Path Integral Methods in Physics & Finance

Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture – 56
Pricing of Options: Binomial Model (1)

Welcome back. So, before the break I discuss the issue of put call parity and there was certain assumptions embedded in that with the two options that we talked about; had identical maturities, and they had identical underlying assets, and they had underlying strike prices as well. And furthermore one important assumption was that, during the life of the options there were European options; during the life of the options there was no income generated or expense for that matter, expense can be taken as negative income or vice versa.

So, there was no income generated from the underlying asset. Suppose we relax that assumption, we relax the assumption that the underlying asset does not generate any income during the life of the option. In other words we make the assumption that the underlying asset gives a certain dividend or a certain pre expected cash flow is arises from the mere holding of the underlying asset. How does the put call parity gets modified in that case? It is very interesting.

(Refer Slide Time: 01:33)

PUT CALL PARITY (DOLLAR RETURN)

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K
INVEST	$-Ke^{(-rT)}-D_0$	$K+D_T$	$K+D_T$
TOTAL	$-c-Ke^{(-rT)}-D_0$	$K+D_T$	S_T+D_T

So, in this case again we construct two portfolios; portfolio A and portfolio B with the portfolio A consists of a long call. I will not take up in detail because of again paucity of time, but the methodology is same; we take a long call, the cost of the call is obviously a negative quantity minus c, and the call gives us a payoff of S T minus K in the event that S T is greater than K.

And we invest a certain amount, because it is an investment, so it is a negative cash flow; it is a cash outflow at t equal to 0 and it will generate a positive cash flow when we recover the proceeds of the investment at t equal to maturity. What is the amount that we invest?

We invest an amount K e to the power minus r T that is present value of the excess price plus the present value of the dividend that we expect to receive from the underlying asset during the life of the options that is D naught. D naught I repeat is the present value of the dividend that we expect to receive during the life of the option contract.

Obviously, the first component K e to the power minus r T will give us, because we are investing at the risk free rate r; the future values which is K, and D naught will give us the future value D T. So, the total cash flows in this case work out to K plus D T, if S T is less than K; and S T plus D T, if S T is greater than K. So, this is as far as portfolio A is concerned.

(Refer Slide Time: 03:15)

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	$-c-Ke^{(-rT)}-D_0$	$K+D_T$	S_T+D_T
PORTFOLIO B			
BUY STOCK	$-S_0$	S_T+D_T	S_T+D_T
BUY PUT	$-p$	$K-S_T$	0
TOTAL	$-S_0-p$	$K+D_T$	S_T+D_T
	$c+Ke^{(-rT)} = S_0-D_0+p$		

Let us move to portfolio B, portfolio B you buy the underlying asset by the stock that cost you a amount of S naught at t equal to 0. Now by virtue of merely holding the stock, you are getting the dividend on the stock and that dividend you can invest, and on maturity of the options you will get an amount equal to D T.

So, the total realizations from holding the stock or buying the stock and holding it will be S T, which will be from the disposal of the stock that you have brought. And the future value of dividend that you have invested, that we have received from the stock during the life of the life of the option.

And you buy a put option that gives that entails the cost of p. So, it is minus p and that gives you pay off of K minus S T, if S T is less than k; otherwise it lapses unexercised. So, again here what we have is the same situation; if S T is less than K, the payoff is K plus D T and if S T is greater than K, the payoff is S T plus D T.

So, again in the first partition we are having the same cash flows between portfolio A and B, and in the second partition we are again having the same cash flow between A and B; in other words the two portfolios A and B resemble each other as far as the cash flows on maturity are concerned.

Whatever state, whatever state the stock price evolves; if it evolves in the first segment that is S T is less than K, the cash flows from the two portfolios are identical and if it evolves in the second segment S T is greater than K, again we find that the cash flows are absolutely identical. And therefore, the cost must be identical and that gives us the put call parity in the form that is given right at the bottom of the slide.

So, if you look at this carefully, if you look at this new put call parity carefully; you find that the effective cost of holding the asset is reduced by the present value of the amount of dividend.

In other words, where you are actually spending an amount; in other words it can be interpreted as where you are actually spending an amount of S naught for acquiring the asset, you are partly financing that S naught by the present value of the amount of dividend that you are expecting to get during the life of the options.

(Refer Slide Time: 06:07)

PUT CALL PARITY (YIELD)

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K
INVEST	$-Ke^{(-rT)}$	K	K
TOTAL	$-c-Ke^{(-rT)}$	K	S_T

There can be another situation which, which may arise when the dividend is not given in terms of a dollar amount, not given in terms of a money amount; but it is given in terms of a percentage, yield, a continuous compounded yield. If that is the situation, in that case the put call parity again can be arrived at in a similar manner.

(Refer Slide Time: 06:35)

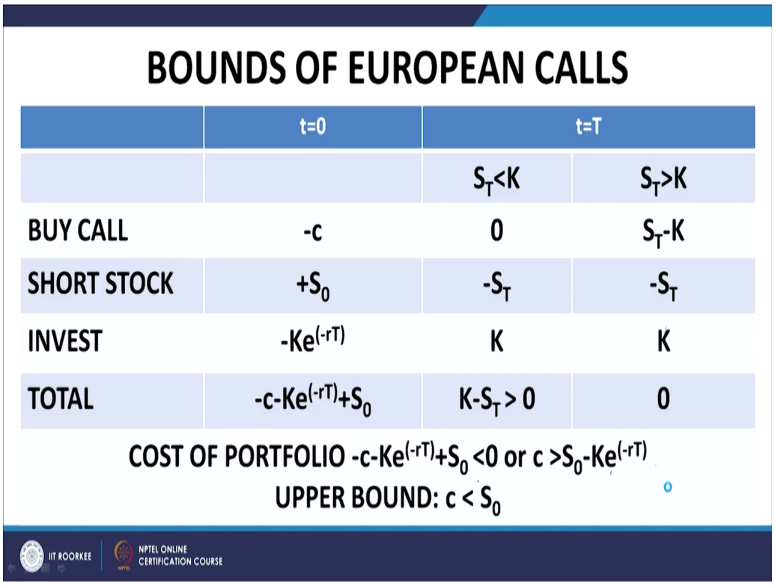
	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	$-c-Ke^{(-rT)}$	K	S_T
PORTFOLIO B			
BUY ASSET	$-S_0e^{(-qT)}$	S_T	S_T
BUY PUT	-p	$K-S_T$	0
TOTAL	$-S_0e^{(-qT)}-p$	K	S_T
	$c+Ke^{(-rT)} = S_0e^{(-qT)}+p$		

The only difference would be, let me only highlight the difference here; that instead of buying one unit of the asset here in this situation, you buy only e to the power minus q T units of the asset, where q is the continuous compounded yield that you are talking about, that the asset is going to generate if you are going to hold onto the asset.

