Path Integral Methods in Physics & Finance
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Lecture - 46
Propagator Properties in Minkowski Space

Welcome back. So, let us continue from where we left off. What I will do now is I will work out explicitly the expression for the Feynman propagator. The other approaches or the other scheme of singularities would be done similarly. And I leave with them as the exercises for the viewers.

But this is the most important propagator that appears in the context of quantum field theory. We will work that out explicitly.

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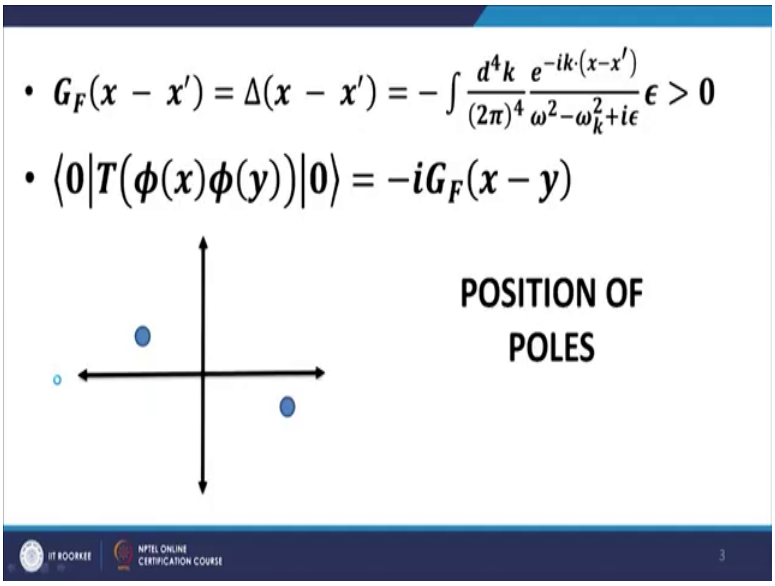

The scheme of poles is given in this diagram. We have a pole here in the second quadrant and we have a pole here in the fourth quadrant in the case of Feynman propagator. The expression for the Feynman propagator is given in the upper right hand corner of your slide. And this propagator is related to is usually written with the suffix F to identify it as the Feynman propagator.

And it is related to the time order product by the with the pre factor of minus i. When the Feynman propagator is written in the form which is given in the expression at the top of your slide which is so far. And in the future I will be denoting by delta x minus x dash.

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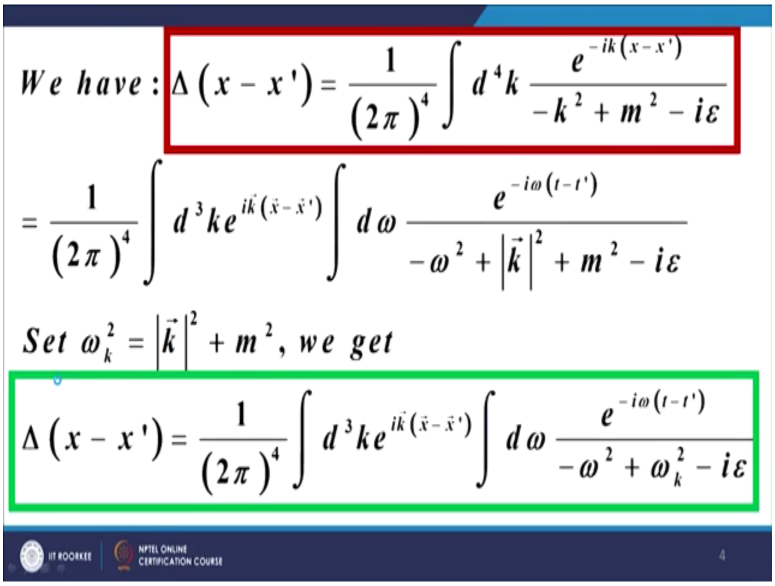

What we are going to do here is fundamentally; we are going to split up the propagator into the time component and the space components. And we will do explicit integration over the time components and that is the object of this exercise.

So, let us first split this propagator into two parts. We introduce omega k square equal to mod of k square that is the space like part plus the plus m square. The space part of the wave vector and plus m square we write as omega k square.

Using this expression we can and omega 0. So, I am sorry; k 0 we write as omega and we write k 0 square as omega square and this is what we get the expression that is here in the green box.

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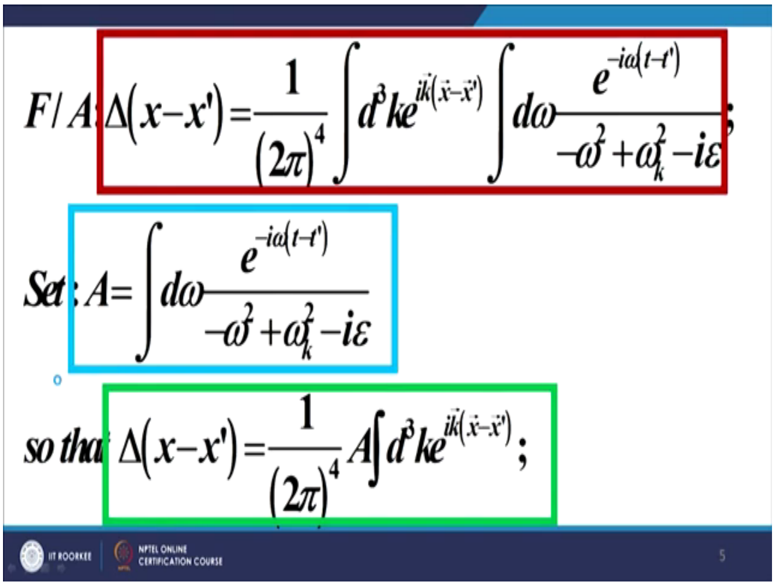

This is what is brought forward from the previous slide. We set to you know to reduce the amount of writing continuously. We split up this integral into a component A which is to be evaluated which we are going to evaluate. And the space portion which is left as a pre factor or as a separate factor here. For the moment we shall be concerning our self ourselves with the evaluation of the factor A.

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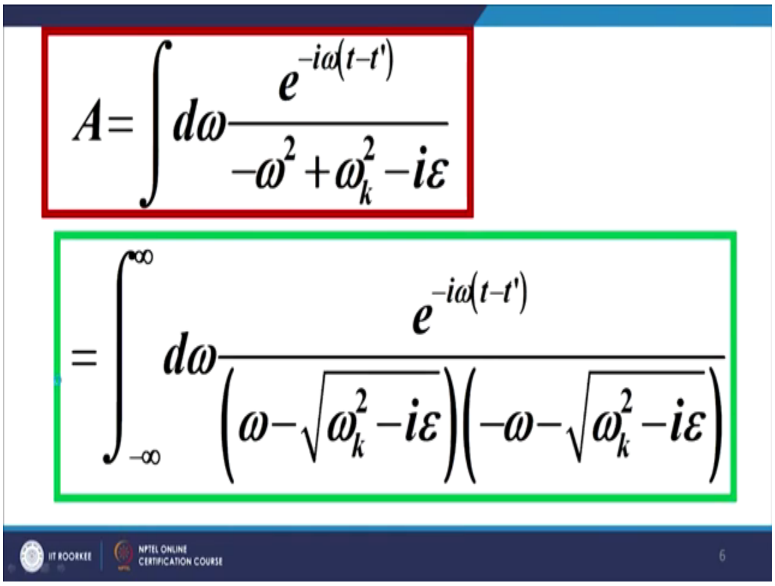

So, A is given by this expression in the red box here. What we do is; we factorize the expression in the denominator and we get this expression which is here in the green box.

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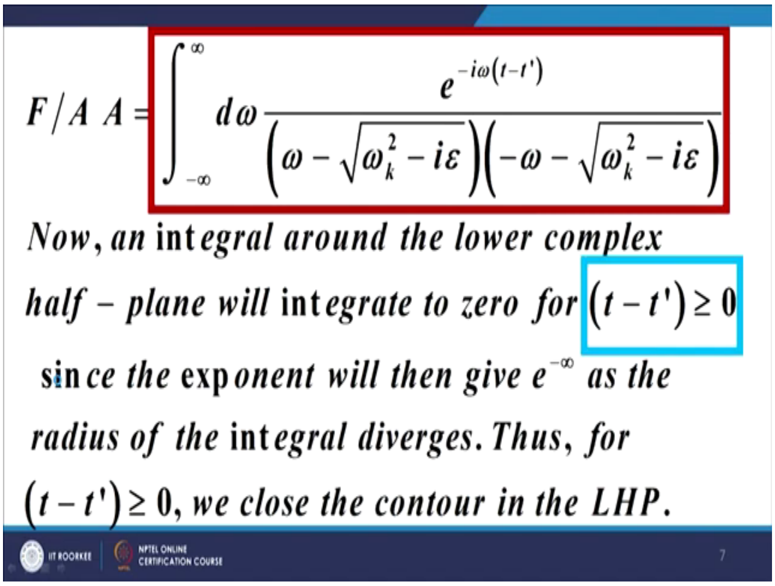

Now, if we integrate this around the lower complex half plane for t minus t dash greater than 0. Now, let us first do the case t minus t dash greater than 0 ok. Let us first do that case t minus t dash greater than 0; as per the rule that we evolved just before the break if t minus t dash is greater than 0 greater than equal to 0.

