Path Integral Methods in Physics & Finance
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Lecture - 50
Spinor Fields Path Integral

Welcome back. So, now, we take up the Fermi Dirac situation, Fermi Dirac scenario. As you know Klein Gordon equation deals with particles scalar particles, that is particles having no spin and massive, they have non-zero massives. Now we talk about Fermi Dirac particles which have spin half particles and which are usually called fermionic particles.

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So, we shall develop this theory right from scratch and we shall work onto the fermionic path integral that is our agenda for the current lecture.

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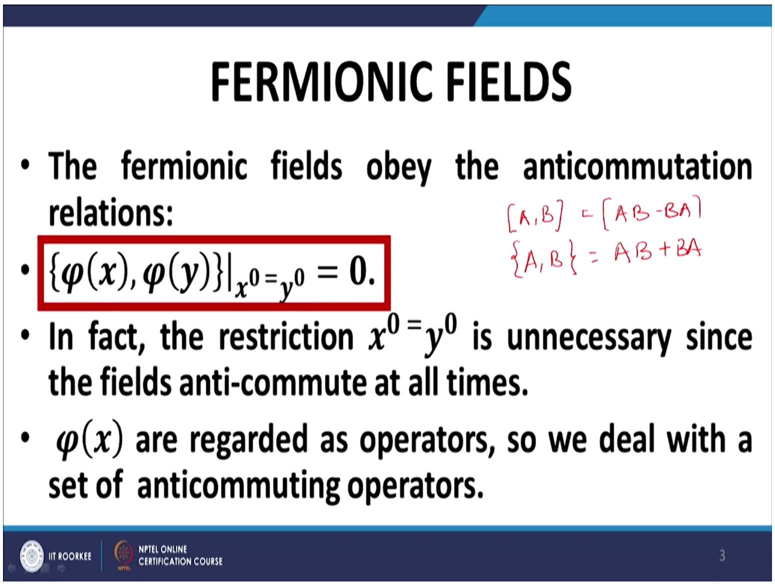

So, let us proceed. Now for the fermionic fields they obey the anticommutation relations. remember we are so far a customed with the commutation relations now, we talk about anticommutation relations.

Recall that commutation relations are given by if you have commutation brackets between A and B these are two operators, then they are equal the this is equal to A B minus B A. But when we talk about anticommutators, we represent them by curly brackets A comma B these are again operators remember they are called anti commuting operators and A B plus B A.

So, in the case of fermionic fields, the they obey the anticommutation relations phi x phi y at a given point in time is equal to 0. Indeed, you may also afford to omit the subscript x 0 equal to y 0; because this relation has to be met at all points of time.

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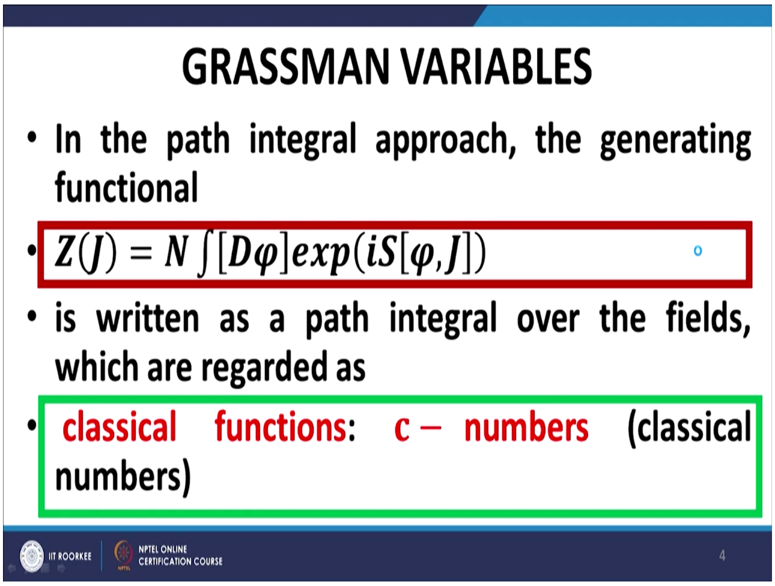

Now, the important things here as I mentioned before in the earlier lecture also, when we talk about the path integral or the generating functional for the full green functions, which is usually the expression given in the red box in the middle of your slide, the phi set appear here are classical numbers are c numbers they are not operators.

So, when we deal with the corresponding path integral or the corresponding generating functional in the context of fermionic fields of fermionic particles, we need to replace them by fermionic or anticommuting variables not anticommuting operator and these anticommuting variables are called Grassman variables.

They were first introduced into the literature by Grassman through a paper some time probably in the 19th century. And they have been extensively used for the management of fermionic fields which obey in the Pauli exclusion principle. We will see how the interesting this relation this Grassman variables naturally lead to the Pauli exclusion principle in fact.

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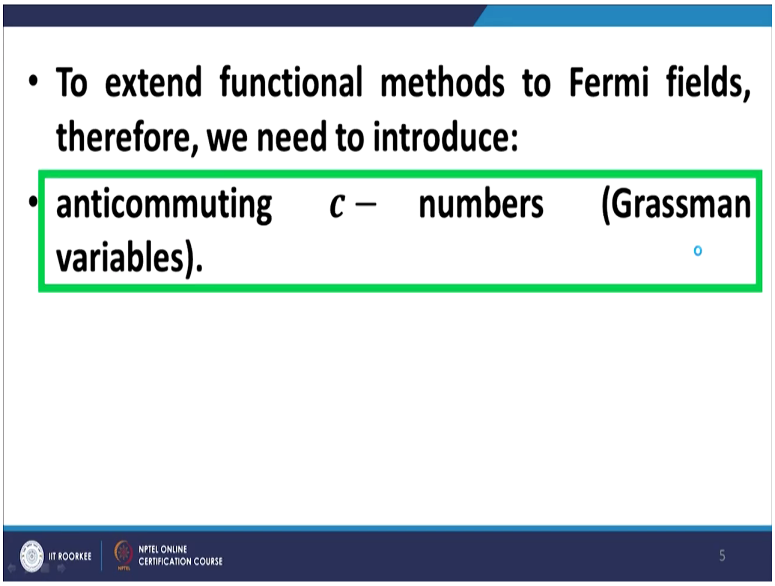

So, we extend the functional methods to Fermi fields through the Grassman anticommutings c numbers are anticommuting variables remembered not operators.

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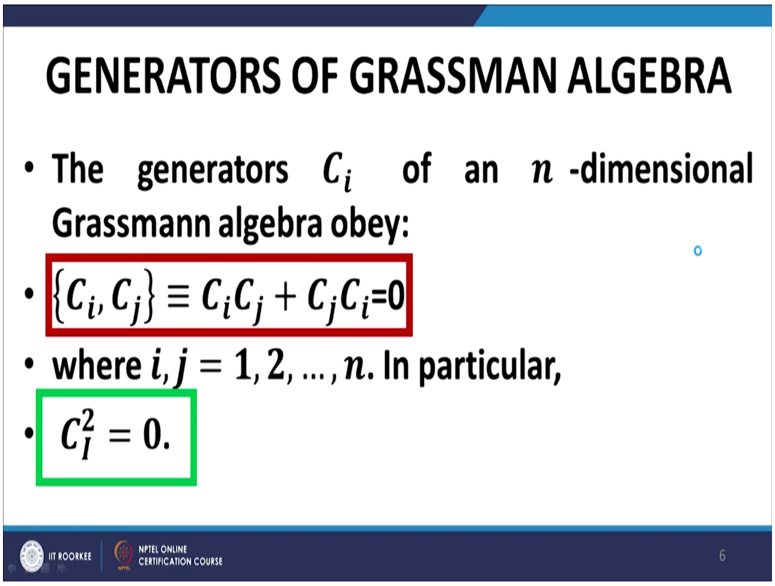

The generators of an n dimensional Grassman algebra of a this commutation relationship anticommutation relationship I am sorry obey this anticommutation relationship which is given in the red box here. And as a corollary to this anticommutation relationship for i equal to j it immediately follows that C i square is equal to 0 for i equal to 1 to up to n, n dimensional algebra.