So, instead of buying one unit of the asset in portfolio B, you buy only e to the power minus q T units of the asset. Why? Because e to the power minus q T units of the asset would translate to one unit of the asset at the point of time of maturity of the options; and that one unit of the asset when disposed of in the market will generate S T amount which is the then prevailing market price, the prevailing market price at the maturity of the options.

Other than that, the analysis is absolutely parallel and this change is reflected in the put call parity relationship that is shown in the bottom of this particular slide.

(Refer Slide Time: 07:38)



Quickly running through the bound of European calls, again we invoke the arbitrage principle; we have, if you look at this combination, you take a long call. If you take a long call, the payoff is S T minus K; if S T is greater than K, 0 otherwise you have a short stock.

Short stock means, you borrow a stock and then you sell it in the market at t equal to 0; in anticipation that the stock price will go down at a later date and you will buy the stock from the market and replenish it to the original owner that is what is called shorting of the stock.

So, you short a stock; short means borrow the stock from somebody and sell it in the market, you get a cash inflow equal to the current market price of the stock which is S naught. So, this is plus S naught, and because you are going to replenish it by buying it in the market at a future date and the price at that future date is going to be S capital T the maturity date of the option.

So, at that point it is going to be a cash outflow. So, it is minus S T and minus S T here, and this is independent of the state of nature; whether S T is less than K or S T is greater than K and you invest an amount of money K e to the power minus r T.

And because you are investing this, you will receive a cash flow on redemption of this investment which is again independent of the state of nature. So, the total cash flow that you receive if S T is less than K is K minus S T which is positive, because S T is less than K; if S T is less than K, obviously K minus S T is positive and it is 0, atK, if S T is less than K.

In other words what is happening here that, at maturity there is one situation, there is one situation or a set of situations where you will get a; there is a possibility there is a probability that the stock price could end up in the first segment of the partition and thereby you could get a positive cash flow.

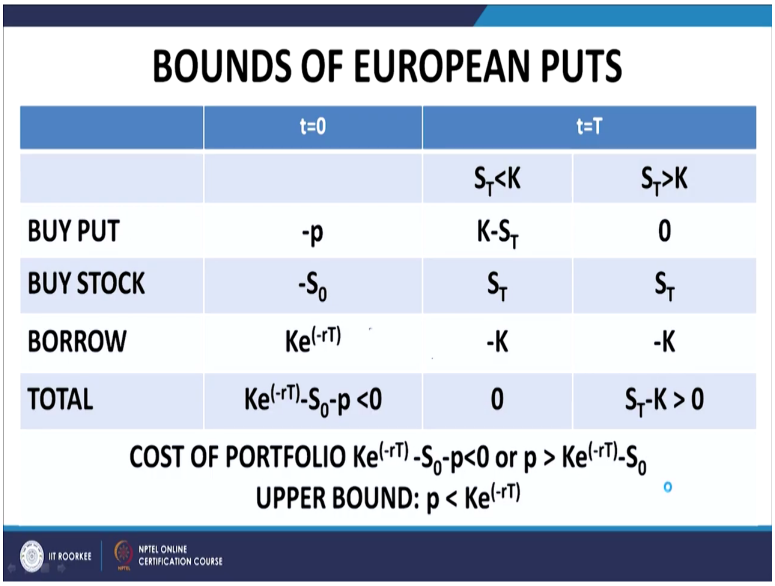
Or and the worst case scenario is that, if the stock price ends in second partition, you will get a zero cash flow. In other words there is no situation whatsoever in which you can get a zero net cash flow or a negative net cash flow rather; either you get a zero cash flow or you get a positive cash flow.

There is at distinct finite probability that you could get a positive cash flow and the worst case situation is that, you get a zero cash flow. And what does that means? That means that, the cost of this portfolio has to be positive or the cash flows has have to be negative and that gives us this relation that c has to be greater than S naught minus K e to the power minus r T.

So, this is the; this is the lower bound on the call price; European call remember, lower bound on the call price. And obviously, what is c? c is the right to buy the assets; so obviously the value of the call cannot exceed the value of the asset, and therefore c has to be less than S naught, this is the upper bound.

So, the lower bound on the call option is c call option is S naught minus K e to the power minus r T; and of course 0 call cannot be negative of course, call cannot be worth negative, it will be, it is an asset, so it cannot be worth negative. So, it has to be the maximum of S naught minus K e to the power minus r T or 0; and of course the upper bound is c has to be less than S naught.

(Refer Slide Time: 11:36)



BOUNDS OF EUROPEAN PUTS

	t=0	t=T	
		S_T<K	S_T>K
BUY PUT	-p	K-S_T	0
BUY STOCK	-S_0	S_T	S_T
BORROW	$Ke^{(-rT)}$	-K	-K
TOTAL	$Ke^{(-rT)}-S_0-p<0$	0	$S_T-K>0$