We close the contour by a semicircle lying in the lower half plane. So, assume for case 1 we take t minus t dash greater than 0 and we close the contour by adding a semicircle to the contour in the lower complex half plane.

We know that this lower this semicircle or the integral over this semicircle in the lower half plane will of this integrand will contribute 0 or will tend to 0 in the limit that r or the omega diverges. So, we close the contour by a semicircle in the lower half plane.

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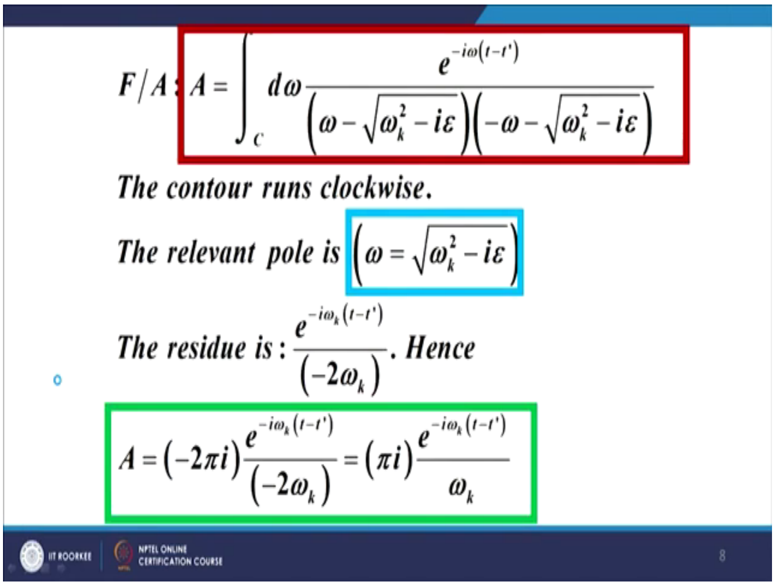

So, this is what we have; the pole if we close the contour in the lower half plane we will enclose the pole which lies in the fourth quadrant. And the pole in the fourth quadrant is given by the expression in the blue box.

So, we have to work out the residue at this particular pole. And when we work out the residue at this particular pole we find the residue to be the expression that is given here e to the power minus i omega k t minus t dash divided by minus 2 omega k. Please note this is in the limit that epsilon tends to 0.

Now because the contour has been transferred in a clockwise direction it is being traversed in the clockwise direction there will be a minus sign involved. And the value of the integrand value of the integral by Ketch’s formula will be equal to minus 2 pi i into the residue at the enclosed poles.

Please note; there is only one pole enclosed in this particular contour and that is the pole in the fourth quadrant at which the residue is this quantity. And therefore, the value of this integral is given by minus 2 pi i e to the into this residue which simplifies to the expression given in the green box at the bottom of your slide.

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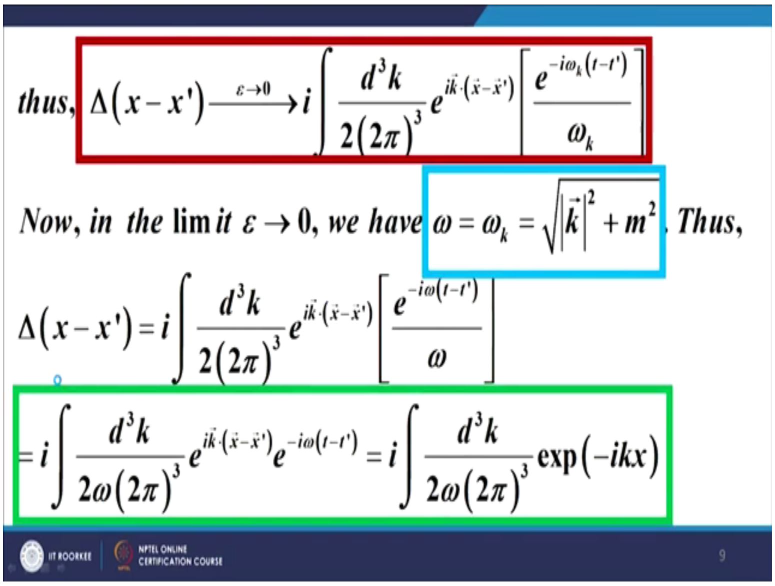

Now, substituting this value in the expression for the propagator. And also noting that omega in the limit that epsilon tends to 0 omega is equal to omega k is equal to under root mod k square plus n square making this substitution that omega is equal to omega k.

And using the value of a that we have already arrived at; we get this expression for the propagator which is here in the green box right at the bottom of your slide.

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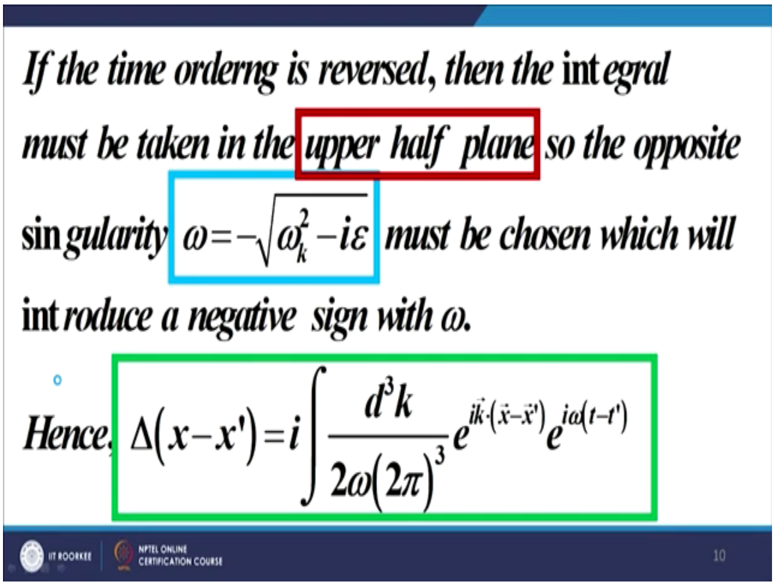

Now, what happen if the time ordering is reversed? In other words t minus t dash is less than 0. What happens if t minus t dash is less than 0? If t minus t dash is less than 0 the semicircle that we use for closing the contour will lie in the upper half plane. And because it is semicircle in the upper half plane. And importantly this is the integral over this semicircle will give a 0 will give no contribution to the overall integral as in the previous case.

But when we close the contour with the semicircle in the upper half plane. The pole that we enclose the singularity that we enclose is in the second quadrant. And that is given by this expression omega is equal to minus under root omega k square minus i epsilon.

And the residue we can work out at that point. And because now this is in the contour clockwise direction there will be no minus sign here. But there will be a minus sign attached with omega when we at the end of the day we complete the calculations you will end up with a minus sign with omega.

And the whole value of the propagator will take the expression given in the green box at the bottom of your slide. Please note the difference the difference between the 2 expressions. The difference between the 2 expressions; if you look at the expression for the propagator when; t is greater than t dash and the expression when t is less than t dash is the sign of the omega other than that it remains the same.

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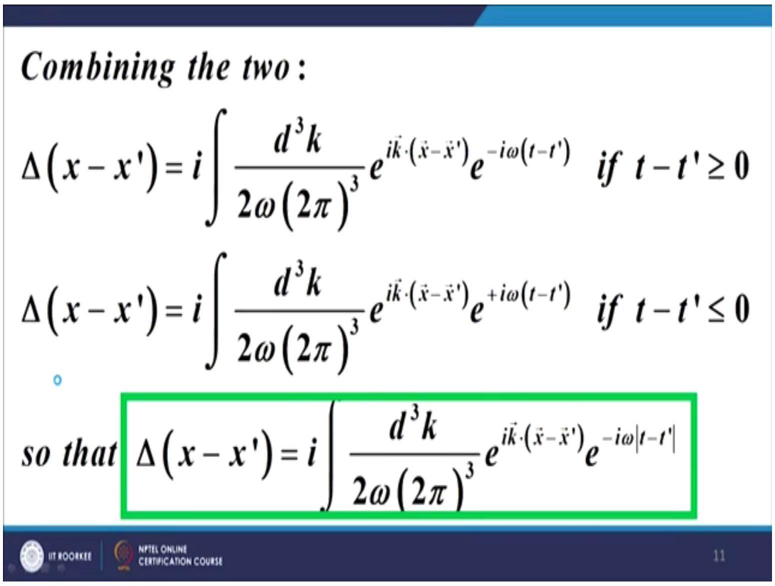

So, combining the two we can write the expressions for the propagator in the form which is given in the slide. And which can be combined together to one form to one equation which is given in the green box at the bottom of your slide. Where; we have introduced the mod of t minus t dash in the exponent of the omega exponential right.