So, this is given in the green box at the bottom of your slide generators of an n dimensional algebra obey. In general for i n equal to j they obey the relationship given in red box and for i equal to j they obey the relationship given in the green box here.

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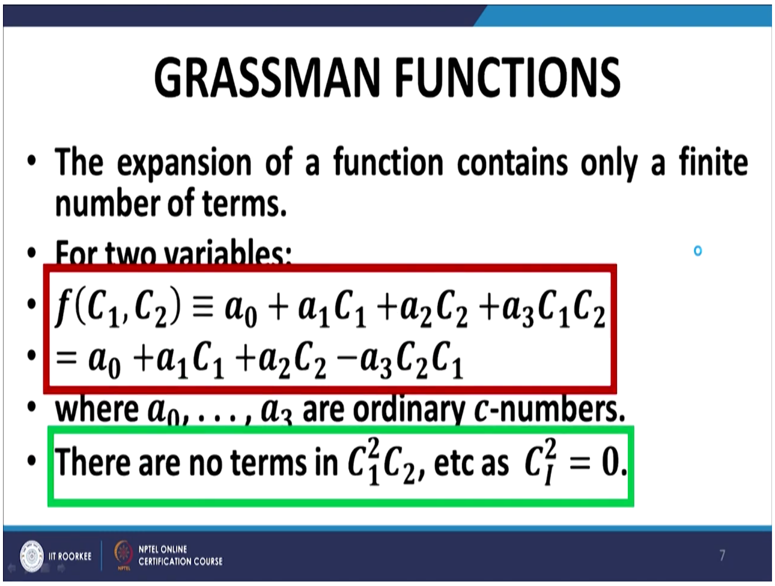

Now let us introduce the concept of Grassman functions. In the important thing is the expansion of a Grassman functions a polynomial sort of expansion in of the Grassman functions contains only a finite number of terms. For example, in the context of the expansion of a function of two variables, function of two variables we can have only an expansion of the form that is shown in the red box.

Either first expansion or the second expansion which is in fact, in first expansion obtained by reversing the order of the last term. You will immediately notice that any further expansion vanishes because of the condition C I square equal to 0. For example, any at the next term would either involve C 1 square or C 2 square or and both of them would vanish and therefore, our expansion of a of a function of two Grassman variables can comprised of only these four terms.

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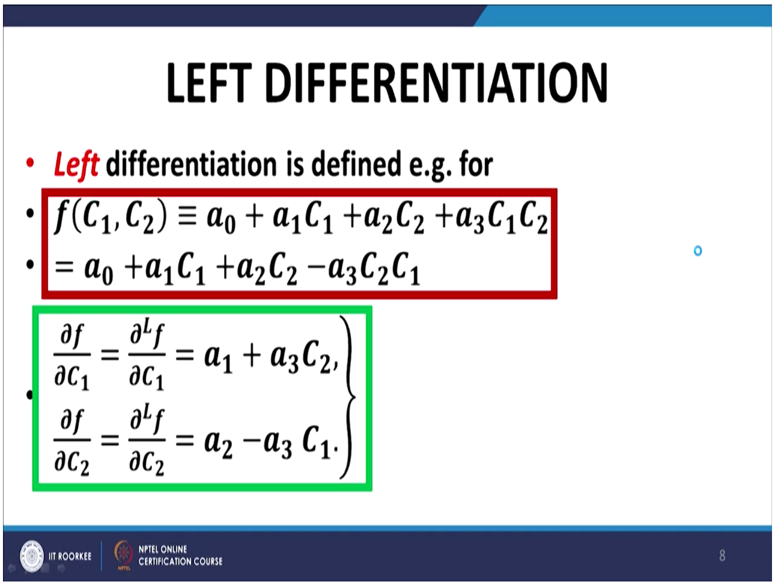

Now, we introduce the concept of differentiation. Now differentiation in the context of Grassman variables or Grassman functions of anticommuting variables can take the form of two types it can be left differentiation, it can also be right differentiation. Usually in the absence of explicit mention, it is assumed that the differentiation is left differentiation.

So, let us start with left differentiation. If you are given a function of the form that is in the red box here which is what was there in the previous slide and which is in some sense a general function of two variables. Then we define the left differentiation as if you look at it the a 0 term is constant.

So, it goes differentiation of C 1 with respect to C 1 gives us 1. So, the differentiation of the second term with respect to C 1 gives us a 1 C 2 is independent assumed independent of C 1.

So, the third term gives us 0 and C and the fourth term gives us a 3 C 2 a 3 C 2 the derivative of C 1 with respect to C 1 is 1.

Now, and similarly if I want to work out this is the derivative of f with respect to C 1 the left and left derivative of f with respect to C 1. Let us now work out the left derivative of f with respect to C 2. For this purpose we need to write the write the last term in the form that is given in the second equation plus a 3 C 1 C 2 needs to be written as minus a 3 C 2 C 1 using the anti commuting property.

And then of course, we can differentiate as we did in the earlier case and what we get is a 2 minus a 3 C 1. So, the results are tabulated here in the green box at the bottom of the slide.

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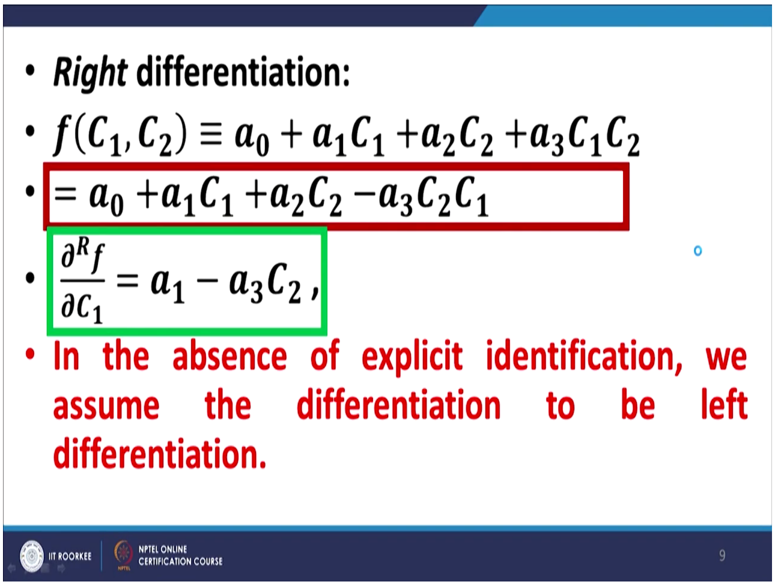

Similarly we can define right differentiation for the purpose of right differentiation. For example, for the purpose of right differentiation with respect to C 1 we will need to write the last term the fourth term in the reverse order and then do the differentiation backwards operating from the back to the front.

We from the right to the left and therefore, we get a 1 minus a 3 C 2 as the right derivative of f with respect to C 1. Now properties of the derivative, properties of differentiation you can clearly see that the first derivative with respect to C 1 gives us a 1 plus a 3 C 2 we have done it just now.