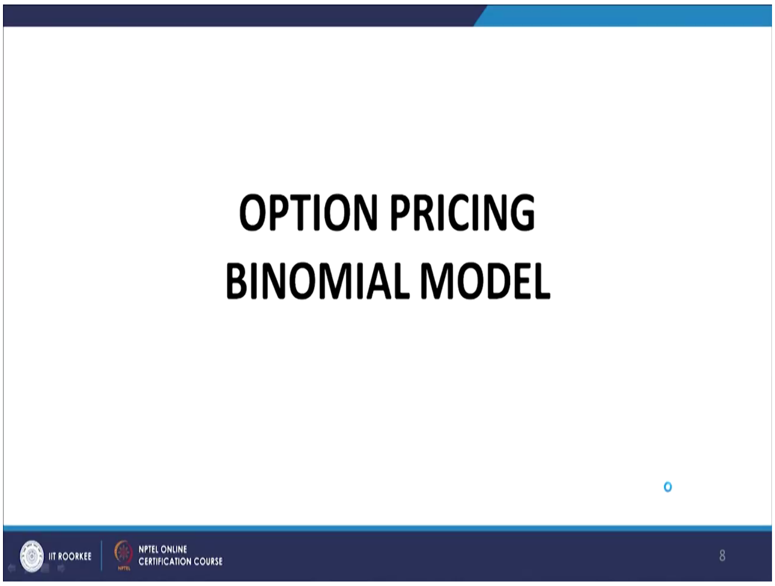
COST OF PORTFOLIO $Ke^{(-rT)}-S_0-p<0$ or $p>Ke^{(-rT)}-S_0$

UPPER BOUND: $p<Ke^{(-rT)}$

Similarly, working on similar lines, we can arrive at the lower bound for puts; the lower bound for puts is K e to the power minus r T minus S naught or 0 whichever is larger. So, this is the lower bound, but remember the upper bound in the case of put is. What is the put? Put is a right to sell the asset.

So, because it is a right to sell the asset, the value of the put option cannot exceed the exercise price which you are going to receive under the exercise of the put option. But that excess price that you are going to receive, you are going to receive at the maturity of the option; therefore as of today the put cannot be worth the present value of that, the put cannot be worth more than the present value of that excess price and this represents the upper bound of the put option.

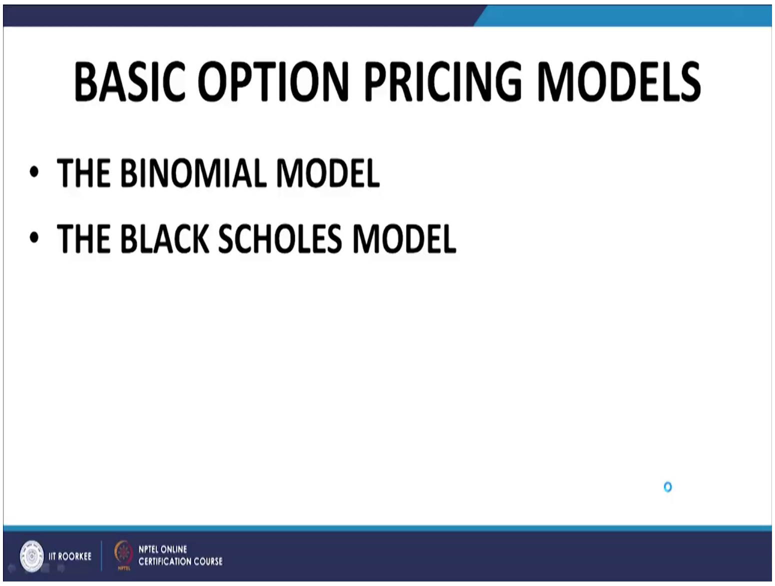
(Refer Slide Time: 12:33)



Now, we come to option pricing. This is now the strategy as per as I plan, my plan as far as the remaining topics is concerned is that, we start with option pricing; we discuss the simplest model which again mirrors the single step random walk in a sense, it is based on the single step random walk.

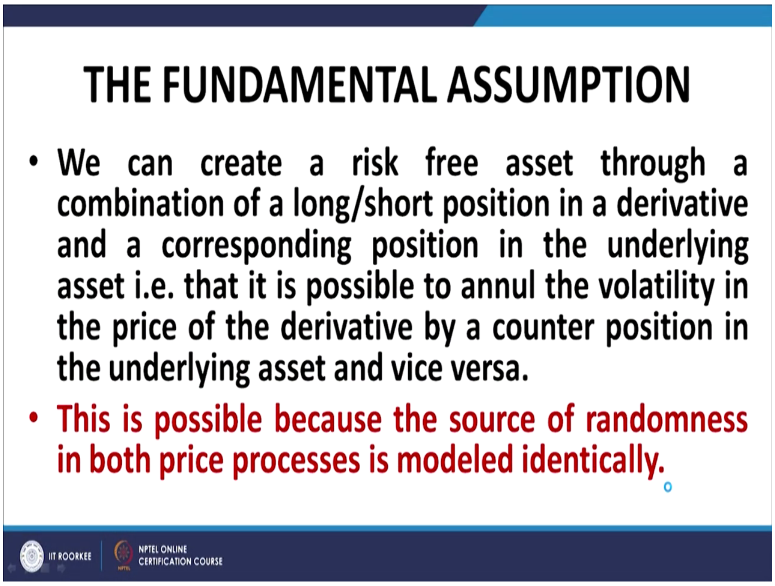
From there we move to the Black Scholes model, which moves to the continuous framework and then we talk about the path integral, formulization or path integral approach which extends the Black Scholes model.

(Refer Slide Time: 13:19)



So, let us start with the simplest discrete model; at the elementary level option, pricing models can be classified into the binomial model and the Black Scholes model. The binomial model is the discrete model and the Black Scholes model works in the continuous environment; it is based on Ito's lemma.

