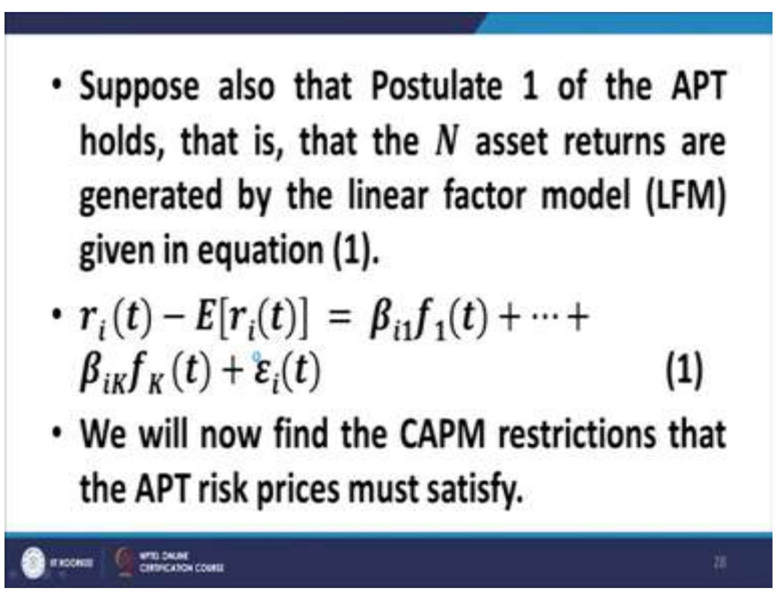

Now, suppose also that in postulate 1 of the APT, that suppose also that postulate 1 of the APT holds, that is that the N asset returns are generated by a linear factor model given by equation number 1. Then what do we have? We have ri t minus the expected value of ri t is equal to beta i1, f1 plus beta i2, f2, i is the security, please note and 1, 2, 3, are the various risk factors, that contribute to the total systematic risk.

So, beta i1 f1 plus beta i2 f2 and beta ij is the sensitivity of the security i to the risk factor j plus beta ik fk t plus the idiosyncratic risk term E or epsilon i t. We find the CAPM restrictions rather that the APT risk prices must satisfy.

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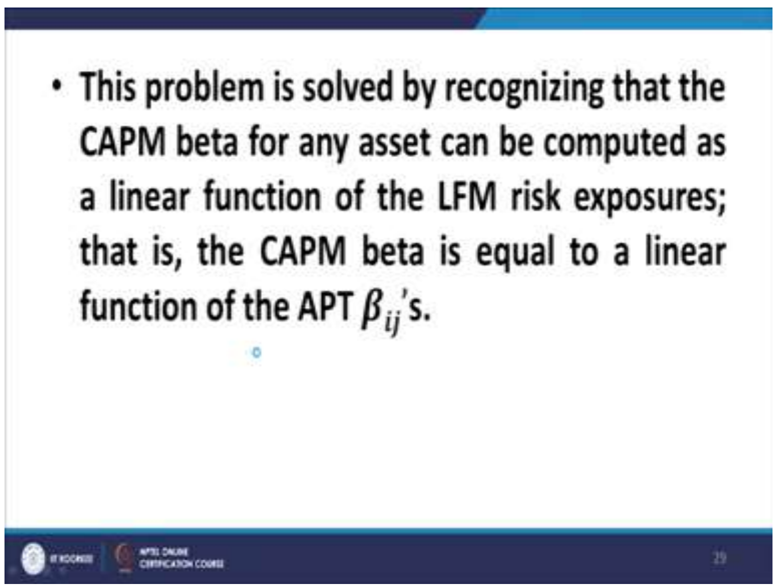

And this problem is solved by recognizing, that the CAPM beta for any asset can be computed as a linear function of this linear factor model risk exposures, that is the CAPM beta is equal to a linear function of the APT beta ij s.

