Path Integral Methods in Physics & Finance
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Lecture – 27
Relativistic Path Integral

Welcome back. So, in the last lecture towards the conclusion, I had started discussing the path integral for a point particle in a relativistic metric. We wrote down the action for the point particle in the Minkowski's space with a signature of plus 1 minus 1 minus 1 and minus 1 as the expression that is given in your slide, bottom of your slide.

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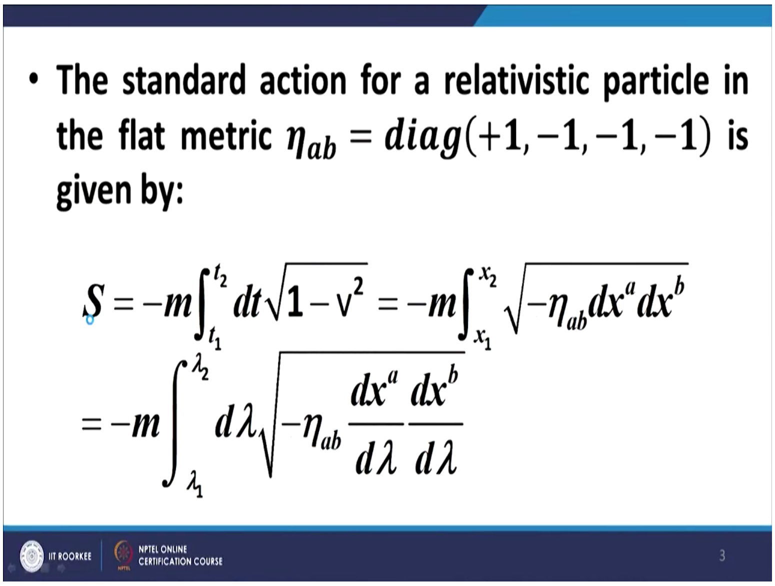

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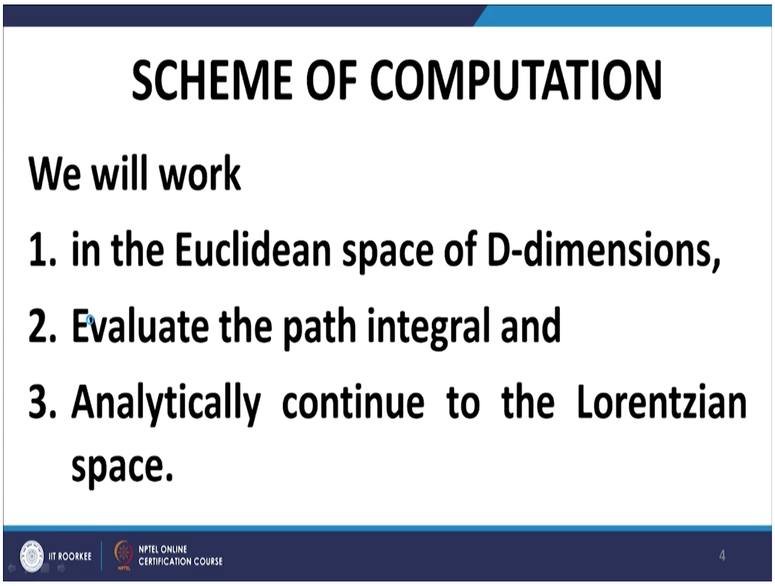

And, then we will outlined the strategy that we are going to follow for the computation of the path integral. The first step would be to transfer the problem to the Euclidean space of appropriate dimensions in this case, 4 dimensional space. But, we keep that in that issue open, we use a D-dimensional space, work out the path integral in that space and then by a weak rotation we transfer the problem back to the or the solution back to the Minkowski's space. So, that is how we will proceed.