(Refer Slide Time: 13:30)

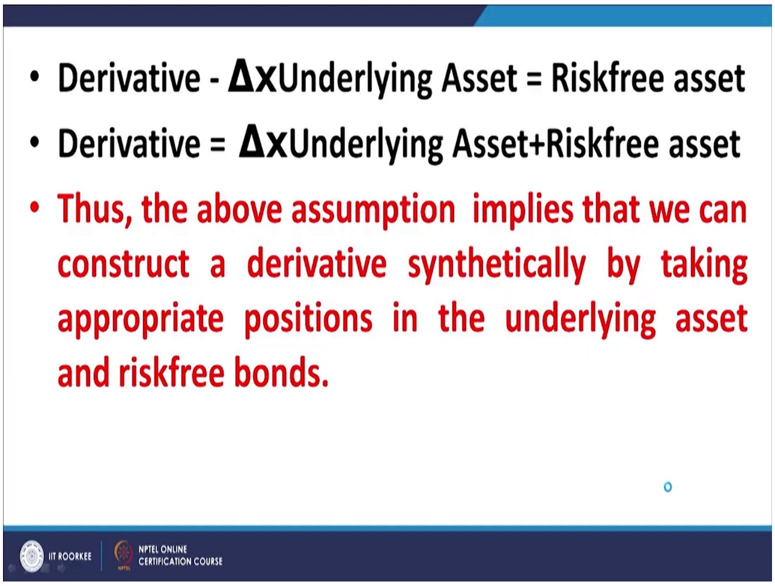


But we will come back to that, let us start with the Binomial model. Now what is the fundamental premise? In fact, the fundamental premise of both these models is that, we can create a risk free asset, we can create a risk free asset; but it might taking a combination of the derivative, long or short position in the derivative and a corresponding position in the underlying asset.

In other words say, let me repeat this is very fundamental and this forms the underlying of the both these models, the Binomial model and the Black Scholes model; that we can create a risk free asset consisting of a combination of a long shot or short position in the derivative asset and a corresponding position in the underlying assets.

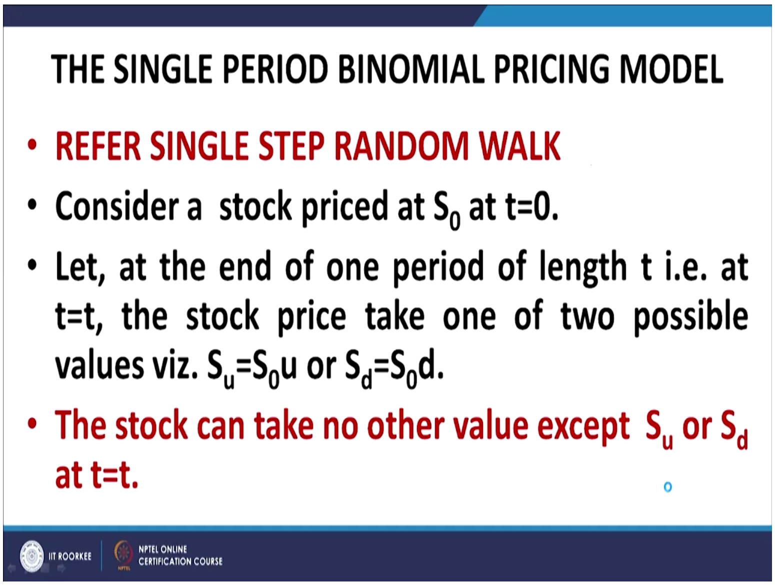
The two together, the two together constitute a risk free asset; in other words it is possible to annul the fluctuations, annul the randomness in the derivative asset or you may say in the underlying asset by taking a position in the other asset. So, why that is possible? It is possible because we are modeling that randomness in on the basis of the same phenomenon or the same mathematical structure as we will be a apparent gradually.

(Refer Slide Time: 15:05)



So, we can write derivative minus delta into underlying asset is equal to risk free asset; we can also write derivative is equal to delta into underlying asset plus risk free asset. In other words we can also synthetically create derivative assets by taking positions in risk free assets and the underlying asset.

(Refer Slide Time: 15:27)



So, let us start with a single period Binomial pricing model, recall this is based on the single step random walk. In the single step random walk, you would recall that we are, we discussed the unbiased random walk; but we also assume that the size of the jumps to be identical.

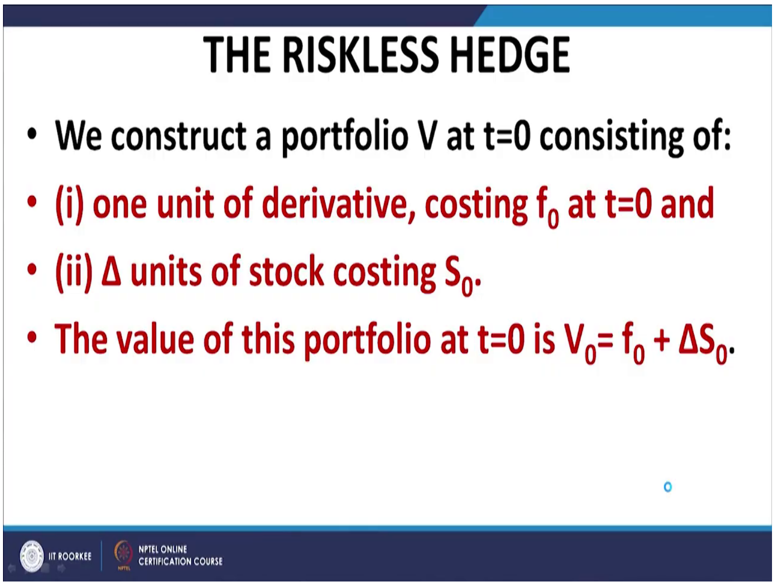
Here we modify that situation a bit, we continue to assume that the walk is unbiased inverse the; in other words the probability of up and down jumps are equal, but we assume that the amplitudes, in other words the steps, the upswings and the downswings are different.

So, let us start, we consider a stock which is priced at S naught at t equal to 0; at the end of one period of time, one time step which is of length t. Remember it is a one time step, but it is of length t; whatever measures you are adopting on that basis you can define t, but it is one time step. One time step in the sense that, it is only at the end of this time step that the stock price makes a move.

So, at the end of that time step, the stock price makes a move, it either moves up to S u or it moves down to S d; S u can be written as S 0 u in terms of fractions or S d can be written as S 0 d.

So, the because it is a binomial model, we assume that these are the only two values that is stock price can take; and we also assume that in the period 0 to t, the stock price cannot move, the stock price will not show any dynamic behavior. The dynamic behavior will arrive only when the clock strikes t equal to t at which point it will make a random jump either upwards or downwards with the given amplitudes.