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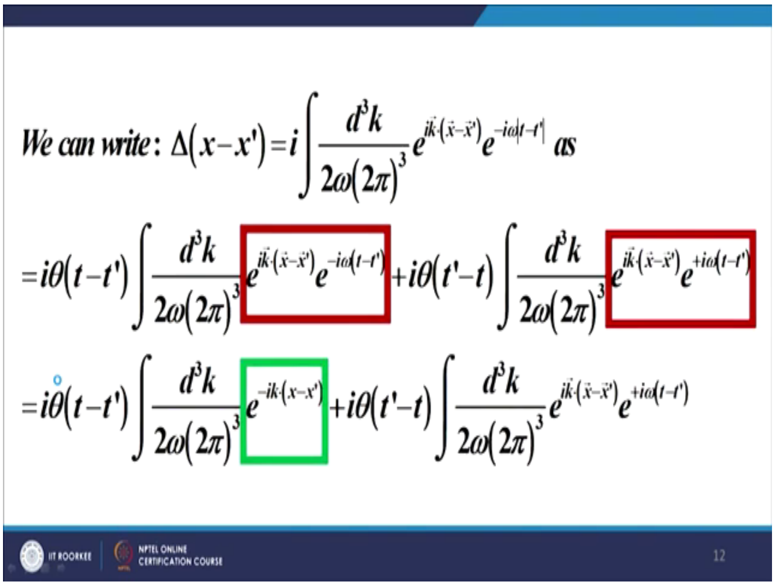

Now, we try to combine the two for the first expression relates to when t is greater than t dash and that gives me this expression the expression in the red box. And when t dash is greater than t I get the expression in the red box to the right.

So, for t greater than t dash I get the which is with. Now t greater than t dash is managed this condition this constraint is managed by this unit step function the Heaviside function which operates as a pre factor to this integral. And similarly t dash greater than t is managed by a introducing this Heaviside function as a pre factor to the second integral.

So, when t is greater than t dash this integral will take a non zero value and this will return this particular expression. Because this theta t minus t dash will return a value of 1 and we will have this integral. And in the second case this will be first in the second case when t dash is greater than t theta t minus t dash will give a 0 contribution it will be equal to 0 and therefore, the whole integral will vanish.

And this theta t dash minus t will give me 1 and therefore, this second integral will contribute a nonzero value. So, that is how this scheme operates.

So, this is what we have and this is nothing, but a four product and that is written as a four products here. The expression in the red box here is nothing, but the four product so we can write it as the expression in the green box here.

But we for the moment we cannot do so in the case of the second red box. This does not look like a four product in the matrix that we are talking about. Because both omega and k this phase component and the time component of the wave vectors are carrying positive signs.

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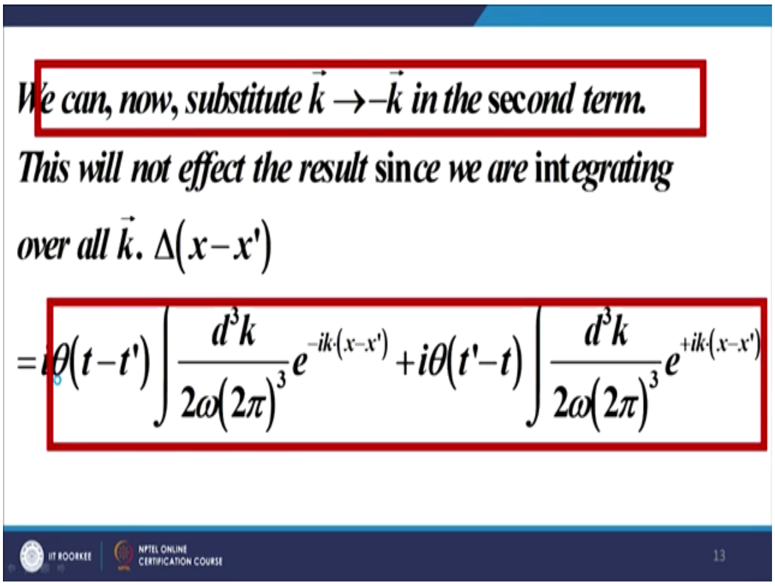

So, to remedy this situation what we do is. We now substitute k as minus k and in the second term k we substitute or we transform k to minus k in the second term. Obviously, this is not going to effect the integral. Because the integral is over all values of k positive and negative.

But it enables us to write this in the write the second expression also in the form of a four vector as shown in the red box. Or the right hand side term of the red box at the bottom of your slide. So, one is having minus i k x minus x dash and the second is having plus i k x minus x dashed.

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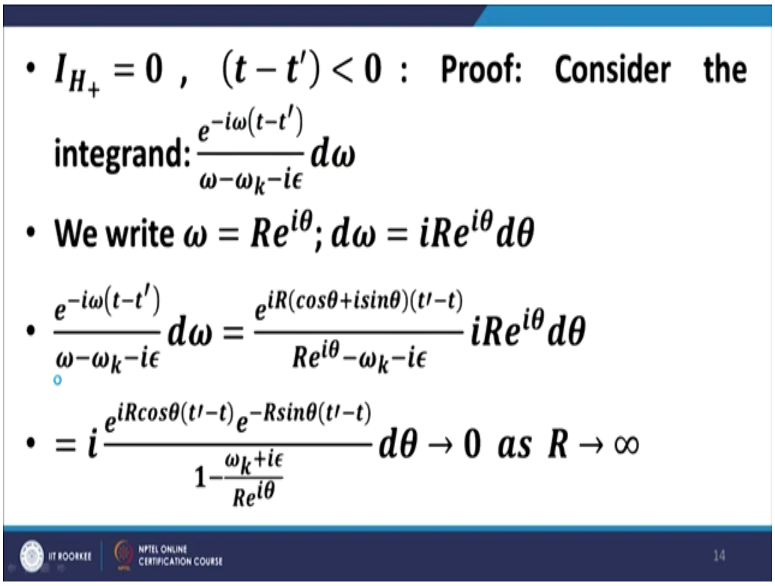

If this is a proof before I get into it so this is the expression that we have for the Feynman propagator; where the omega integration has been carried out and we are left with the space integration. Integration over the space components of the wave vectors and please note this integrand is in the form that is and that is Lorentz invariant in special relativity so that is also taken care of.

Now, this is a proof. In fact, I have proved this earlier also this is the proof that; when t minus t dash is less than 0 then we close the contour with a semicircle in the upper half plane because we want sin theta here. If you look at it we are having t minus t dash is less than 0. Please keep track of this.

We are having e to the power minus R sin theta so we need sin theta to be positive in this case. And therefore, we close the contour with a semicircle in the upper half plane when t minus t dash is less than 0.

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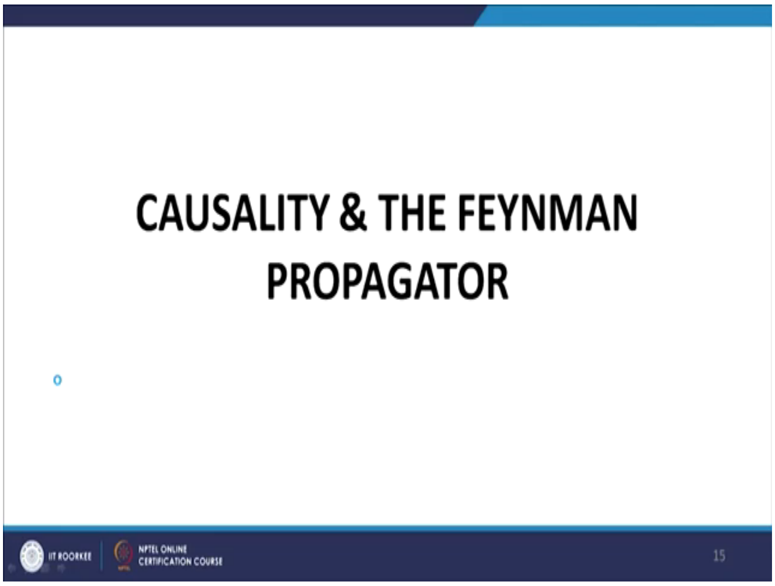

Now, this is a very interesting issue of causality; in the context of the Feynman propagator.

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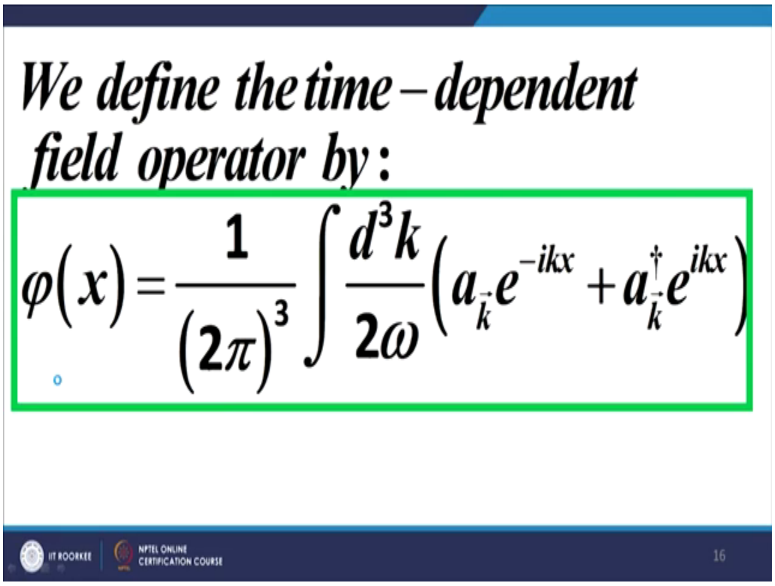

Whether Feynman propagator violets causality that is an interesting issue that we need to address. The time dependent field operator phi x is in terms of the creation. And annihilation operators can be written in this form in a Lorentz invariant fashion it can be written in this form.

Please note all quantities appearing here are Lorentz invariant. So, this is the Lorentz invariant expression for the field operator.