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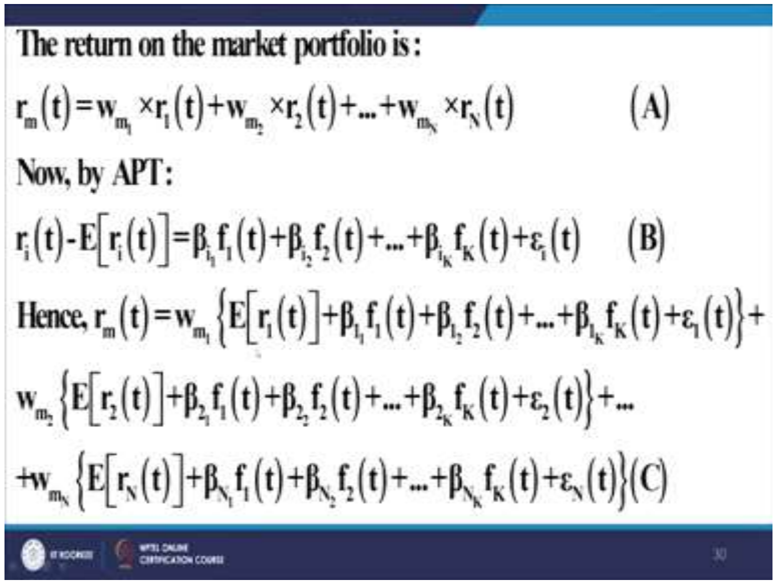

How do we do it, this is very interesting. The return on the market portfolio rm t is given by the weighted average return of its constituent securities. So, we have equation number A, it is straightforward. What does it say? It says that, the return on the market portfolio is equal to the weighted average returns of the constituent securities. Also by APT equation number 1, of the APT, we have that ri t minus the expected value of ri t is equal to beta i1, f1 t plus beta i2, f2 t plus beta ik fk t plus epsilon 1 t.

And similarly, for r2 t similarly for r2, r3 t and so on. For all these values, for all the securities security 1, security 2, security 3, and all these N securities, we have this expression, which is represented by equation number B. So, ri t minus expected value of ri t is equal to f beta i1 f1 t plus beta i2 f2 t up to beta ik fk t plus the idiosyncratic risk. Therefore, if you substitute this value of ri t in equation number A, from equation number B, what we get is equation number C.

It is a slightly extended equation, but I repeat, what we have simply done is, we have simply substituted the value of r1 t, r2 t, r3 t up to rN t. As obtained from equation number B, in equation number A, we get equation number C.

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$$r_m(t) = \left[w_{m_1} E\left[r_1(t)\right] + \ldots + w_{m_N} E\left[r_N(t)\right]\right] +$$
$$\left[w_{m_1} \varepsilon_1(t) + \ldots + w_{m_N} \varepsilon_N(t)\right] +$$
$$\left(w_{m_1}\beta_{1_1} + w_{m_2}\beta_{2_1} + \ldots + w_{m_N}\beta_{N_1}\right) f_1(t) +$$
$$\left(w_{m_1}\beta_{1_2} + w_{m_2}\beta_{2_2} + \ldots + w_{m_N}\beta_{N_2}\right) f_2(t) + \ldots +$$
$$\left(w_{m_1}\beta_{1_K} + w_{m_2}\beta_{2_K} + \ldots + w_{m_N}\beta_{N_K}\right) f_K(t) \quad (D)$$

Now, rearranging equation number C, we simply rearrange the terms of equation number C, we have done nothing else and we get equation number D. In essence we have isolated the coefficients of f1 t, f2 t, f3 t, up to fk t and we have captured the other the epsilon terms have been clubbed together and the expectation value terms have been clubbed together.

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Setting $\beta_{m_j} = w_{m_1}\beta_{1_j} + w_{m_2}\beta_{2_j} + \ldots + w_{m_N}\beta_{N_j}$

for $j = 1, 2, \ldots, K$

$$r_m(t) = \left\{E\left[w_{m_1} r_1(t) + \ldots + w_{m_N} r_N(t)\right]\right\}$$
$$+\varepsilon_m(t) + \beta_{m_1} f_1(t) + \ldots + \beta_{m_K} f_K(t)$$
$$r_m(t) = \left\{E\left[r_m(t)\right]\right\} + \varepsilon_m(t) + \beta_{m_1} f_1(t) + \ldots + \beta_{m_K} f_K(t) \quad (E)$$

The beta m j is equal to w m1 beta 1 j plus w m2 beta 2 j plus so and so up to w mN beta Nj, please note 1, 2, 3, is the security number, is the first suffix is the security number and the second security is the factor identity, or the risk factor identity. So, there are two surfaces, one for first suffix is the security number and the second suffix, or the suffix of the suffix is the you see factor number. So, j is equal to from 1 to k. Now, when we substitute this expression in equation number D, what we get is equation number E. So, putting all this together, what we end up with equation number E.