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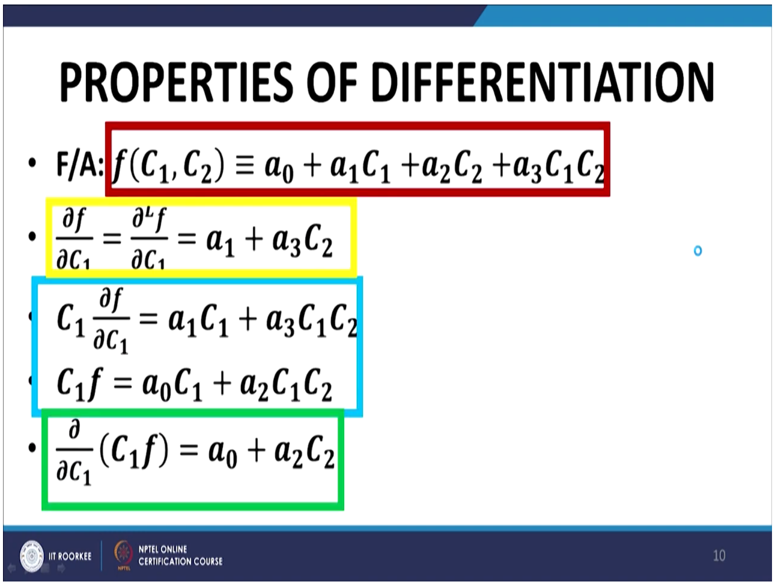

Therefore, C 1 into the left derivative of f with respect to C 1 gives us a 1 C 1 plus a 3 C 1 C 2 and C 1 f on the other hand gives us a 0 C 1. If you can if you multiply throughout by C 1 it gives us a 0 C 1, a 1 term vanishes because C 1 square is equal to 0 plus a 2 C 1 C 2 you are multiplying by C 1 from the left and again in the fourth term also vanishes. So, clearly we have the derivative of C 1 f with respect to C 1 gives us a 0 plus a 2 C 2,

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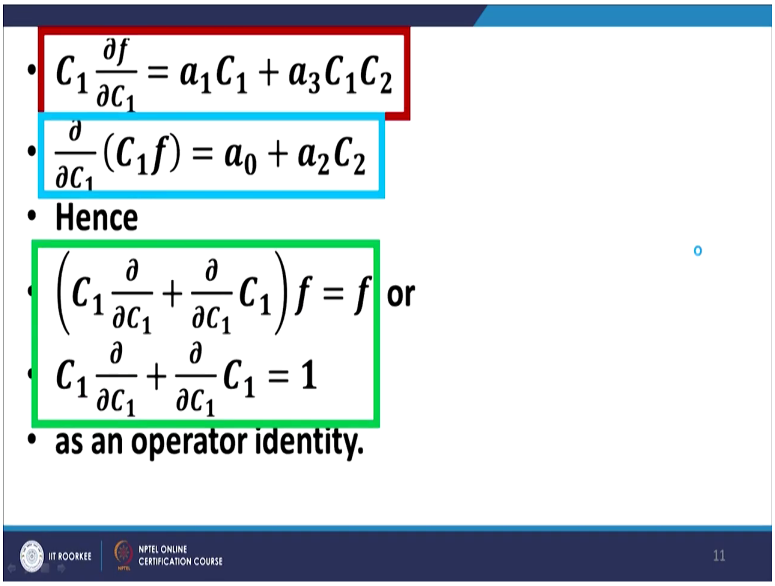

And that leads us to a very important relationship C 1 we have got this from the previous slides the expression in the red box and the blue box that is what we have derived in the previous slide.

And from this what we find is that the expression that we have gives us that C 1 into the derivative with respect to C 1 plus derivative with respect to C 1, C 1 is equal to 1 this is an operator identity this is a very important in relationship connecting the differentiations with respect to C 1 with the corresponding variables C 1.

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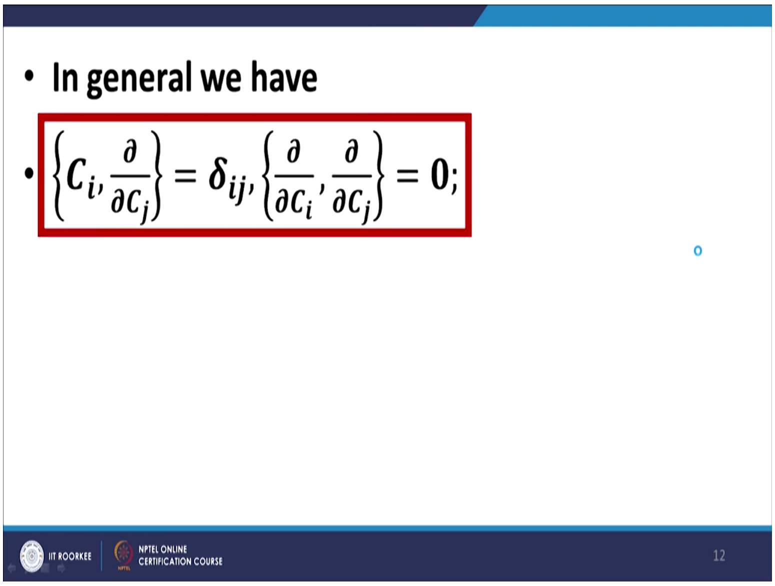

In general, what we have is derivatives anticommute among themselves. The anticommutator of C i with its own derivative is equal to 1. The anticommutator of C i with the derivative of any others generator is equal to 0; the anticommutator among generators is equal to 0 derivatives of generators is equal to 0.

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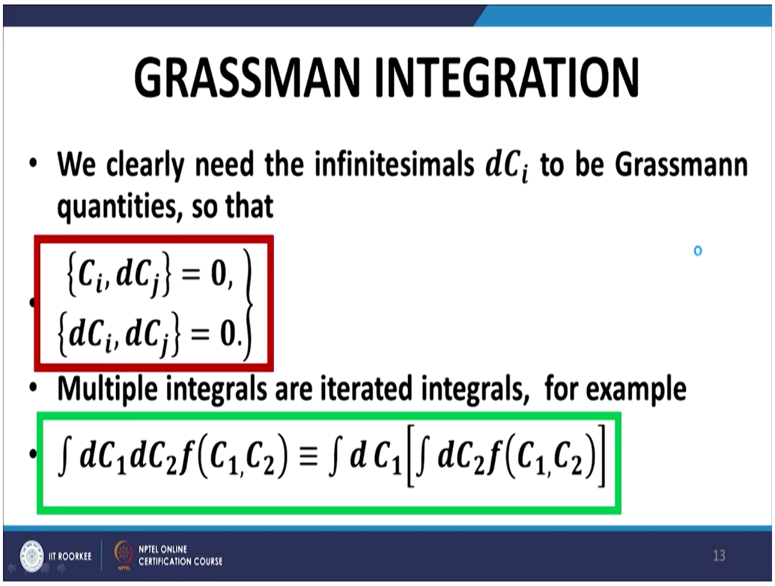

Now we talk about Grassman integration. For the purpose of Grassman into integration we start with the relations that the infinitesimals with d C i must talk obey the Grassman relations must be Grassman quantities. Therefore, we must have the relations that are given in the red box at the in the middle of your slide; the anticommutator of C i with d C i will be 0 and we will anti commutator of C i d C j with also be 0.

In other words it for all j equal to or an equal to i, the anticommutator is vanish similarly the amp anti commutator between d C i and d C j also vanish. Multiple integrals are interpreted as iterated in integrals as is the case in usual calculus and we have integral of d C i d C 1 d C 2 f C 1, C 2 is equal to d C 1 integral d C 2 f C 1 C 2.