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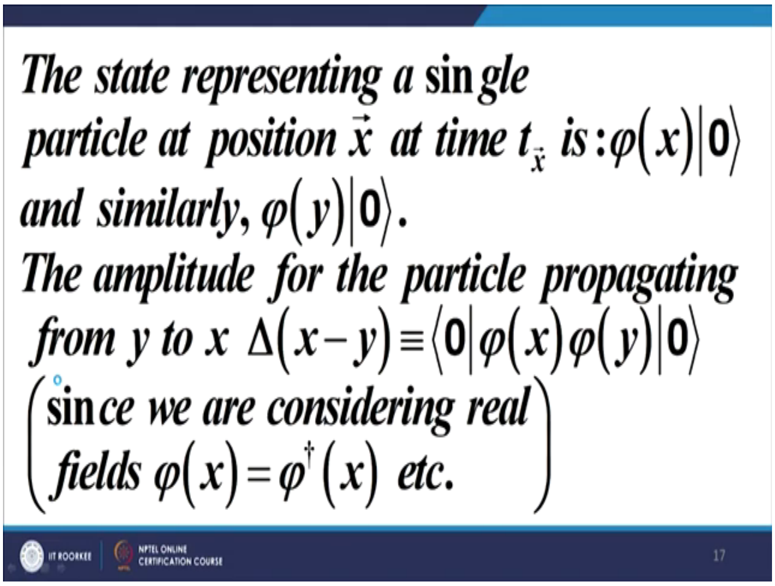

Now, the state representing a single particle at position x at time t can be obtained by operating with this field operator on the vacuum state. And similarly if you want to create a particle at position point space time point y; you operate with this field operator on the vacuum state.

And the amplitude of a particle going from a point y to a point x is given by the propagator which is nothing, but the time ordered product in fact, given by this expression. We are assuming that the time ordering is implicit in this expression. And please note since we are considering real fields so phi x is equal to phi adjoint x.

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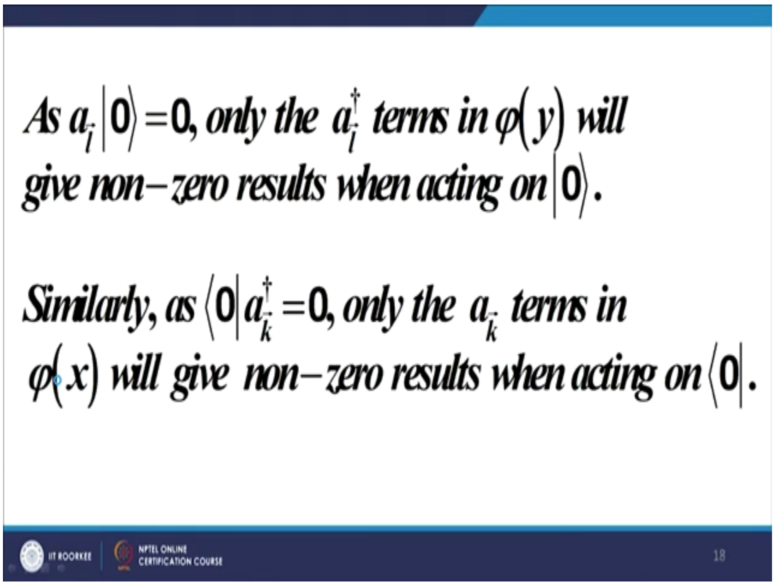

Now, if the annihilation operator operates on the vacuum it returns 0 because there is no point in the vacuum, no particle in the vacuum. So, the annihilation operated by definition the annihilation operator that is a definition of the vacuum state by the annihilation operator operating on the vacuum state gives us 0.

Therefore, only the creation operators contained in phi y will give non zero results acting on the; please note y is to the right here, y is to the right x is to the left. So, only annihilation only creation operators in phi y acting on the ground state will give nonzero results. Annihilation operators in phi y acting on 0 will give us 0.

Similarly, for the case of phi x only annihilation operators operating on the bar 0 will give us nonzero results. And creation operators acting on this bar 0 in phi x will give us a 0; so that is important.

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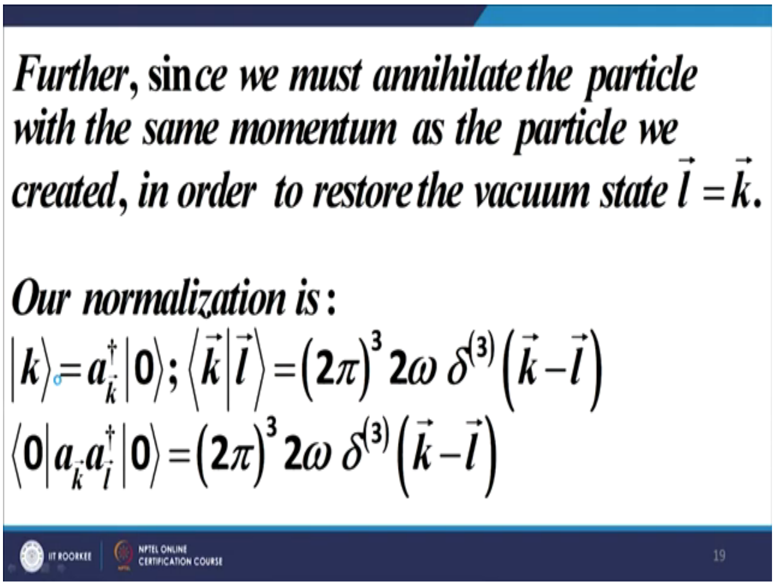

Now, if we want to recreate the ground state; we must annihilate the particle with the same momentum as the particle we created. If we in other words l must be equal to k, if we want to recreate the recreate the vacuum state; first by creating a particle and then annihilating a particle; that means with the both the particles must be of the same momentum.

That leads us to the normalization conditions which are given at the in the last two equations of the slide for the creation and annihilation operators. The and the states and the quantum states labeled by k and l.

These are the normalization schemes that follow as a result of what is stated above. That we need we must have that the annihilation of the particle must be of the same momentum as the moment of the particle that we have created in order that the ground state be restored.

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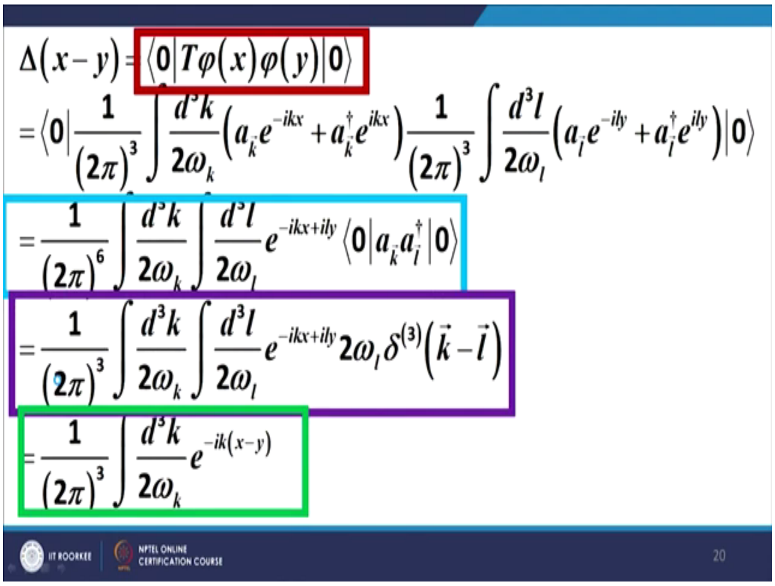

Using all these conditions now we simplify the expression for the our propagator and what we get is the expression that is given here in the green box. After simple calculations and introductions of and normalization and we end up with the result that we have in the green box here at the bottom equation of your slide.

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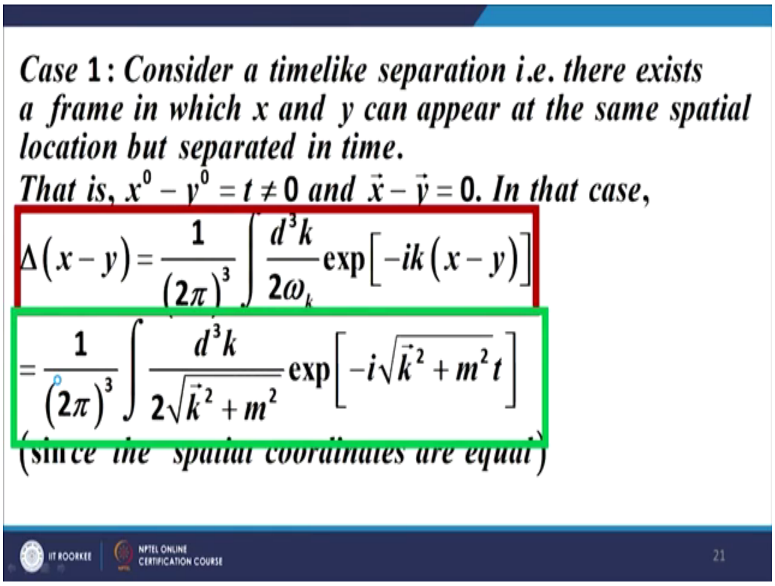

Now, we consider two different cases the case of a space like separation and the case of a time like separation between x and y. The case of a time like separation is relatively simple so let us crack it first and then we will talk about the case of the space like separation.

For the time like separation what happens is; we can have there can be a frame where the spatial location of x and y coincide. That means, they are occurring at the same place same point in space, but they are separated in time. This is consequence of special relativity that.