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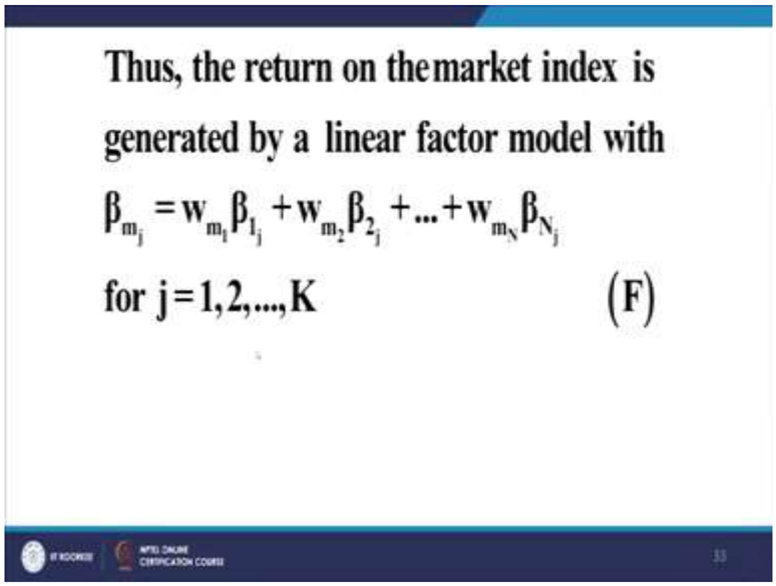

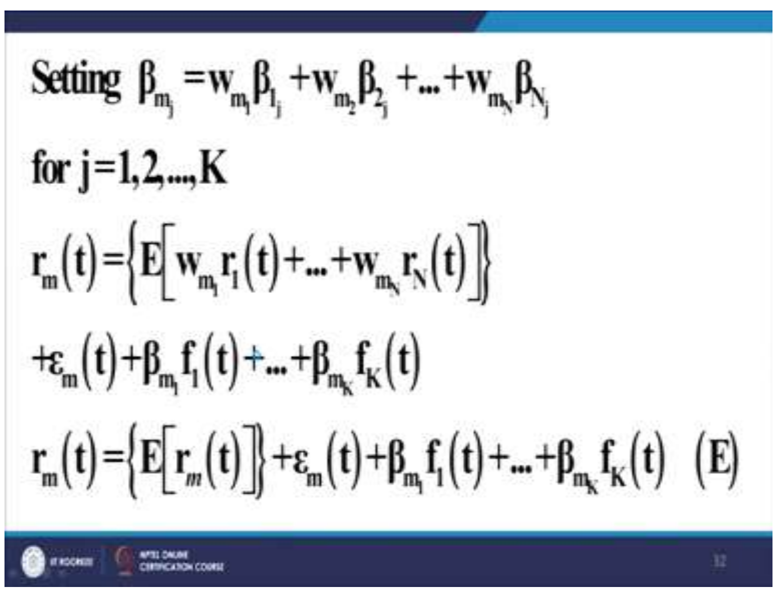

Now, this equation number E, what does it tell us? It tells us, that the return on the market index is generated by a linear factor model with the various betas, that are given by equation number F, you can see here in equation number E, if you compare this equation with the linear factor model equation you find that it is valid term to term. And therefore this shows that the market return is also generated by a linear factor model with the betas, which are given by the expression f.

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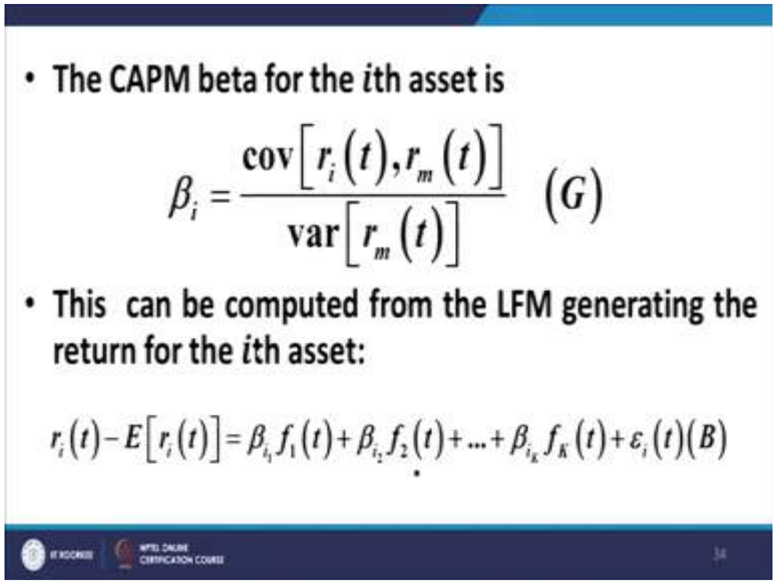

Now, the CAPM beta for the ith asset is given by the expression, which is here in equation number G. We know that very well, beta of the is a regression coefficient the CAPM beta is a regression coefficient. So, it is given by the covariance between ri and rm divided by the variance of the market square divided by the variance of the market.