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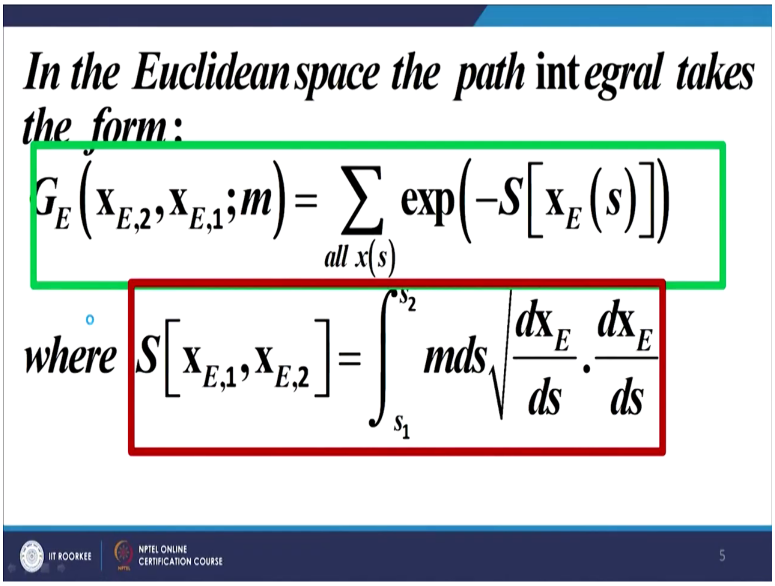

So, let us write down the expression for the action of a point particle in Euclidean space, it takes the form of the expression that is given in the red box at the bottom of your slide.

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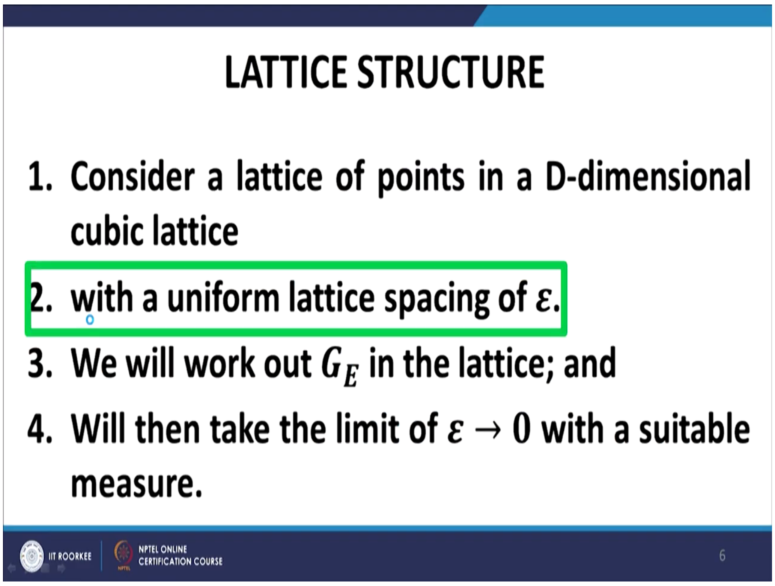

Now, we introduce a lattice structure, we introduce discretization as is usually the case and we have been following that prescription in a lot of cases that we have investigated during the progress of this course. So, we introduce a lattice of points in this D-dimensional space. The uniform lattice spacing we assumed to be epsilon.

So, we shall progress to the continuum scenario or the continuum limit with the with the limit epsilon tending to 0. So, we start with working out the path integral that is G in Euclidean space G E in this particular discretized space time.