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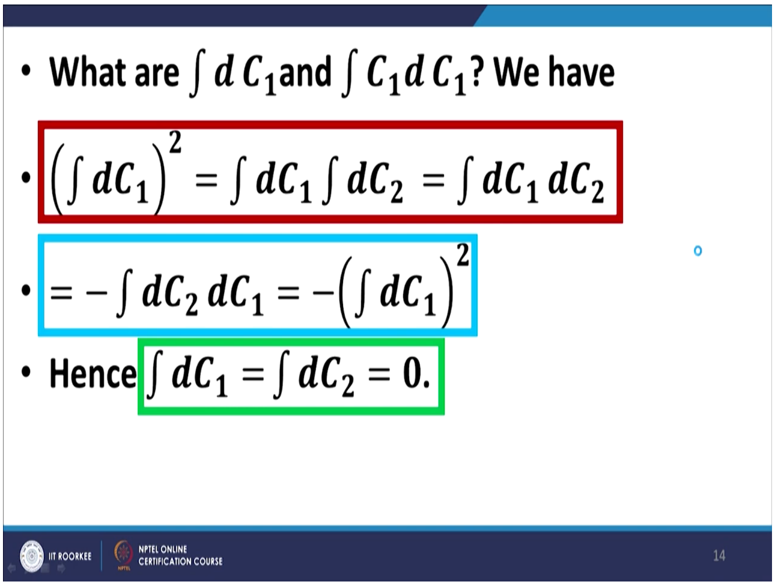

Now, what are integral d C 1 that is very interesting. Let us start with integral d C 1 square we write it as integral d C 1 integral d C 2 with the condition that we shall put d C 1 equal to d C 2 at the end of our calculations. This is nothing, but using the anticommutator, we can write this as d C 1 and d C 2 integral in the with in the is using the property of multiple integrals.

I can write it as integral d C 1, d C 2 that becomes integral d C 2 minus d C 2 d C 1 using the anti-commutator and now I put d C 2 equal to d C 1. So, what I get is integral d C 1 minus integral d C 1 whole square. Let me retrace the steps integral this is important integral d C 1 square is equal to integral d C 1 integral d C 1 put C 1 put 1 equal to 2 with the condition that will replace it later.

So, that becomes integral d C 1 integral d C 2 the in because of the property of multiple integrals. I can write it as integral d C 1 d C 2 which I can replace that integral d C 2 d C 1 with the minus sign due to anticommutation and now I can write 2 equal to 1 and I get minus integral d C 1 square. And this at the therefore, I get integral d C 1 square is equal to minus integral d C 1 square at the end of the day and that implies that integral d C 1 is equal to integral d C 2 is equal to 0.

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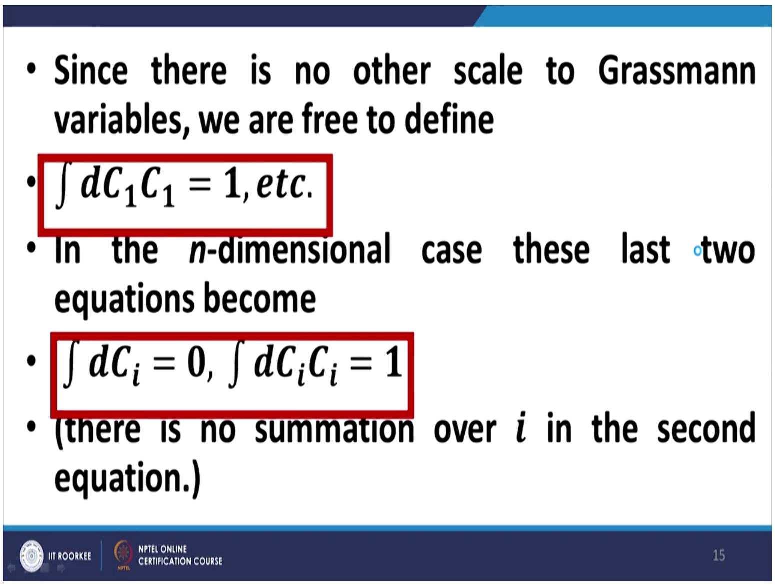

Since there is no other scale no other scale to Grassman variables we choose we choose integral d C 1 C 1 equal to 1 and so, on integral d C 2 C 2 equal to 1 and so, on. Now we move to the n dimensional case. In the n dimensional case what we what we have is on in analogy with whatever I have said earlier.

We get the relations integral d C 1 is equal to 0 d C i is equal to 0 now where i is equal to 1 2 0033 up to n and integral d C i into C i is equal to 1 for every i between including and between 0 to a sorry 1 to n.

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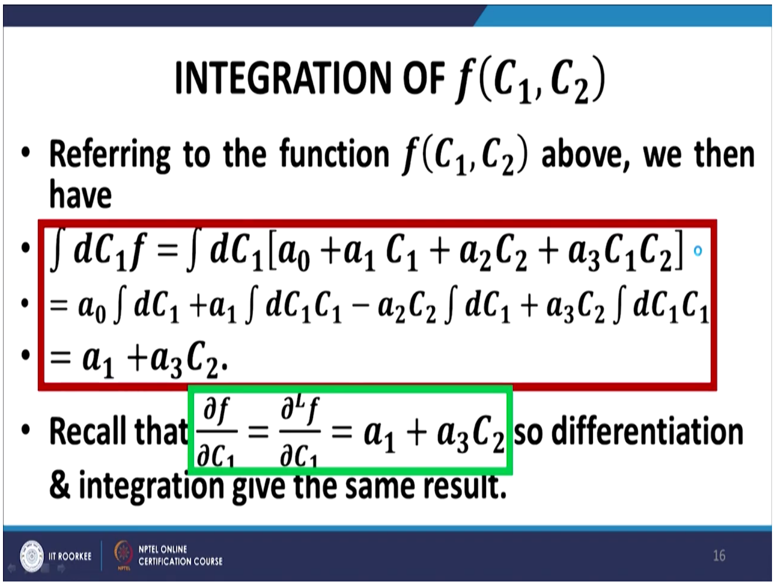

Now integration of f C 1, C 2 let us see what we get. Integration of f C 1, C 2 remember what is f? f is equal to a 0 plus a 1 C 1 plus a 2 C 2 plus a 3 C 1 C 2 that is the definition of f C 1 C 2.

. So, what we get is integral d C 1 of f integral of f d C 1 in other words is equal to integral d C 1 into a 0 plus a 1 C 1 plus a 2 C 2 plus a 3 C 1 C 2 this is equal to a 0 integral d C 1 plus the d C 1 distributes over. All these terms of the function f and now we use the property d integral d C 1 is equal to 0 integral C 1 d C 1 is equal to 1 integral d C 1 in the third term is also 0 integral d C 1 C 1 in the fourth term is 1.

So, what we are left with is a 1 plus a 3 C 2 and now recall that the derivative of f with respect to C 1 was also a 1 plus a 3 C 2. So, at least as far as f C 1 C 2 is concerned the general function of two Grassman variables integration and differentiation give us the same result.

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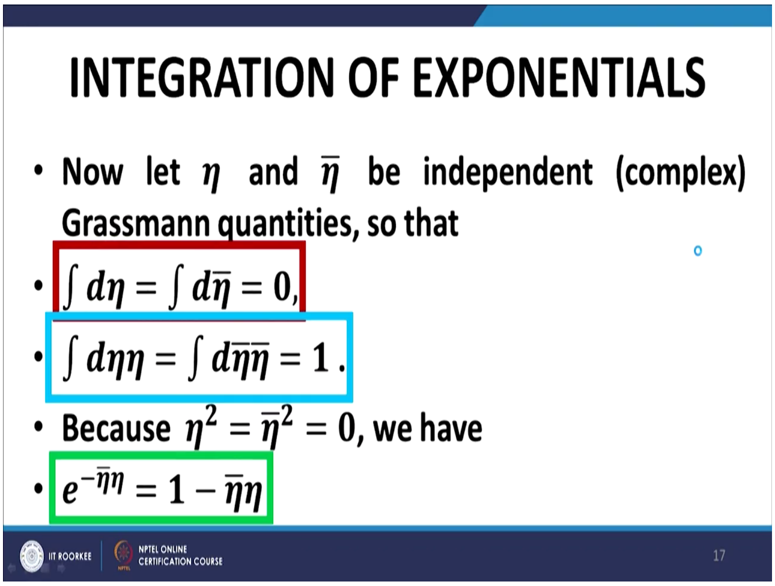

Now, we come to integration of exponentials. Let eta and eta bar be independent complex Grassman quantities, eta and eta bar be independent complex Grassman quantities. So, we have integral d eta is equal to 0, integral d eta bar is equal to 0 and d eta integral d eta is equal to 1, integral d eta bar eta bar is equal to 1.