If two space if two events in space time are time like separated then we can find a Lorentz frame in which they occur at the same point in space, but they are separated in time. So, we assume that they are separated in time we write x 0 minus y 0 equal to t. And we write because they are occurring at the same point in space the space coordinates coincide.

Using the fact that the space coordinates coincide and the time coordinates the time is represented t represents the time difference between the two. We can write the expression for the propagator in the form that is given in the green box. Where under root k square plus m square is nothing, but the k 0 the time component of the wave vector.

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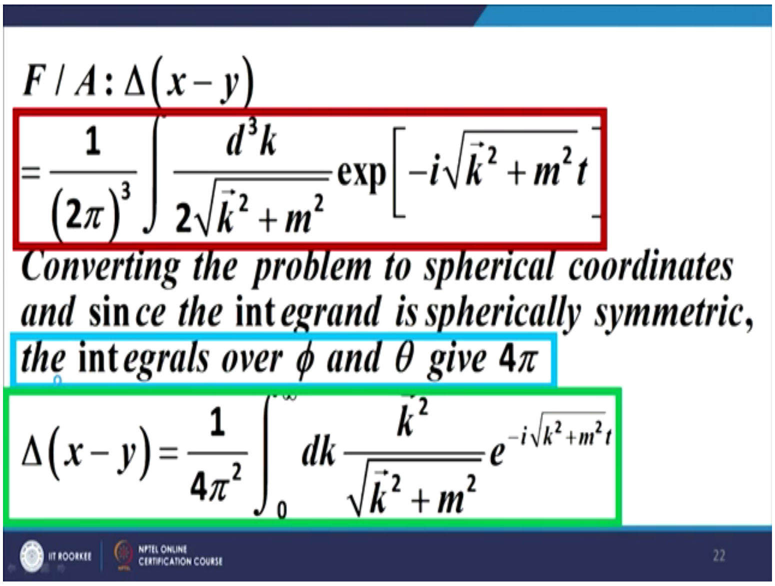

So, we now have to integrate this expression that is our problem now. We have already done the omega integration or the k 0 integration; we now have to integrate it over the space components of the wave vector. We convert this problem to a spherical coordinates and since the integrand is spherically symmetric.

If you look at it the integrand is clearly spherically symmetric. Under root k square plus m square is spherically symmetric and the integration element is also spherically symmetric. So, the integrals over phi and theta give 4 pi and in that 4 pi cancels with the pre factor and leaves as a factor of 1 upon 4 pi squared.

And the k integral remains the out of 3 components the 2 components 3 phi component and the theta components are integrated over they give us 4 pi that 4 pi is adjusted through the pre factor. And what remains is shown in the green box at the bottom of your slide.

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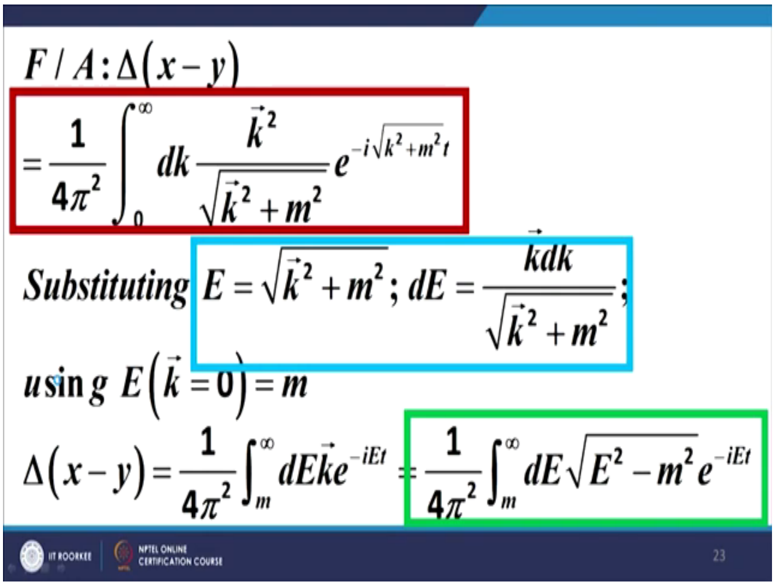

This is what we have from the previous slide. We make this substitution E is equal to under root k square plus m square d E is equal to this expression k d k. When you and now the point about the integration limit the integration limit is from 0 to infinity for k.

But k the minimum value of E corresponding to k equal to 0 is equal to m. So, when you are making this substitution of variables the lower limit will shift from 0 to m the upper limit of course, remains unchanged infinity. And the integral is given by the expression which is here in the green box right at the bottom of your slide.

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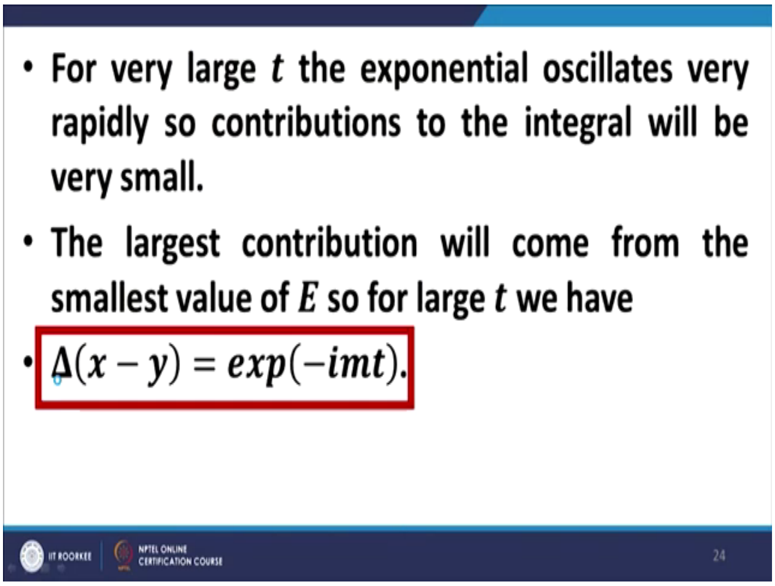

Now, at very large t; what happens if you look at go back to it this exponential term the imaginary i factor here in the exponential means that this integral is oscillatory. And as t increases as t increases the frequency or the oscillations become more rapid and the overall contributions to the integral will diminish.

Now, nevertheless as t becomes large they exponential increases, but even then or keep not withstanding this particular fact. That as t increases the contributions to the integral becomes less the significant contribution will come when E is when E is as small as possible.

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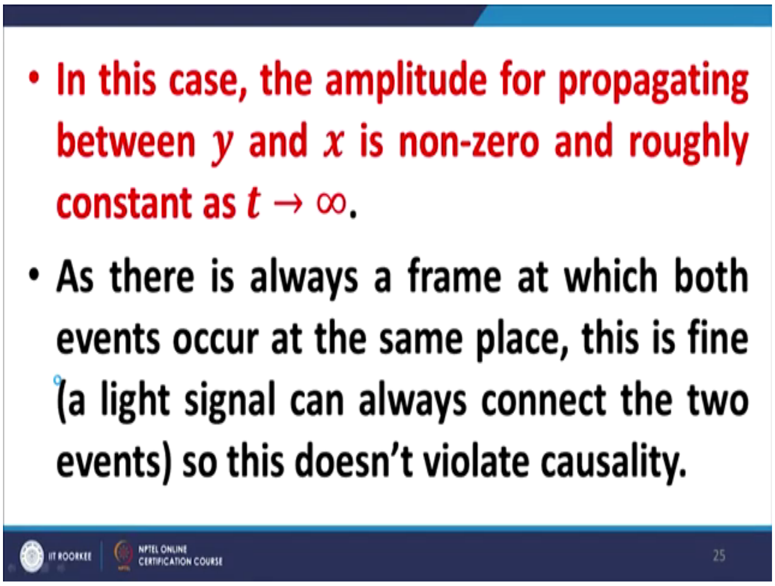

When E. And when is E as small as possible when E is as small as possible means E is equal to m that is the smallest value that E can take. So, to that approximation we can write delta x minus y is equal to exponential minus i m t. Here if you have E is equal to m and this factor is a significant contributor. We have integral on simplification E to the power minus i m t. That is what we have here for the propagator. This is the approximate value for large values of t.

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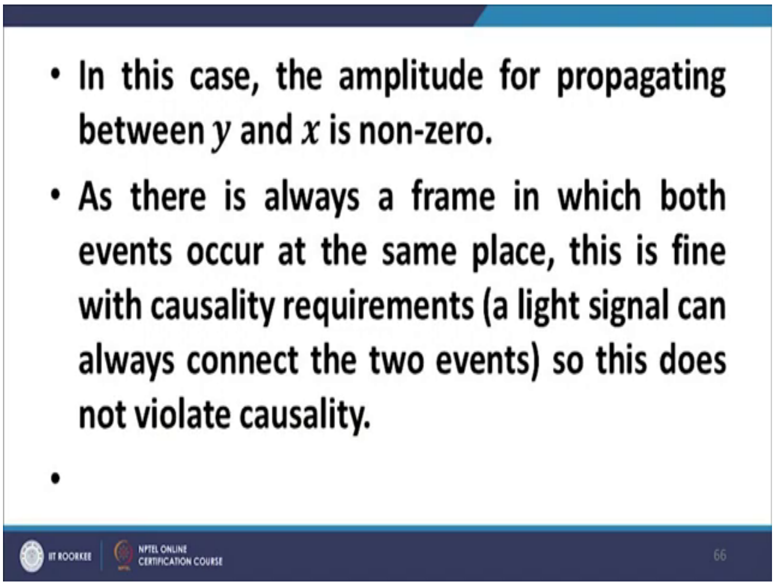

Now what does it say? So, the we are talking about time like separation and we find that the value of the propagator between y and x is nonzero. Now, the important thing is that in the case of events which are time like separated you can always find a frame in which the events occur at the same place, but they are separated in time which is not a violet. And therefore, the fact that and therefore, the fact that the propagator between y and x is nonzero does not violate relativity.