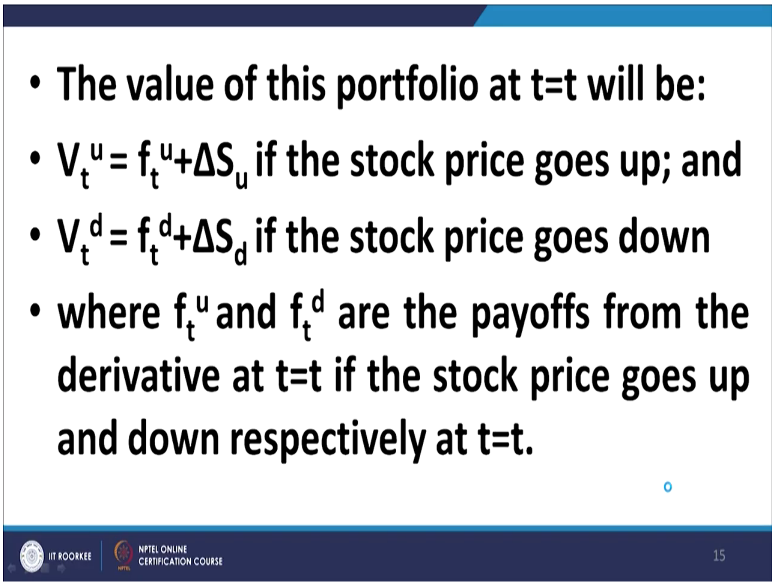
(Refer Slide Time: 17:31)



What we do now is, we construct a riskless hedge. We construct a risk riskless hedge by taking one unit position or a long position in one unit of the derivative which we are proposing to value. We assume that the current value of the derivative is f naught t equal to 0 and we assume that derivative will be annulled by taking a position of delta units in the underlying asset, which is the stock, and the current price of the stock or current value of the stock is S naught.

So, the total value of the portfolio obviously, is V naught is equal to f naught plus delta into S naught; delta is the number of units of the stock, S naught is the price per unit of the stock. So, delta S naught is the stock component, f naught is the price per unit of the derivative; and we are having one unit of the derivative. So, the total value of the portfolio is f naught plus delta S naught.

(Refer Slide Time: 18:30)



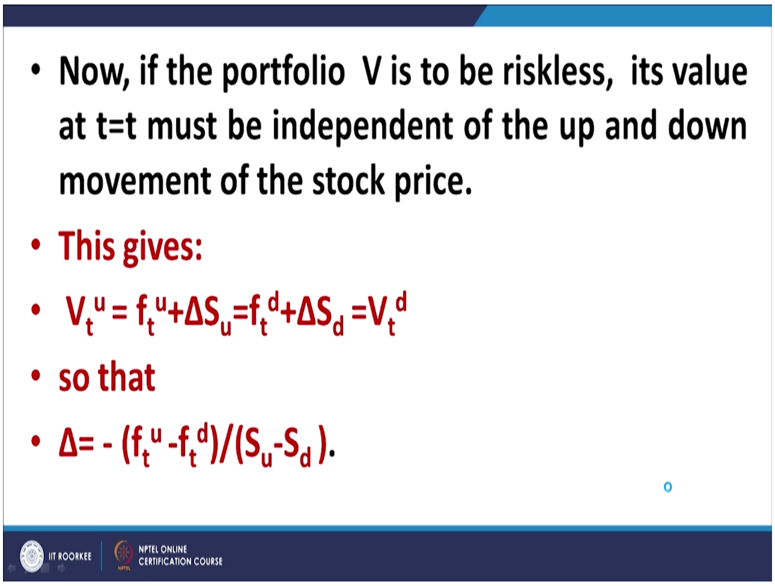
Now the value of the portfolio at t equal to t, at the end of step when it has made this jump either up; when the stock prices made this jump either upwards or downwards, let us see what the value can be.

Now, because if derivative is a the price of the derivative is a function of the price of the underlying asset; because the stock price has moved either up or down the value of the derivative will also change. Let us assume that if the stock price moves up to S u, the derivative price moves to f u and if the stock price moves down to S d, the derivative price moves down to f d.

That being the case the value of the derivative if the stock price moves up sorry, the value of the portfolio; I am sorry the value of the portfolio at t equal to t if the stock price moves up, it will given by f u plus delta into S u, because f u is the new price of the derivative or new value of the derivative corresponding to an up jump of the stock price. And similarly if the stock price moves down, the value of the portfolio will be equal to f d plus delta S d, right.

So, this is the situation as far as the values of the portfolios in the event that the stock price goes up and the stock price goes down.

(Refer Slide Time: 20:05)

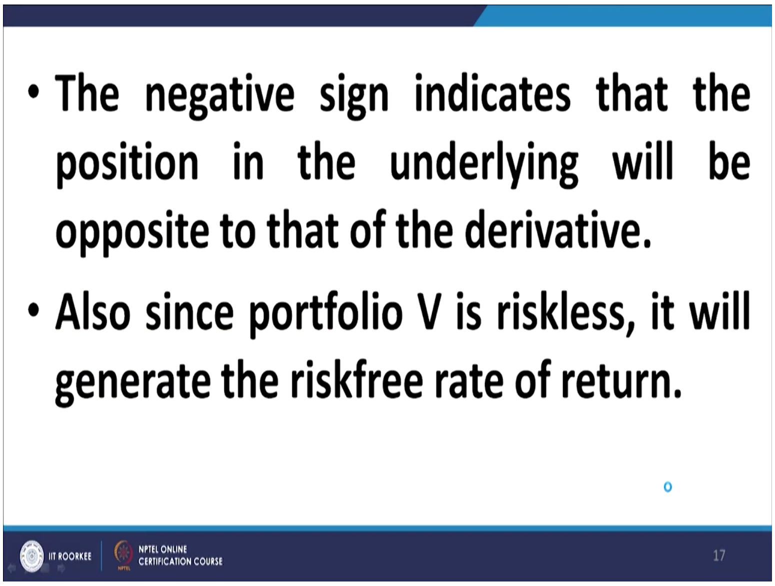


Now the important thing arise is that we have postulated, we have conjectured that we have we have assumed, we have created this hedge in such a way that we want the hedge to be riskless. And if the hedge will only be riskless; if the value of the portfolio at t equal to t is independent of the up movement or down movement.