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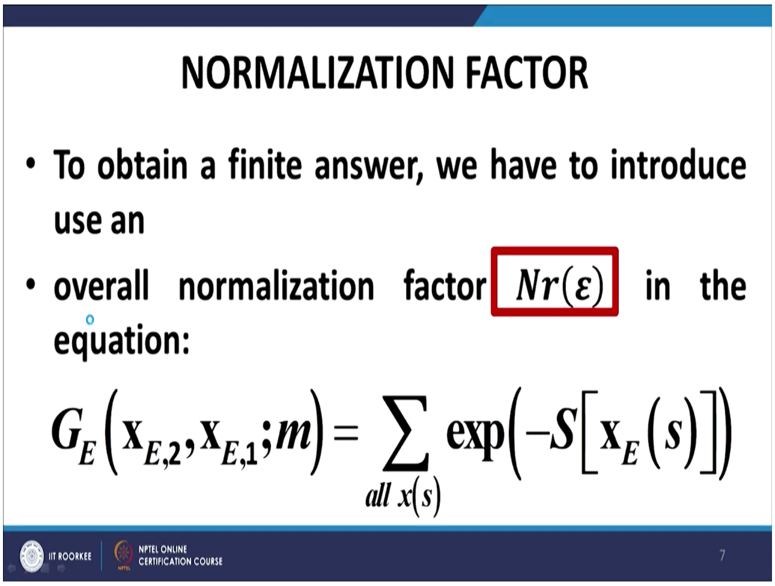

To obtain a finite answer to our problem, we will all of course, need a normalization constant. And, we defined the normalization constant in such a way that this Euclidean this in the left hand side here gives you the continuum path integral, the right hand side gives you the discretized path integral. So, we introduce this factorial and we write the expression in the form which is given in the green box.

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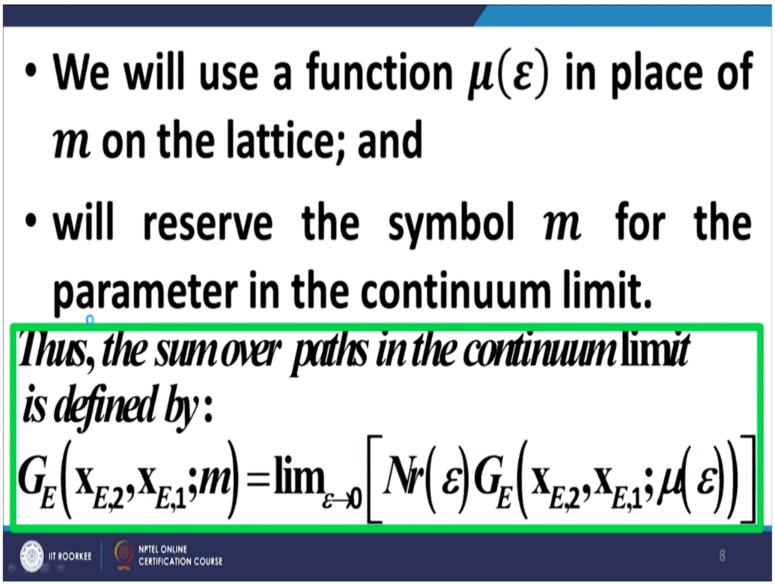

Hm The we make a summation over the path in the continuum limit continuum limit defined by the limit, as I told you epsilon tending to 0; the lattice spacing approaching 0 or in infinitesimal lattice spacing with a normalization constant attached to our path integral.

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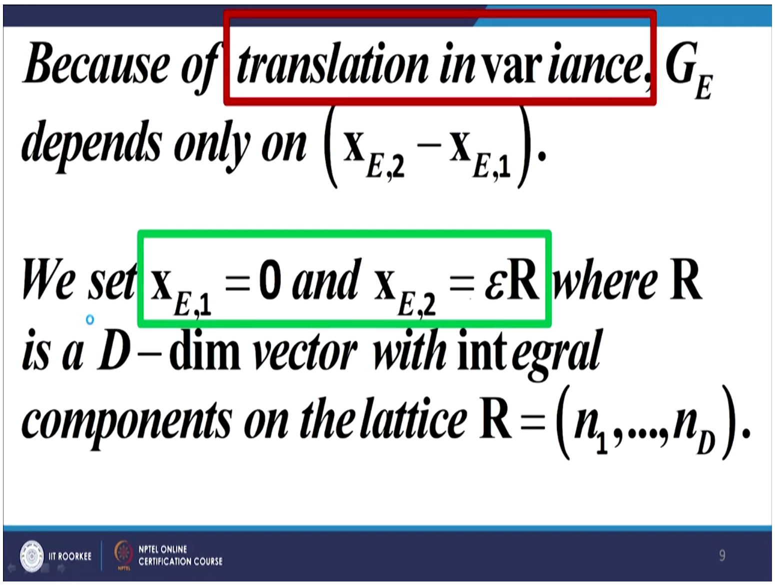