So, this can be computed from the, from the linear factor model generating the return from the ith asset, that is we use this equation, which is equation number B, for substituting for ri t in equation number G, I repeat you substitute the value of ri t from equation number B in equation number G.

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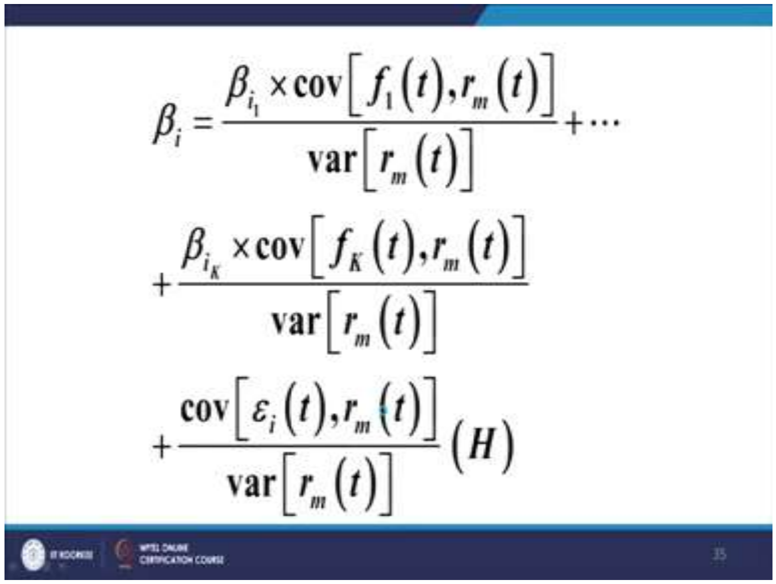

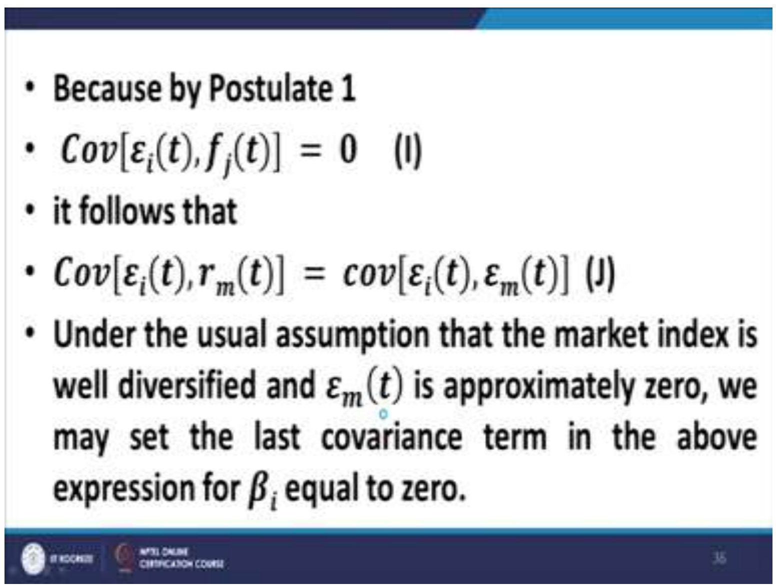

What we get is? This equation H. And in an equation H, we do some simplifications. What are the simplifications? Because of postulate 1, what we have is that, the idiosyncratic risk is independent, or uncorrelated within all the factor, all the factor risks. Therefore, what we get is f i and r m, if you look at this f 1 and the epsilon term, you look at this epsilon term here, the last term in equation H, what we find is, when you use this expression epsilon i with fj is equal to 0. What we are left with is the expression j. If you use E, if you use the fact that the covariance between apply epsilon i and fj is equal to 0.

And then what you find is that, the relationship between epsilon i and rm t, because you see r m t is a linear combination of f1 t, f2 t, f3, up to fk t and then there is an epsilon term. But all the factor terms, the covariance between all the factor terms and epsilon i t is 0 by in virtue of equation number 1. So, the only term that is left in rm is the epsilon term.