But eta square is equal to eta bar square is equal to 0 therefore, we have e to the power minus eta bar eta is equal to 1 minus eta bar eta the for the remaining terms will be 0 because of the doubling of or the squaring of either eta or eta bar.

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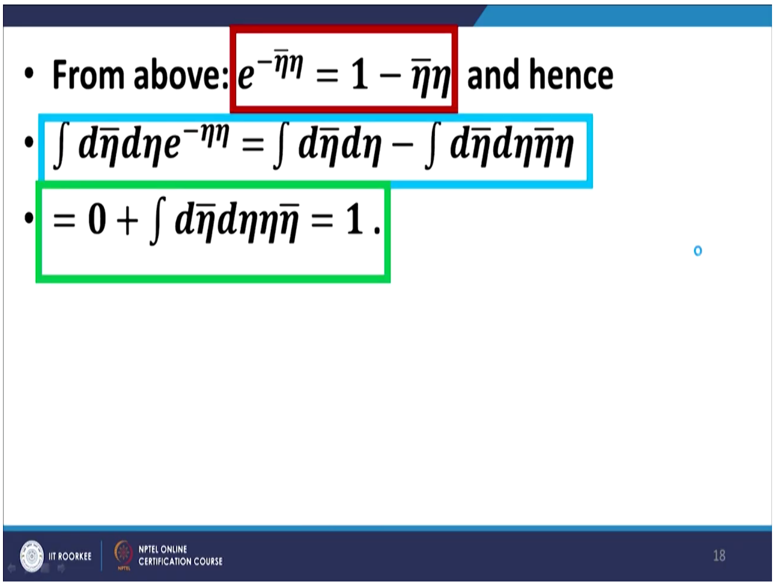

So, what we have is e to the power minus eta bar eta is equal to 1 minus eta bar eta. And therefore, the integral of d eta bar d eta e to the power minus eta bar eta is equal to integral d eta bar d eta minus d eta bar d eta bar eta the first term is clearly 0 and the second term is clearly 1. So, what we have is the integral of e to the power minus eta bar eta with respect to eta bar and eta is equal to 1.

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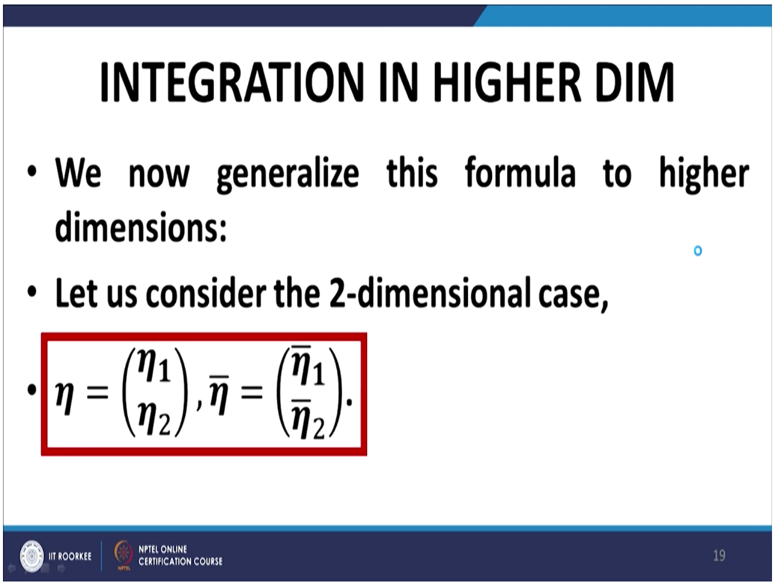

Integration in higher dimension we now generalize these formula to higher dimensions let us consider the 2 dimensional case. We write eta as vector eta 1 eta 2 and eta bar as the vector eta bar 1 eta bar 2, column vector eta 1 eta 2 and the column vector eta 1 bar eta 2 bar.

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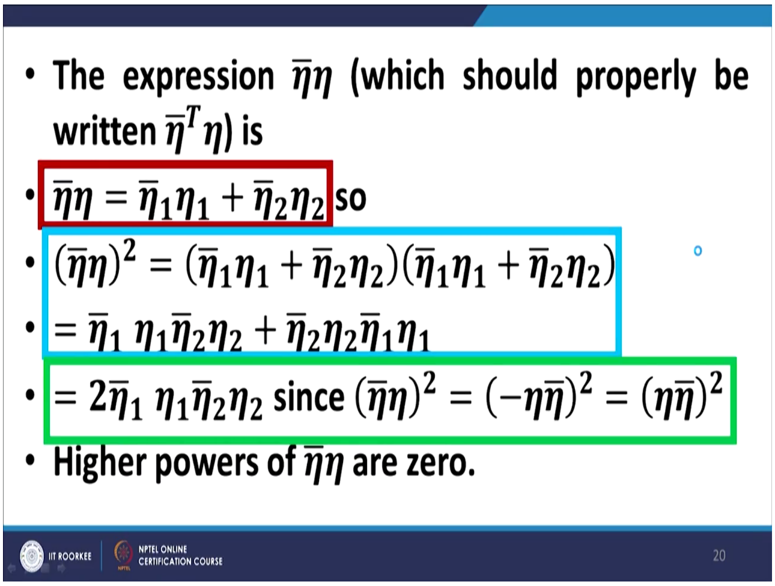

The expression eta bar eta is essentially eta bar transpose eta is and it gives us eta 1 bar eta plus eta 2 bar eta simple matrix multiplication and eta bar eta square is equal to eta bar eta 1 into eta bar eta 2 multiplied by the same expression. Again and when we simplify this expression the first term and the fourth term vanish and we are left with eta 1 eta 1 bar eta into eta 2 bar eta and the cross terms are there.

And we have eta 2 bar eta 2 and eta 1 bar eta 1 and these two terms because of the anticommutator operating twice they add to each other. And we have two eta 1 bar eta one eta 2 bar eta 2 because as you can see the anticommutator operates twice because of the squaring and the higher powers are obviously, 0.

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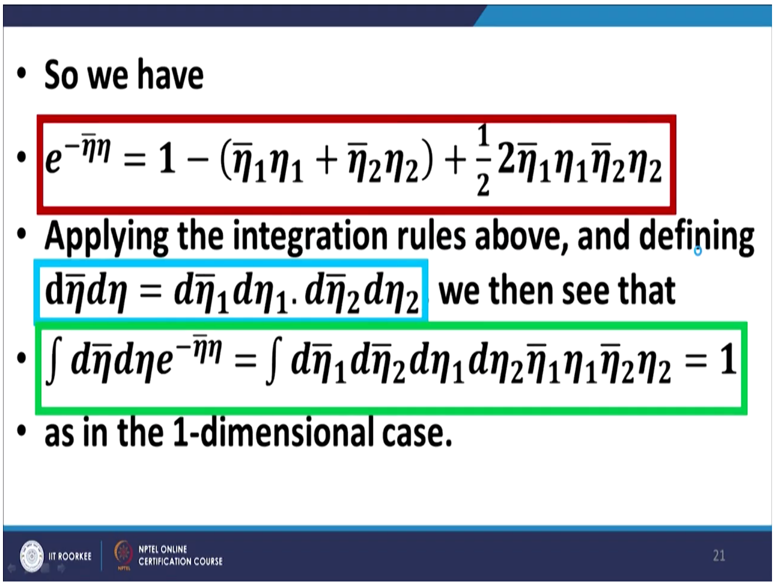

Therefore, what do we have for the e for the exponential minus eta bar eta is equal to 1 minus eta 1 bar eta plus eta 2 bar eta plus 1 by 2 into 2 eta 1 bar eta, eta 2 bar eta. Now we when we apply the integration rules that we have defined earlier and that is eta integral d eta while d eta bar d eta is equal and we use this expression d eta bar d eta is equal to d eta 1 d eta 1 d eta 2 bar d eta 2.