In other words whether the stock price moves up or the stock price moves down, the value of the portfolio remains unchanged. What is the condition for that? The condition for that is given here in this slide V t u must be equal to V t and d; because that is the fundamental proposition, that is what we are saying that it we want, we have been able to create a riskless hedge. Risk means what? Risk means fluctuation.

So, if you are talking about riskless, we means there should be no fluctuation; the stock price should not influence the price of the portfolio, then only it can be a riskless portfolio. So, that is, that means what, V t u must be equal to V t d and on simplifying this we get the value of delta as this expression which is given in the bottom equation on this slide.

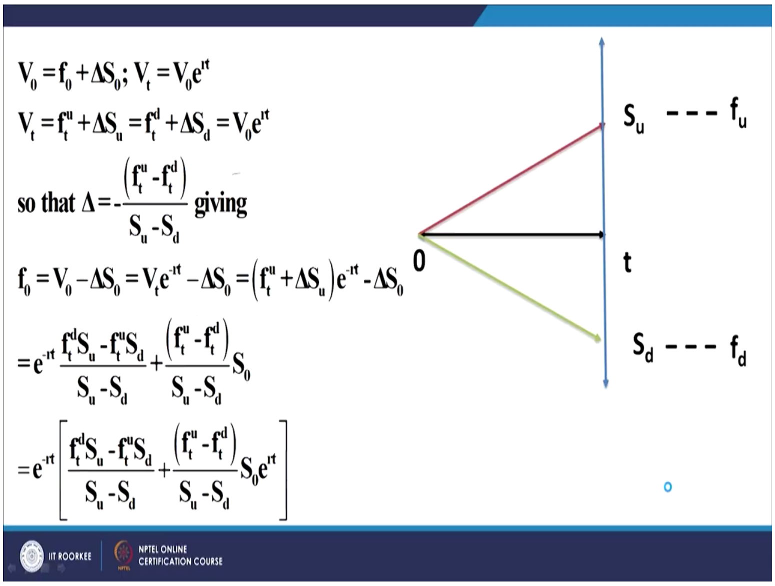
(Refer Slide Time: 21:27)



The negative sign obviously increase implies that, the position in the stock has to be opposite to the position in the derivative. If you are long in the derivative, the position in the stock will be short and vice versa.

Now the important, now the next important thing; the next important thing is we have created this portfolio, we have created a situation where this portfolio is riskless; that means what, that means this portfolio is going to generate risk free rate of return. It is not that a risk free portfolio will generate zero return; even a risk free portfolio will generate positive return and that return is called the risk free return.

(Refer Slide Time: 22:18)



So, that is the next point; that means what, that means V t, V t must be equal to V 0 into e to the power r t. Remember r is the continuous compounded risk free rate, so we can write V t is equal to V 0 e to the power r t. Now simplifying this, we can write V t is equal to f u plus delta S u is equal to f d plus delta S d is equal to V 0 e to the power r t.

And we also know the value of delta, we worked out is given by this expression; delta is equal to f u minus of negative of f u minus f d divided by S u minus S d. Substituting this value of delta and simplifying a little bit, what we get is the equation right at the bottom of your slide. It is simple algebraic simplification, nothing more; you are simply substituting the value of delta and we are using this expression for V t equal to V 0 e to the power r t.

(Refer Slide Time: 23:28)

$$f_0 = e^{-rt}\left[\frac{f_t^d S_u - f_t^u S_d}{S_u - S_d} + \frac{\left(f_t^u - f_t^d\right)}{S_u - S_d} S_0 e^{rt}\right]$$

$$= e^{-rt}\left[f_t^d\left(\frac{S_u - S_0 e^{rt}}{S_u - S_d}\right) + f_t^u\left(\frac{S_0 e^{rt} - S_d}{S_u - S_d}\right)\right]$$

$$= e^{-rt}\left[q_d f_t^d + q_u f_t^u\right] = e^{-rt} E_Q\left[f\left(S_T\right)\right] \text{ where}$$

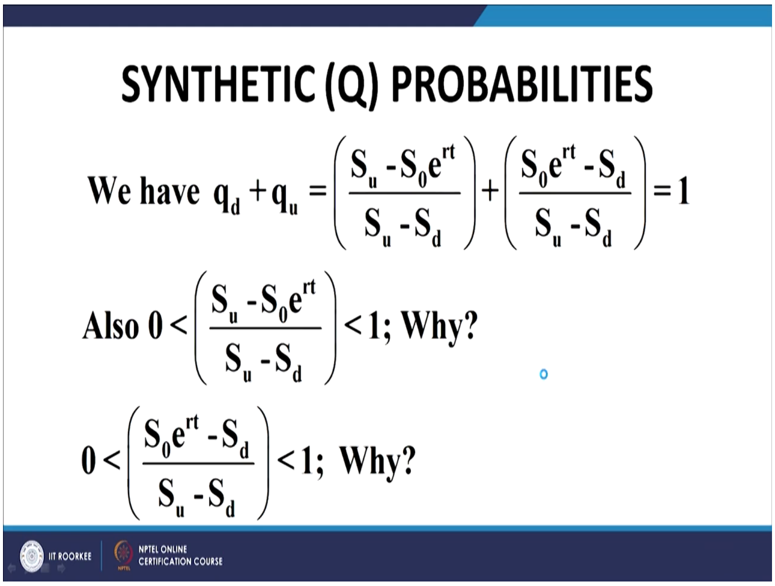
$$q_d = \left(\frac{S_u - S_0 e^{rt}}{S_u - S_d}\right) \text{ and } q_u = \left(\frac{S_0 e^{rt} - S_d}{S_u - S_d}\right)$$

Again further algebraic simplification gives us, enables us to work out the value of f 0. And what we find is that, f 0 can be written in the form and given in the equation which is written here e to the power minus r t into q d f d plus q u f u.

And now we can write this as, we can write you remember q d f d plus q u f u; we can write this as e to the power minus r t which is already there into E q f of S T. What is f of S T? F of S T is the is the functional dependence of the derivative on S T which is f S t is equal to f d, if f S u is equal to S u and f S d is equal to S d f d.