Now, we please note we are designating the coordinates in Euclidean space with a subscript of E. So, wherever E appears, it implies that we are working in Euclidean space. Now, because of translational invariance the path integral must depend only on the difference X E, 2 minus X E, 1. And, in other words we have the freedom to choice to choose the origin which we do as the coordinate E X E, 1 as the origin we choose.

And, we choose an arbitrary point for the purpose of development of the framework as X E, 2 as epsilon R, where R is an arbitrary point on the lattice, arbitrary node on the lattice with the coordinates n 1 n and n D and epsilon of course, as I mentioned is the lattice spacing.

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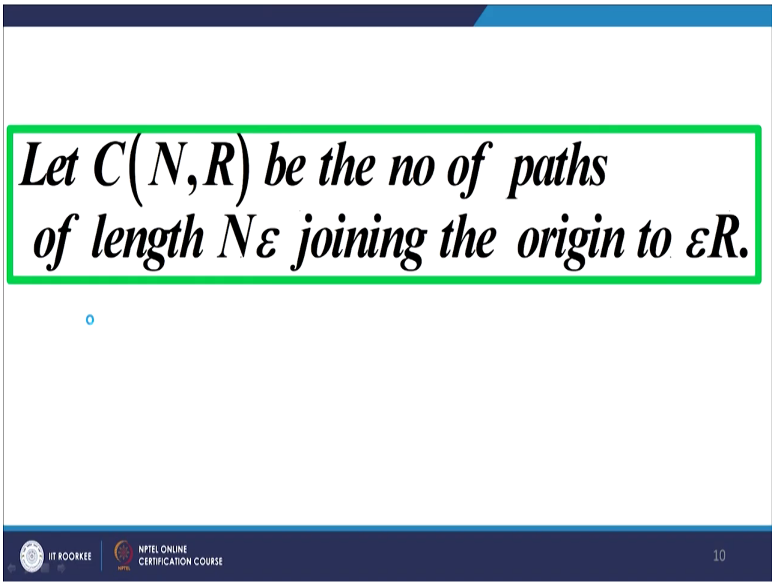

Now, C N, R is the number of paths of length N epsilon of length, please note this it is not R epsilon, it is N epsilon; the where N can be any number. So, we have we are covering all the possible paths which can start from the origin and join the point under reference, that is the point with coordinates R epsilon or R in the lattice framework, discretized framework that.

These are the lengths of the paths joining the origin, various paths not single pass various paths, each will have a different value of N naturally. And, they join the origin to the point under reference which has coordinates which is designated as point R. Each of these, you know each of these coordinates or each of these paths rather each of these paths which I mentioned earlier.