What we see that is, when we integrate this expression we have eta 1 bar eta 2 d eta 1 bar, d eta 2 bar, d eta 1, d eta 2 this is clear. And then we have when we do the integration when we do this integration clearly the first term one term vanishes, the second term also vanishes, the third term also vanishes and what we are left with is the fourth term and the fourth term gives us 1 only.

So, out of these four terms when I integrate this with respect to this this for infinitesimals, the first term vanishes because d eta 1 is the integral d eta 1 is 0 and d eta 2 is. So, the whole thing goes d eta 1 bar d eta 1 will take out to two terms, d eta 1 bar into eta 1 will be 1, d eta 2 bar into eta 2 will be 1.

I am sorry d eta 1 bar with d eta 1 bar will be 1, eta 1 bar with d eta 1 bar will be 1 eta 1 with d eta 1 will be 1, but the other two terms will be will be 0. So, again it is 0 the third term similarly will be 0 eta 2 bar with d eta 2 bar will be 1, eta 2 with d eta 2 will be1, but the other two terms will be 0.

So, again it will be 0 and the only term left is the fourth term which eta 1 bar will join d eta 1 bar give you give us 1, eta 1 will join d eta 1 give us 1, eta 2 bar will join d eta 2 give us 1 and eta 2 will join d eta 2 and give us 1. So, the net result will be 1 here again as in the 1 dimensional case.

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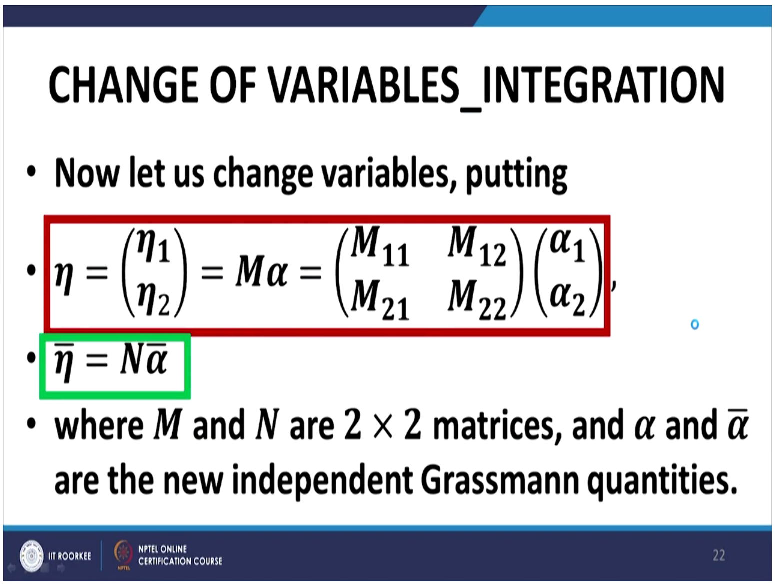

Now, change of variables what happens when you change variables? Let us say eta is equal to the column vector eta 1 eta 2 and this is equal to M into alpha where M is the matrix given by the expression in the red box here; M 11 M 12 M 21 M 22 and then alpha 1 alpha 2.

So, this is the change envisage the in other words the change envisage is eta 1 goes to M 11 alpha 1 plus M 12 alpha 2 and eta 2 goes to M 21 alpha 1 plus M 22 alpha 2 and eta. Similarly, eta bar goes to n alpha bar where n is another similar matrix n is n 11 n 12 n 21 and n 22.

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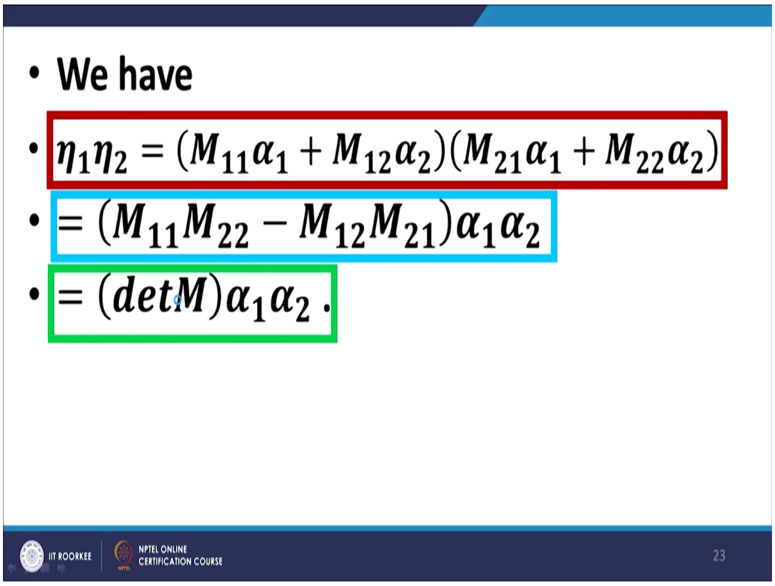

So, what we have is eta 1 eta 2 is equal to M the expression that is given in the red box here. And if you simplify this expression again alpha 1 into alpha 1 gives us alpha 1 square which is one alpha 1 into alpha 1 gives us alpha 1 square which is 0, alpha 1 into alpha 2 is retained.

And similarly alpha 2 into alpha 1 is retained that is nothing, but minus alpha 1 into alpha 2 and alpha 2 into alpha 2 is again 0 alpha 2 square is 0. So, what we end up with here is determinant M into alpha 1 alpha 2. So, eta 1 eta 2 is equal to determinant M into alpha 1 alpha 2 a very important expression we carry forward this one.

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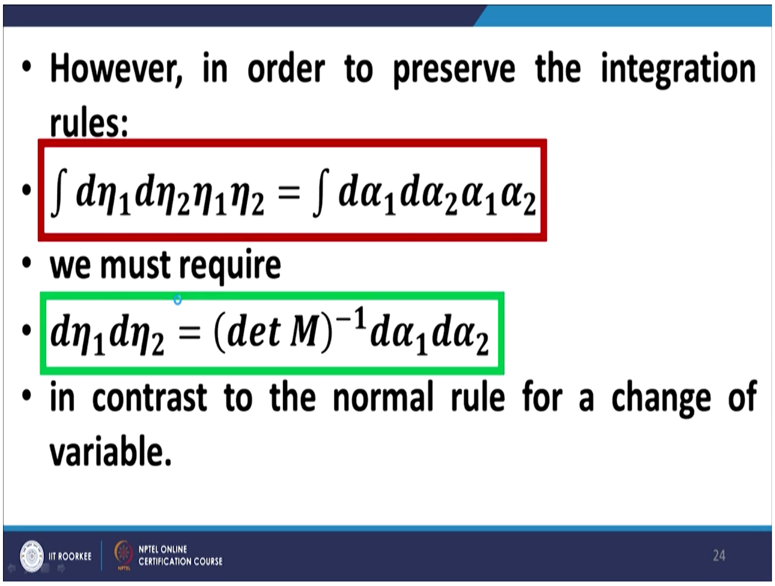

Now, in order to preserve the integration rules what we want is integral d eta 1 d eta 2 eta 1 eta 2 if you recall this is equal to this has to be equal to 1 and this is given as integral d alpha 1 and this has to be also equal to, in the integral d alpha 1 d alpha 2 alpha 1 alpha 2.

And this implies this implies that because eta 1 eta 2 from the previous slide, eta 1 eta 2 from the previous slide is equal to determinant M alpha 1 alpha 2. It clearly follows that d eta 1 d eta 2 must be equal to determinant M inverse d alpha 1 d alpha 2; d eta 1 d another important relationship d eta 1 d eta 2 is equal to determinant M inverse d alpha 1 d alpha 2 and recall eta 1 eta 2 is equal to determinant M alpha 1 alpha 2.