Because there can always be a light signal which can connect the two and causality as a result is not violated.

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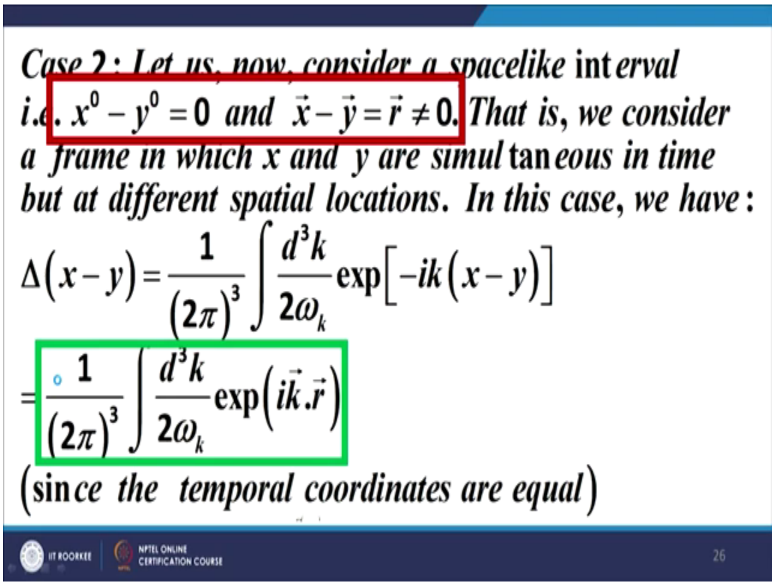

Now, we talk about the second case; when the two; when the two points x and y are space like separated. Now, when the two points x and y are space like separated; we can always find a Lorentz frame in which they are instantaneous in terms of time, but they occur at different points in terms of space. So, we can write x 0 is equal to y 0 in that particular frame. Operating in that particular frame we can write x 0 is equal to y 0, but we have x minus y is equal to r.

The space components x and y is equal to r; the difference the spatial difference between the two. Now in this case when we simplify our expression we get because the time component is the same the time components go out of reckoning.

And what we are left with these space components which are captured by the expression given in the green box at the bottom of your slide.

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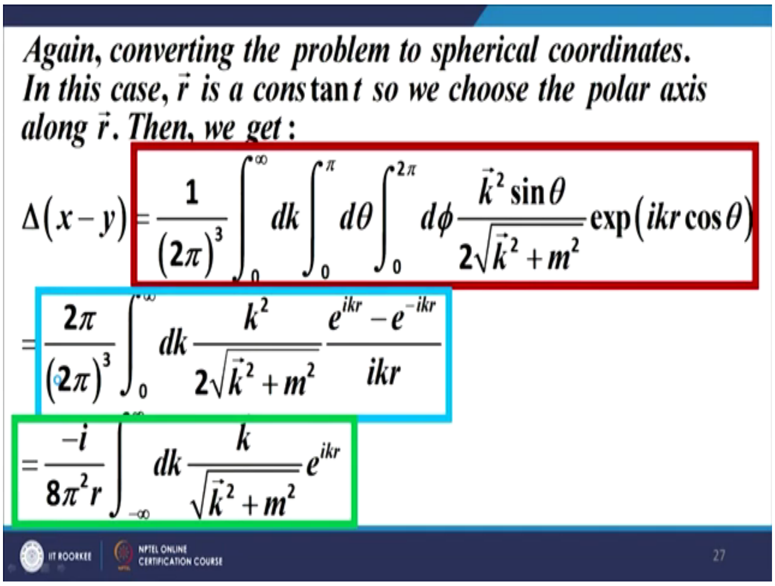

Now, again we convert our problem to special to spherical coordinates. And because r is constant here we can choose the polar axis along r the polar axis we can choose along r.

And when we do; when we do the integration over phi we get a factor of 2 pi here in the numerator the 2 pi is captured here. The integration over theta is slightly more involved because now the integrand becomes an explicit function of theta.

Writing the theta dependence explicitly and then substituting i k r cos theta equal to theta i k r cos theta equal to x. And then doing the integrand we get the expression which is here in the in blue box. And this can be simplified further because by changing by writing k equal to minus k or transforming k to minus k and shifting the limits of integration from 0 to infinity to minus infinity to plus infinity.

And then we get the expression in the green box here. To move from the blue box to the green box we simply do a transformation k equal to minus k. And we get and the limits get from 0 to infinity get from minus infinity to infinity and we can capture both the terms in one expression both the exponential terms in one expression.

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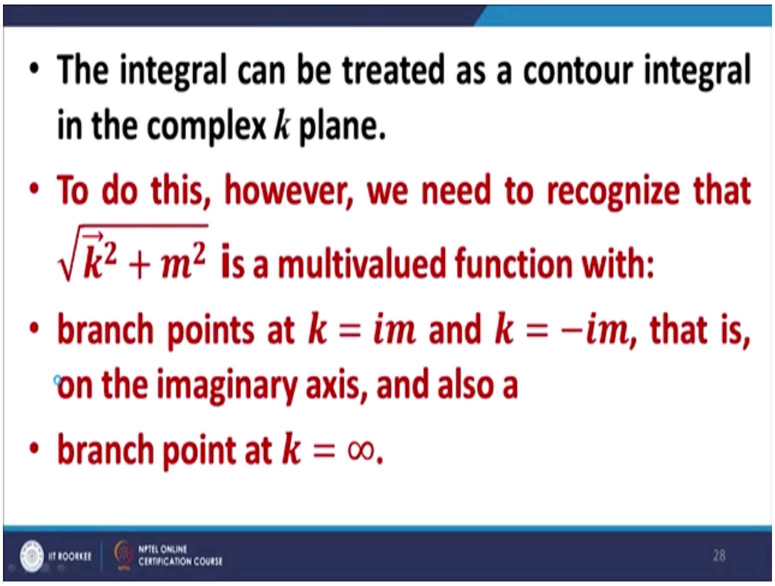

Now, this is we again treat this as a contour integration. Therefore, when we do the integration we have to be very careful about this.

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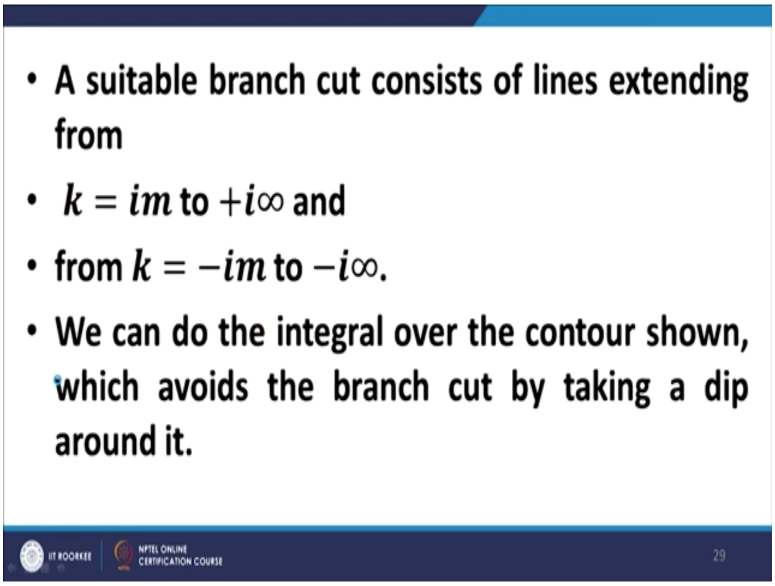

We introduce branch cuts and we do the integral over the contour as shown which shows we avoid the branch cut we show it in the next figure.

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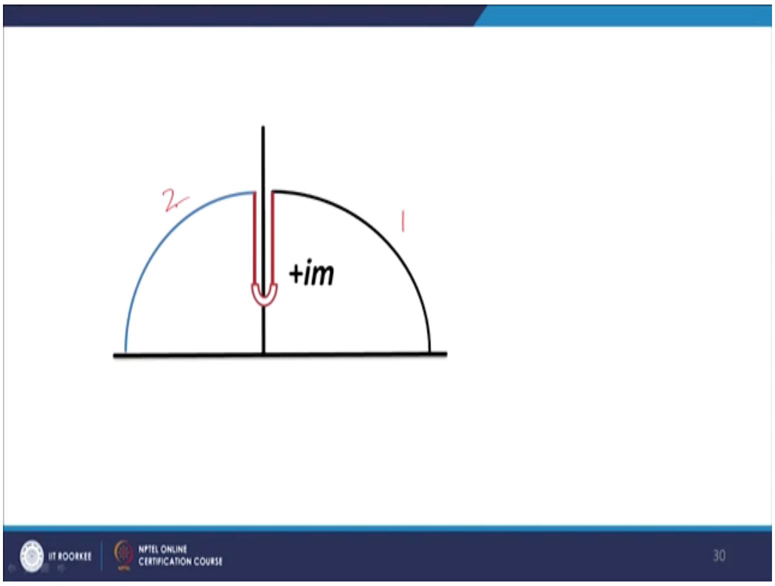

Here is the figure now. You see we want this particular integral. Integral over the over the straight line and we close the contour by the semicircle, but when we close the contour of semicircle we have to do this additional maneuver. Additional maneuver due to there being a branch cut at plus i m.