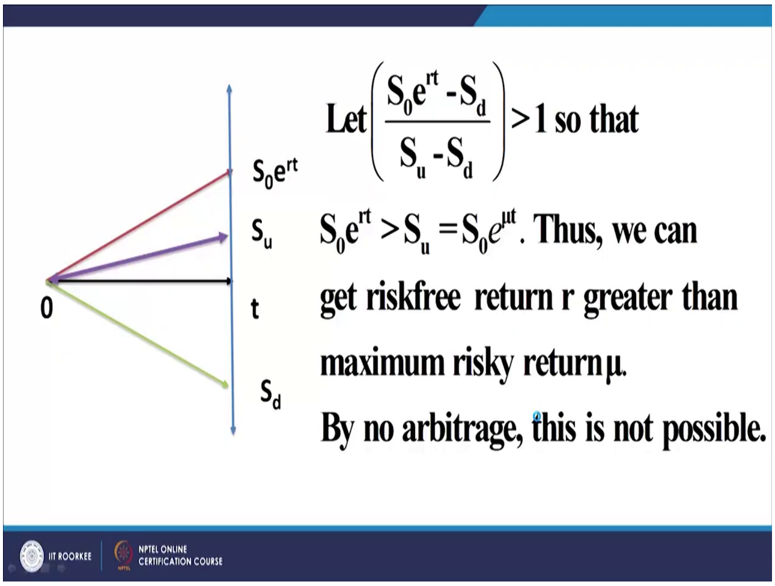
So, that functional dependence is captured here and E Q I will come back to it in a minute; for the moment if you compare the second last and the last equation here, you find expressions for q d and q u as the bottom expressions on the slide.

(Refer Slide Time: 24:54)



Now from the expressions that we have for q d and q u, that we have carried out from the previous slide from this slide; q d we have got here, q u we have got here, q d you have got S u minus S naught e to the power r t upon S u minus S d, and similarly q u we have got here, we clearly find that q d plus q u is equal to 1. And secondly, very interestingly we find that, q u and q d both lie between 0 and 1.

(Refer Slide Time: 25:34)



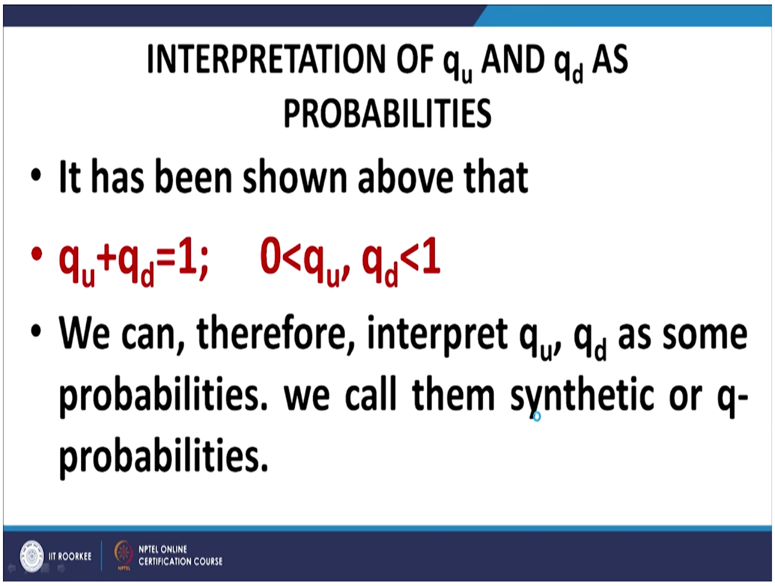
Why is that? Let us see, now let us assume that if possible S 0 e to the power r t minus S d that is this is q u; let us assume that this lies this is greater than 1. What does it mean? If you simplify this, then what it means is S 0 e to the power r t is greater than S u; S 0 e to the power r t is greater than S u. What is S u? S u was the highest price that the stock could take at t equal to t.

Remember the stock could take only two values, either down value S t or up value S u; S u was the highest value that it could take. And remember the stock is risky, it could take either the upper value or it may in the unfortunate case; it may take the lower value as well, S d also. So, if it takes if you are saying that S 0 is e to the power r t is greater than S u; what we are saying is that, the risk free return, the risk free return is greater than the best possible return that the risky asset can provide to you.

I repeat this is fundamental; what we are saying is that the risk free return is greater than, because S u is the maximum value that the stock can take and that means that the risk free return the, that means the return corresponding to S u is the maximum return that you can get on the stock. And that means what; that means that, S, that means the risk free return is greater than the maximum possible, the best possible scenario on the risky asset. Obviously, why would you invest in the risky asset in that case?

Why not invest in the risk free asset? So, again because of arbitrage the pressure, the selling pressure on the risky asset will increase, the buying pressure on the risk free asset will increase, and obviously there is there will be a disequilibrium. And at equilibrium we will have to have that, this the risk free return has to be lower than the maximum return on the risky asset; it must be so, similarly the other inequalities may be proved.

(Refer Slide Time: 28:13)

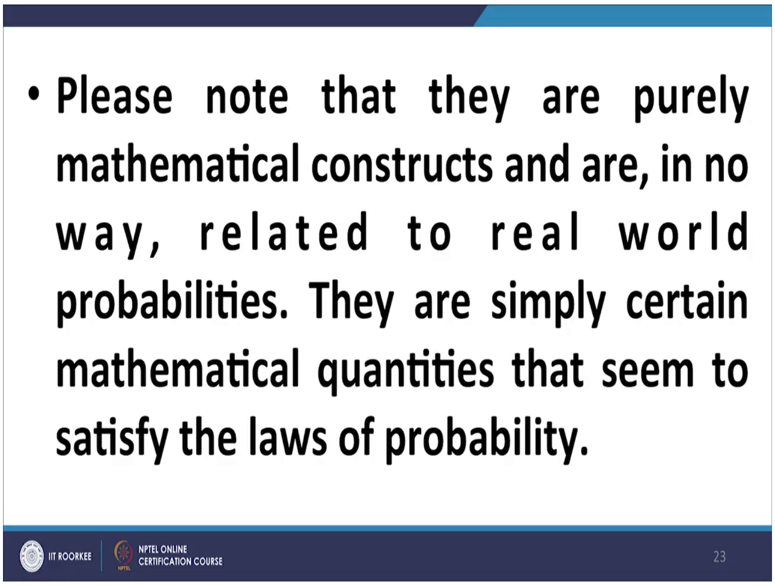


So, what have we got now? We have got q u plus q d equal to 1 and we have also got 0 is less than q u and q d is less than 1. In other words both q u and q d lie between 0 and 1 and the sum of them is equal to 1. What does it mean? It means that we can interpret q u and q d as some sort of probabilities.