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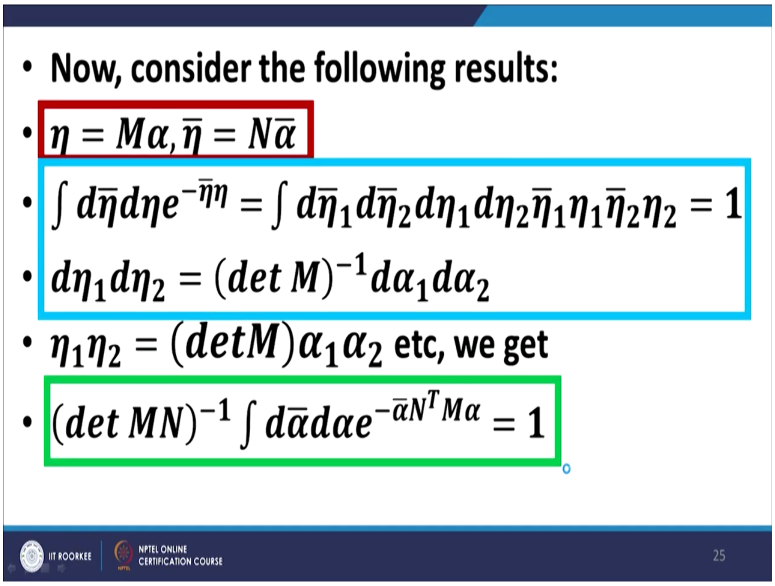

Now, consider. So, we have the following results, we have the following results. Eta is equal to M alpha, eta bar is equal to n alpha bar, eta bar eta e to the power minus eta bar eta is equal to this whole expansion and that is equal to 1 and we also have d eta 1 d eta 2 is equal to determinant M inverse d alpha 1 d alpha 2 and eta 1 eta 2 is equal to determinant M alpha 1 alpha 2.

So, simplifying this what we get is determinant M n inverse because this expression you see this expression is equal to 1. So, if I substitute everything all the etas in terms of the respective alphas in terms of the respective alphas and you can see here e to the power eta bar eta bar is what? Eta bar is n alpha bar. So, the transpose becomes alpha bar n transpose and eta is equal to M alpha which is here.

So, minus eta bar eta is equal to this expression in the superscript of e and all the rest we convert to alphas alpha 1 and alpha 2 and then we write them as alpha bar and alpha. On parallel lines to eta and eta bar using the respective rules, that are here in the blue box. And in the next equation here the equation below the blue box, we get the relationship which is here in the green box this is a very important relationship.

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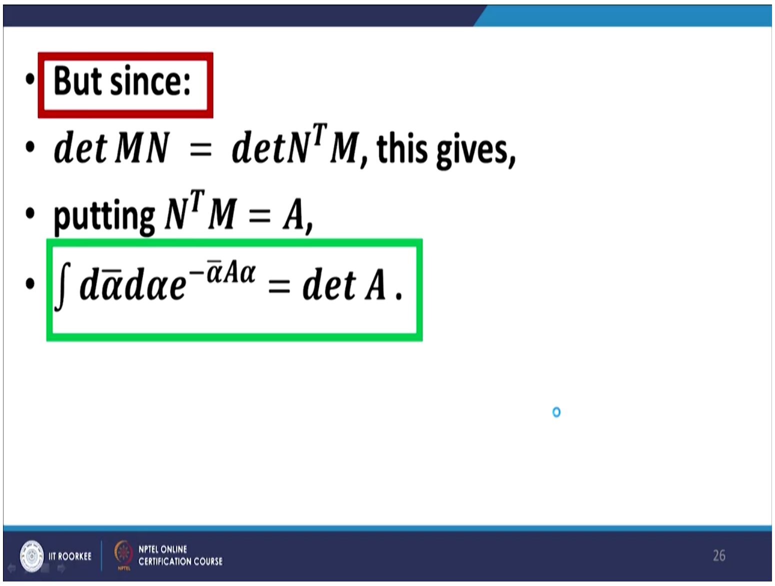

But determinant M N is equal to determined N transpose M this gives us N transpose M is equal to A if I put N transpose M equal to A I get integral d alpha bar d alpha e to the power minus alpha bar A alpha is equal to determinant A.

Determinant integral d alpha bar d alpha e to the power minus alpha bar a alpha is equal to determinant A. Remember look here this is N transpose M. So, we have substituted N transpose M N transpose M as A and that gives us a relation that is given in the green box here

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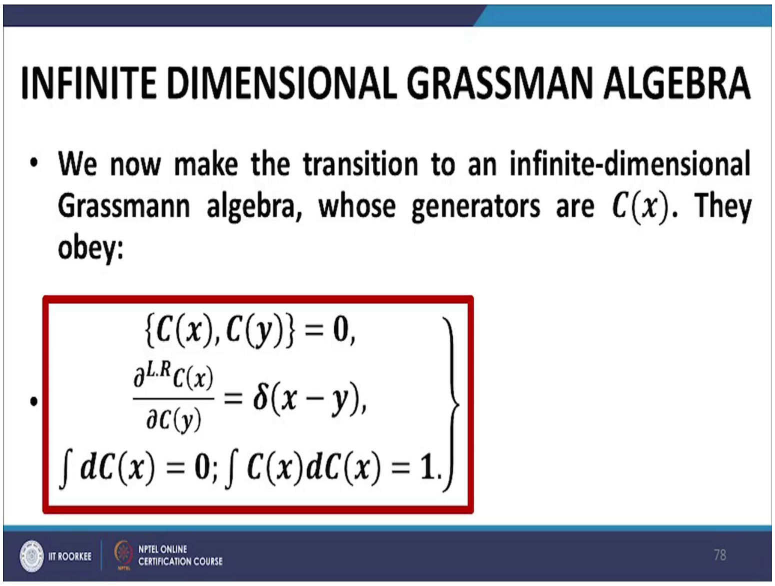

Now we move to infinite dimensional Grassman algebra, we move to infinite dimensional Grassman algebra. The generators of the infinite dimensional Grassman algebra and their derivatives appear there in the follow the relationship that is given in the red box here at the bottom of your slide. The quite straightforward generation generalizations of the expressions, that are given for the finite dimensional case.

Anticommutator of C x and C y C x and C y 0 the left and the right derivatives are C x with respect to C y gives us the delta function direct delta function integral of d is C x is equal to 0 integral of C x d C x is equal to 1 absolutely parallel relationships to what we had for the finite dimensional case.

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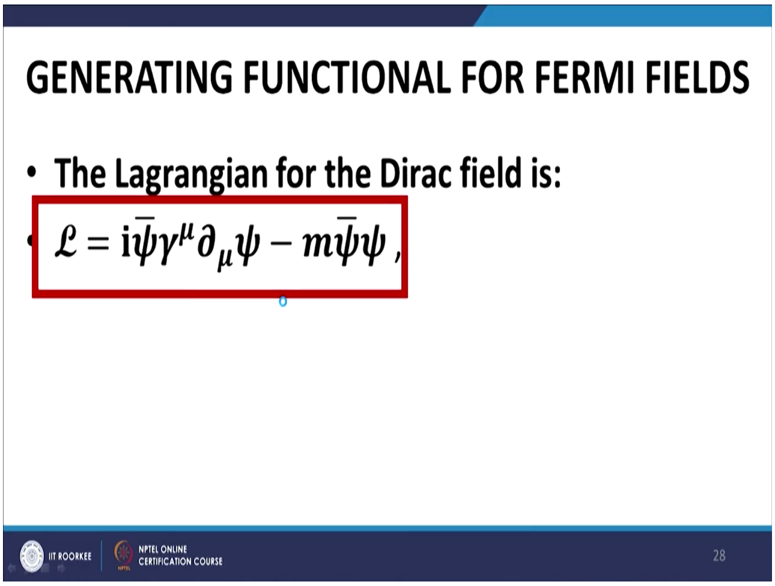

Now the generating functional for the Fermi fields or the Fermi Dirac fields, start with the Lagrangian. The Lagrangian for the Dirac field is given by the expression in the red box.