We have to somehow escape or somehow go somehow move across this branch cut. We do not want to encompass that branch cut within our integration contour. And therefore, what we do is we take a contour of the form which is shown in the diagram. So, this is how the contour integration is to be carried out. We now work out the integration part of this particular contour.

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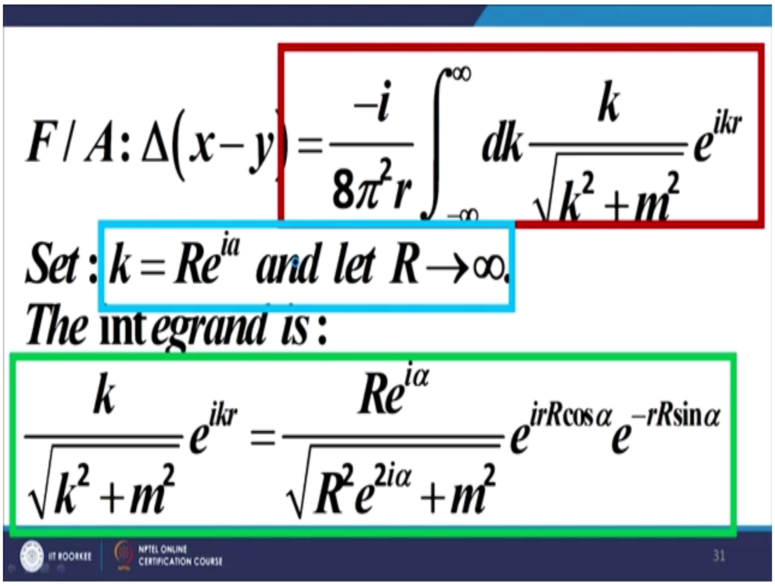

We have this from the previous slide. Let us first do the integration over the semicircles; which semicircles? Let us call it semicircle 1 and semicircle 2 or as the over the parts of the semicircle the part 1 of the semicircle and part 2 of the semicircle. Let us do the integration over these two parts.

So, for that purpose we write k equal to R e to the power i alpha and let R tend to infinity. The integrand becomes what is given here in the green box. And again you find here we have; r into capital R sin alpha. And in the limit R tending to infinity this quantity will go to 0. As you will see in the next slide also.

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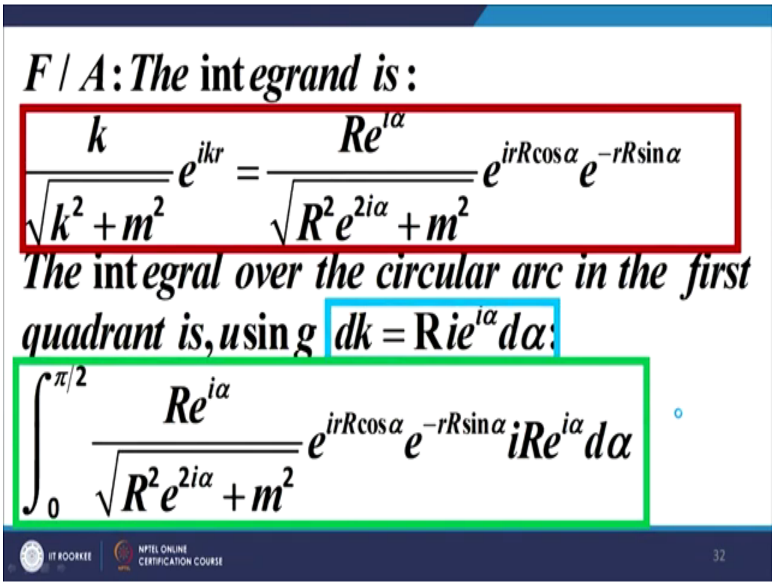

If you integrate over the circular arc in the first quadrant let us work out the first quadrant first. And the first quadrant integral becomes 0 to pi by 2. Nevertheless it does not matter because when you are in the first quadrant sin alpha is going to be positive. And sin alpha is positive means R tending to infinity will this expression will be of the form e to the power minus infinity and that will go to 0.

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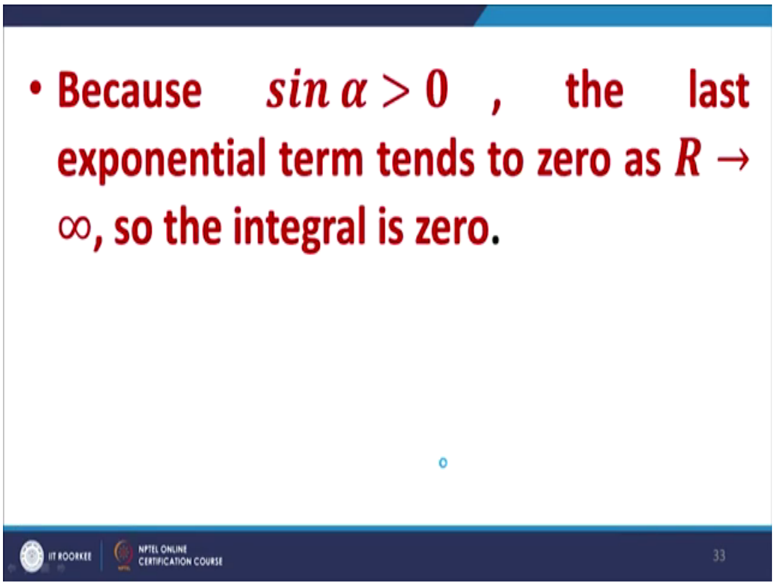

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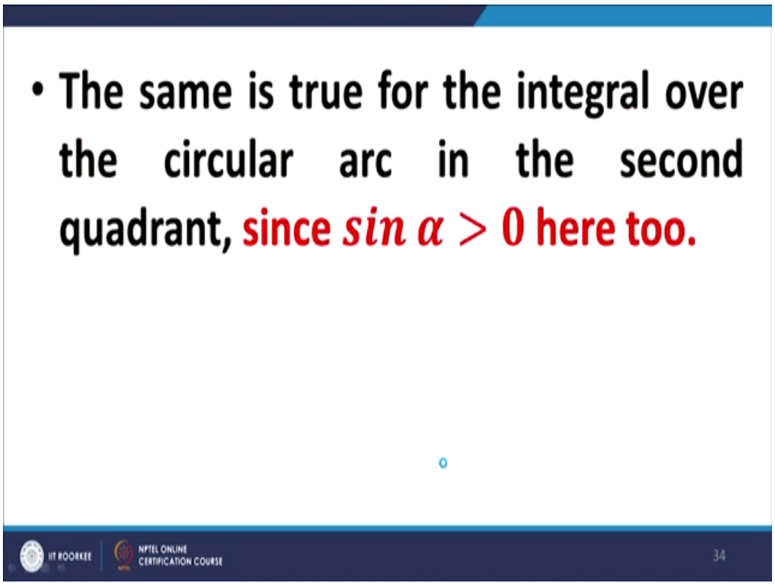

Similarly, the same thing will happen the same thing will happen for the second quadrant or the part of the semicircle that lies in the second quadrant; the larger semicircle that lies in the second quadrant the same thing will happen. Because sin alpha is greater than 0 in this case also as you can see here.

So, the integration over this part of the semicircle of 1 of the semicircle has given a 0 contribution. The part 2 of the semicircle has given a 0 contribution.

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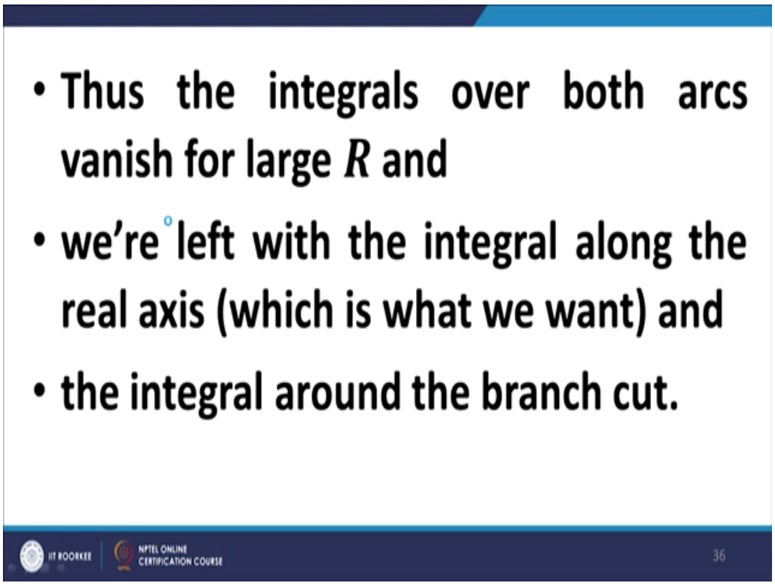

So, the larger semi circles are done with. Now we are left with the integral around the real axis which is what we want and the integral around the branch cut.