So, what we have is that all other terms will vanish because r m, I repeat because r m t is a linear combination of the factors plus the epsilon term, and epsilon i with all these factors has a 0 covariance. So, when you take the covariance of epsilon i with r m t, the only term that survives is the term that is epsilon i t, epsilon m t.

Now, under the usual assumptions that the market index is well diversified and epsilon and t is approximately 0, this term will also vanish, this term will also vanish. So, what we have is epsilon i t, rm t that is this term, the last term in equation H is completely 0, it totally vanishes.

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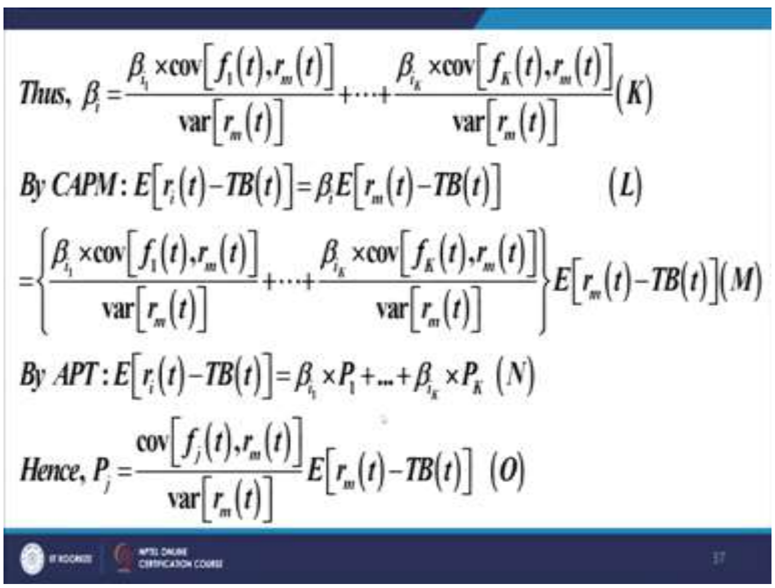

And that means what? That means beta i is equal to the expression, that is given in equation number K, please note the epsilon term has vanished, because of the reason that I explained in equation numbers I, and equation number J. Now, if you look at the CAPM model, if you look at the CAPM model, it is captured by equation number L. Where TB is the treasury bill rate for a majority of T and this captures the risk-free rate. So, in other words the risk-free rate is epitomized by is represented by the TB treasury bill rate, which is the TB term.

So, we have the, we have using this using equation L, substituting for beta i, in equation L what we get is equation number M, simply substituting for beta i from equation K and equation L, what we get is equation number M. And when we compare this equation with the APT equation, which corresponds to the to this particular problem, that is equation number N, where again we are replacing the risk-free rate by the treasury bill rate, what we get is equation number N.

So, comparing equation number N and N comparing equation number M and N, we arrive at the relationship between the various risk prices of the APT. And the CAPM betas as given by the expression, that is equation number O.

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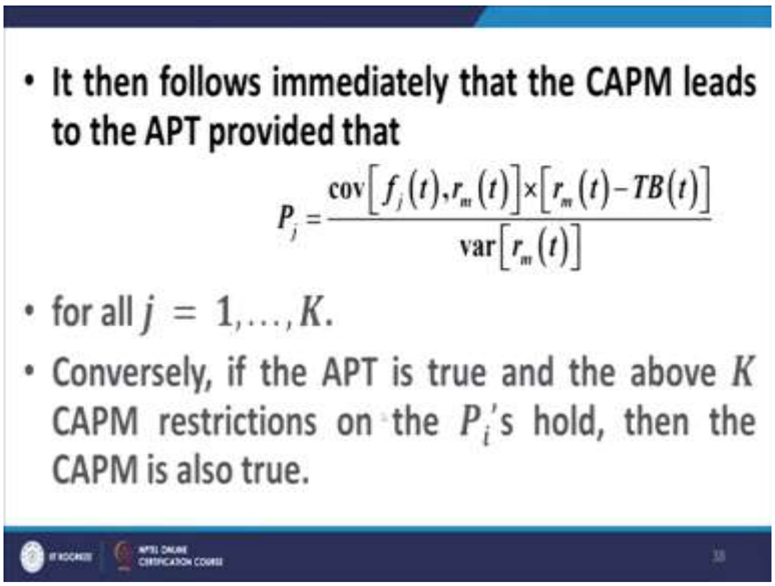

So, from here I will continue in the next lecture. Thank you.