This is quite well known this is where we start our search for the path integral exposition or the path integral expose expression for the generating functional for the Fermi Dirac fields. We start with the Lagrangian and this Lagrangian is well known from the canonical formulation and the Euler Lagrangian equations.

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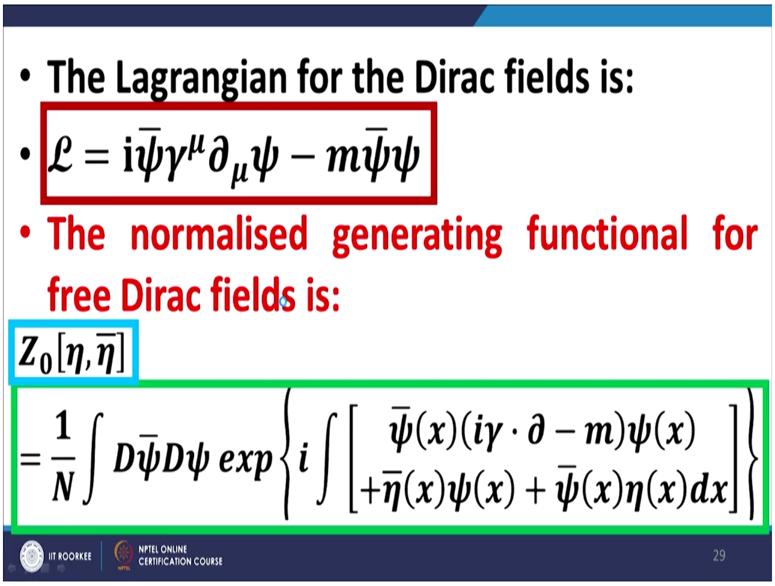

So, this is the Lagrangian for the Fermi Dirac fields that we obtained from the previous slide. And, the normalization that we have here if you look carefully is given by Z 0 and in the normalization we have I am sorry, I will come to the normalization, but before the normalization they normalize generating functional for the Dirac free Dirac field can be written using this Lagrangian in the form which is given in the green box at the bottom of your slide.

This is remember this is the generating functional for the free Dirac field and that is why the suffix is 0 is here, 0 n eta eta bar these are the sources ok. These are the sources and the Lagrangian is given by the middle term i gamma del d minus M where gamma or the gamma matrices Dirac gamma matrices. And so, this is the normalized generating functional for the free Dirac fields.

And, remember this is the Lagrangian the upper equation the red box equation is the Lagrangian, using this Lagrangian. Using the introducing the various sources and using the anti-computing field functions we get the expression for the generating functional which is given here for the free field please note that no interaction terms are there.

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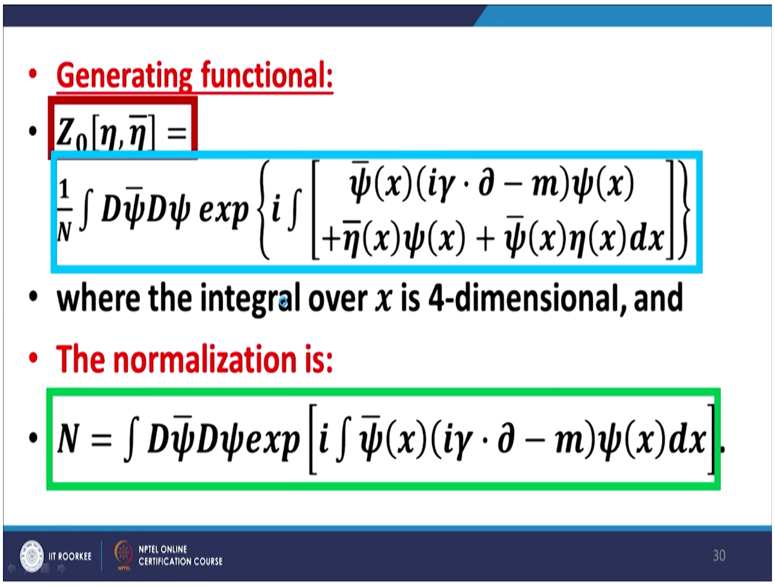

And so, this is the generating functional and the normalizer is given in the green box or the bottom of this slide. Please note in the difference between the two is clearly that the sources are absent.

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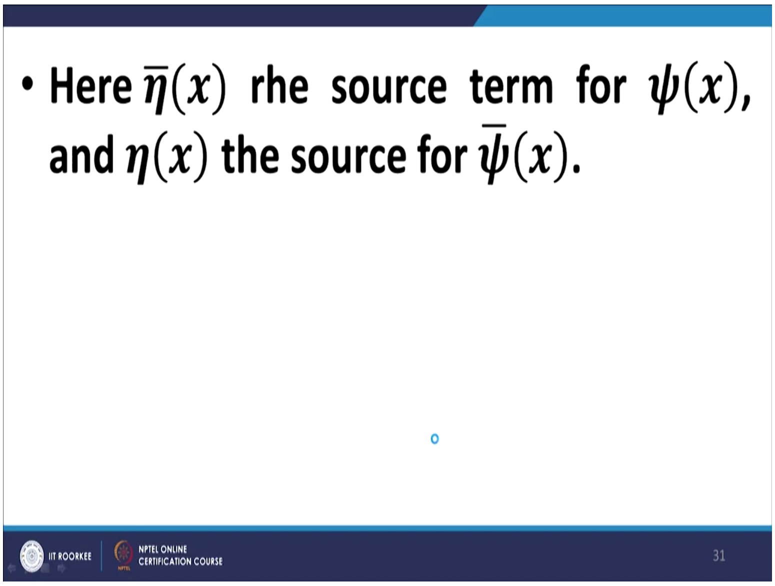

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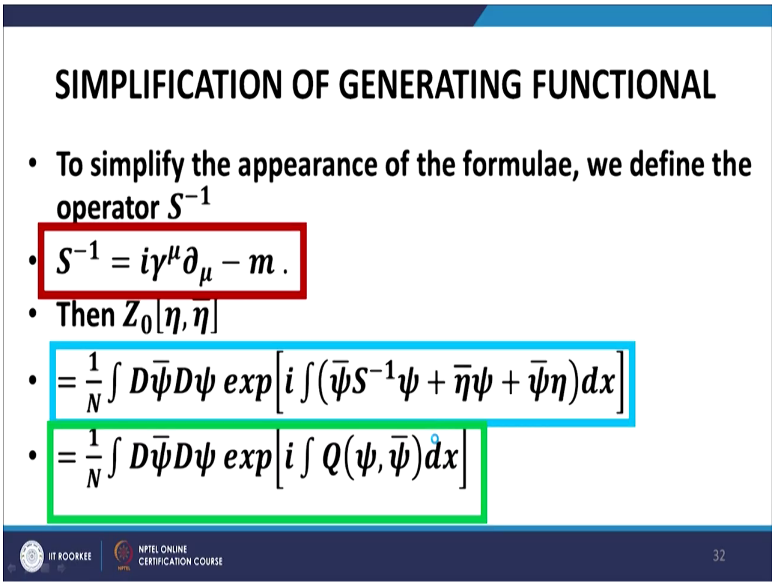

So, the normalization is with respect to the sources setting j equal to 0 gives us the normalizer for the generating functional. And, eta bar t is the source term for phi x and eta x is the source term for phi bar x. So, from here we will continue in the next lecture.

Thank you.