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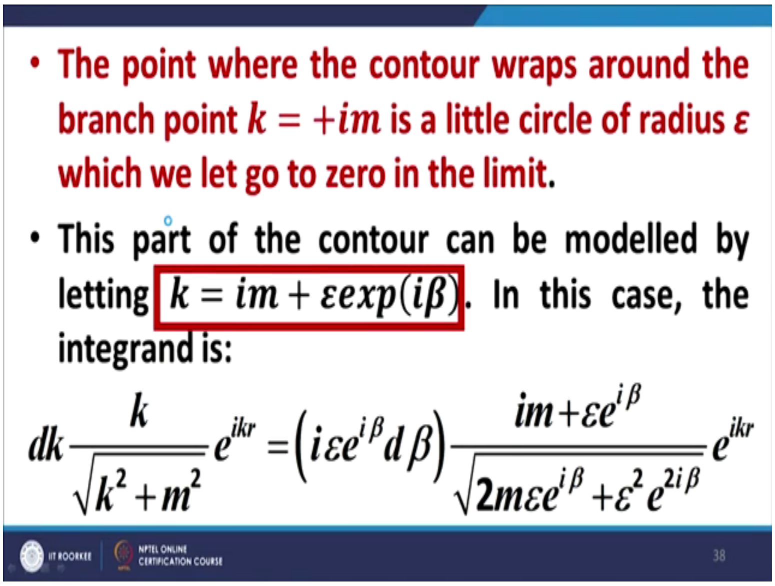

Let us now attack the integral around the branch cut. How are we managing the branch cut? We are managing the branch cut by having a contour which wraps around the branch cut at k equal to plus i m with a little circle of radius epsilon. A small circle of radius epsilon which wraps around the branch cut so that we omit the branch cut from our contour. And to do this integration we write k is equal to i m.

Now the branch cut is at i m so we write the and our circle is a circle of infinitesimal radius around this point. So, we write k is equal to i m plus epsilon exponential i beta. And in this case the integrand becomes when you simplify the whole thing you find as epsilon tends to 0.

As epsilon tends to this whole integral has a factor of under root epsilon. And therefore, the whole integral at the end of the day as epsilon tends to 0 it vanishes.

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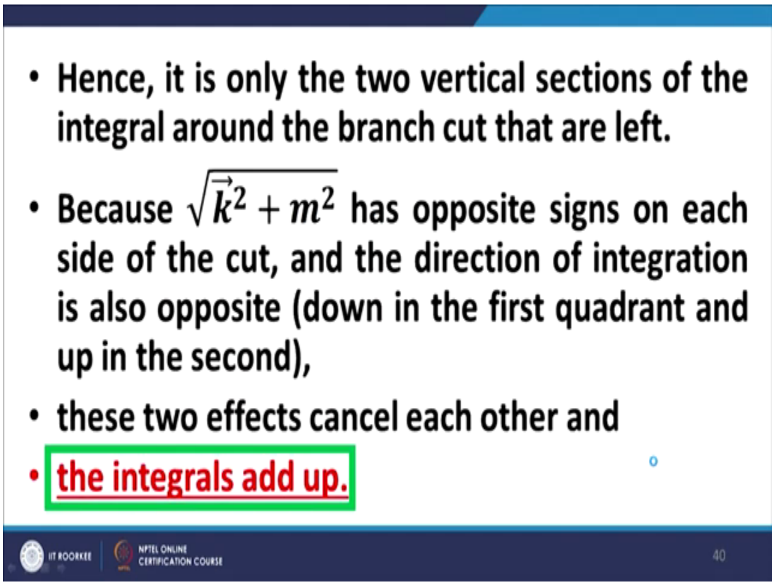

So, now we are left with what? We are left with the we are now left with let me show you. We are now left with the integrals along these two straight lines. Lines which; in a sense move away from i m and towards plus infinity these are the lines that.

Now, the important thing about these two lines if you look carefully is that there will be traversed in opposite directions. If we are moving from in the counterclockwise direction the first line is traversed downwards and the second line is traversed upwards. So, they are being traversed in opposite directions.

And secondly, because the integrand is under root k square plus m square the sign changes when you have when you move from this thing.

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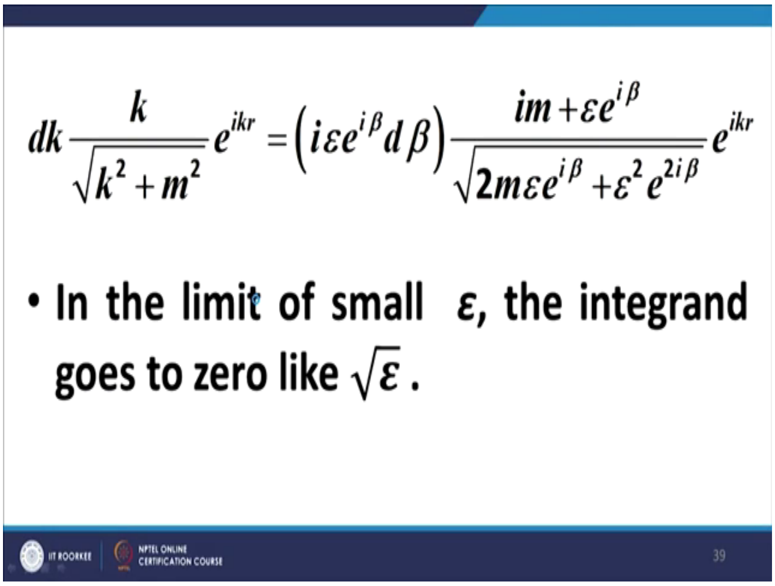

Let me show you. On the opposite sides of the cut the sign of under root k square plus m square will change; that is one part. And the second part is that the straight lines are being traversed in opposite direction. So, these two opposing things cancel each other. And we what happens is that; the integrands add each other they add up.

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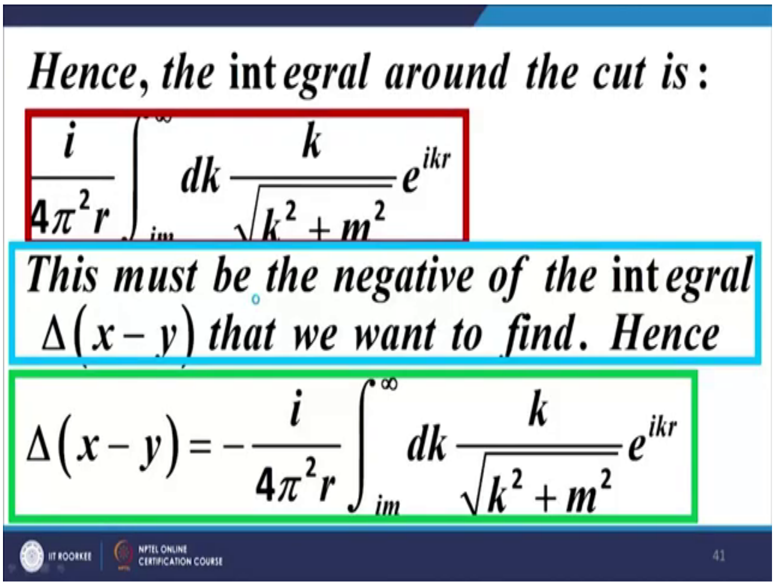

And as a result of which the value of integral around the branch cut becomes the expression that is given in the red box. Remember, we adding the integrals and we are taking the limits from i m to infinity right from the point of the branch cuts to infinity and if we are adding up the two integrals along these two straight lines.

And these two this integral please note must be the negative. Because there are now we have avoided the all the poles there is no pole inside the contour and therefore, the negative of this integral must be equal to the integral of that we want to obtain. Hence we have delta x minus y is equal to this expression; which is given in the green box at the bottom of your slide.

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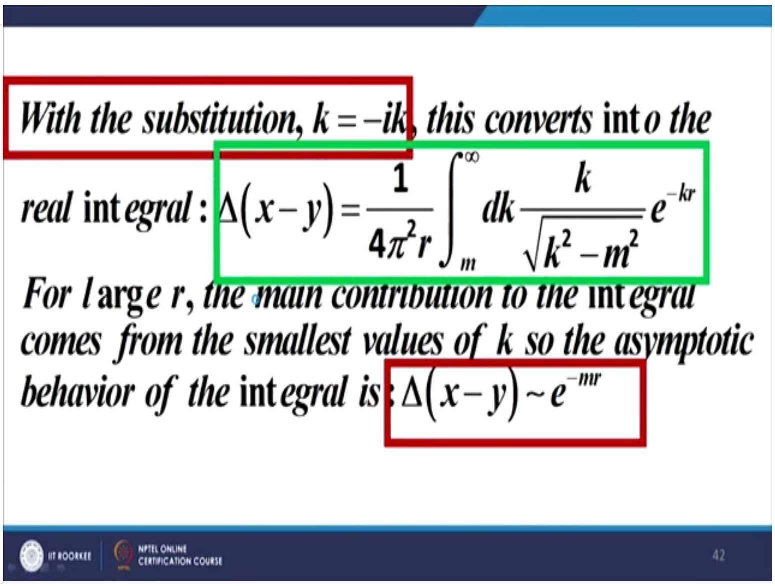

If we substitute k equal to minus i k and this converts it into a real integral which is given at the in the green box in this slide. And for large r the contribution of the exponential predominates and. And the contribution of k is confined to the smallest values which occurs when k is equal to m. And therefore, we get delta x minus y is of the order of e to the power minus r.

We will continue from here. We will discuss the implications of these results or the time like and space like separated events and the corresponding propagators. We shall discuss the implications in the next lecture.

Thank you.