Let us call them for the moment, let us call them synthetic probabilities. For the moment remember, they are simply mathematical constructs, they are not real life probabilities; we have not done any experimentation and then arrived at these probabilities on the basis of experimental data.

What we have, what we have seen is that, while arriving at this pricing of options; we arrived at certain mathematical constructs, certain mathematical quantities which show or which behave in a manner in which probabilities do behave, and therefore we give them the name synthetic probabilities.

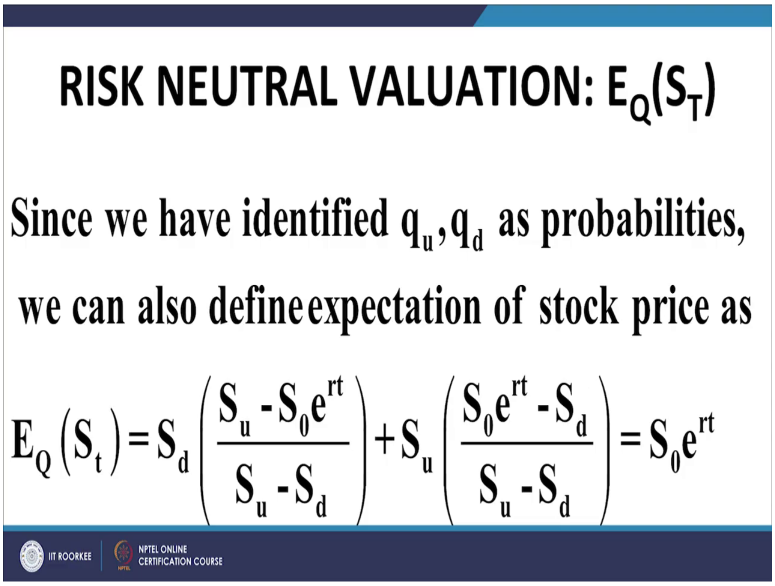
(Refer Slide Time: 29:23)



Let us further explore this; for the moment we have called them a synthetic probabilities, let us further explore this.

Let us work out the expected return on the stock based on the synthetic probabilities. Let us work out the return on the stock based on synthetic probabilities. If you do that what we find is that, we get the risk free return. The expected price of the stock at time t is given by S naught e to the power r t under q probabilities.

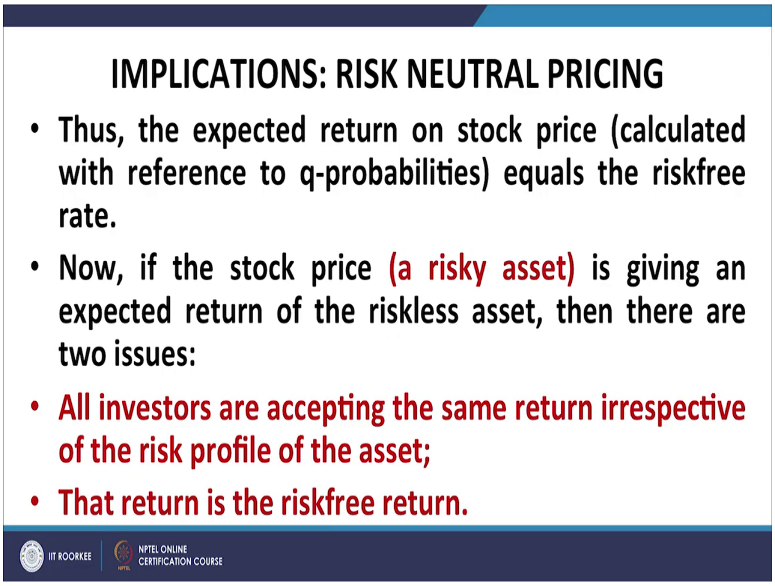
(Refer Slide Time: 29:51)



We also call them q probabilities; we also call them synthetic probabilities for the moment, so the expected value of the stock price at maturity of the option under synthetic probabilities shows that the stock will generate a return of risk free rate.

So, that is what we have for the movement. Now, what does it mean? The meaning is profound, the meaning is very far reaching; the meaning of this is that.

(Refer Slide Time: 30:21)



Now, this please understand the stock is epitomizes the risky asset; the stock is what we are modeling risk with, all the risk is captured by the stock. So, that being the case; when the stock which is the risky asset, which covers or which symbolizes the risk is generating, is generating risk free return.

What does it mean? It means that, the investors are indifferent to, they just do not care about the riskiness of the asset when they are investing; whatever is the riskiness of the asset, because the stock can, there may be many stocks in the market, and but the return that is being generated or the return that is being provided is the risk free asset in each case.

That means that, the investors are investing not on the basis of riskiness, riskiness is no consideration for them; and therefore we call this as a risk neutral world and we call this world, we call this probabilities as risk neutral probabilities. So, that is one fall out, immediate fall out of this concept of the q probabilities; that this q probabilities represent risk neutral probabilities, they represent probabilities in a risk neutral world.

In a world where all investors are going to invest purely on considerations of return; they are all in different to risk, they do not care about risk and they base their investment decisions purely on return.

That is number 1, number 2 what happens, what happens as if as a corollary of number 1. As a corollary of number 1, let us assume there are two assets; asset a and asset b. Asset, let us assume that asset b provides a higher return than asset a, although asset b may have much higher risk; but the investor is not concerned with risk, so he will immediately invest in asset b whatever may be its risk profile.

And the implication of this is that, due to arbitrage the return that would be provided in each case, on each asset, on every asset would be simply the risk free asset; we will continue from here.

Thank you.