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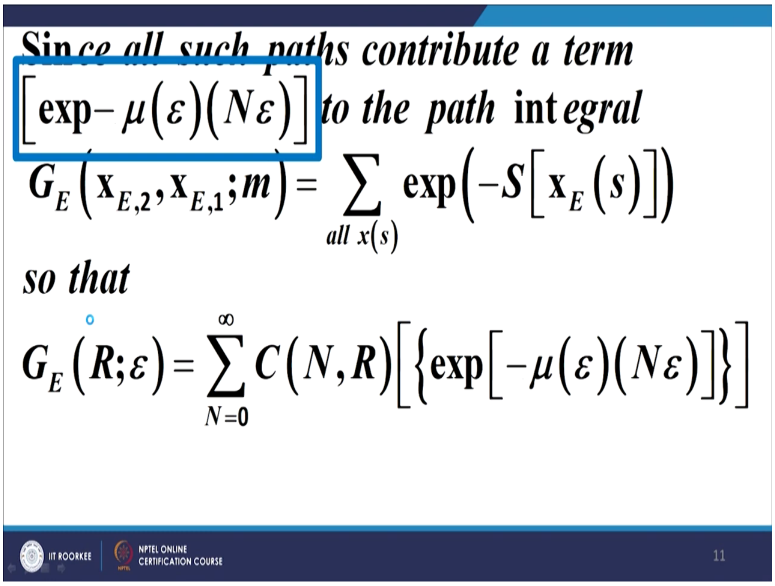

These paths which are of length N epsilon will contribute a term exponential minus mu E into N epsilon. This is this is the equivalent of E to the power minus i s, that we have E to the power i s that we have in the conventional path integrals that we have been dealing with. So, this is exponential, because you would recall that mu epsilon is the term which will transform itself to the mass of the particle, when we move from the from the discrete version to the continuous version.

So, mu epsilon represents the mass and N epsilon represents the length of the path. So, exponential minus mu epsilon into N epsilon or minus mu epsilon N epsilon is in a sense reaction. And, exponential of this is the weight factor attached to each path or a path which is represented by N epsilon which has a length of N epsilon in the summation of the path integral or in the formation of the path integral, constitution of the path integral.

So, this is the weight of each path that goes into the path integral. So, we can write this as we can write our path integral as because, N epsilon is the number of paths of length N epsilon. C N I am sorry C N, R is the number of paths of length N epsilon. So, we multiply this whole expression by C N, R and this is what we and we sum over all possible values of N and we get the value of the path integral.

Let me repeat N epsilon is the path of length N epsilon that joints the origin to the point R. It is weight weighted by the mass and then exponentiated, we get the weight factor that goes into the path integral. So, exponential minus mu epsilon N epsilon is the weight factor that goes into the path integral.

And, because there are C N, R paths of length N epsilon joining the origin to the point R. Therefore, this whole thing represents paths of length N epsilon which contribute to the path integral. And, to get the total path integral we have to integrate or summations, do a summation with respect to all the values of N, all the possible values of N. So, that is how the scheme operates.

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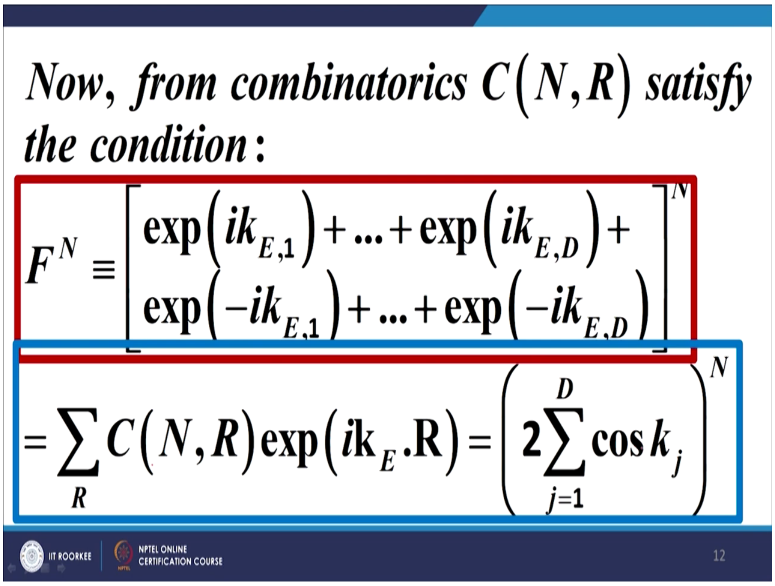

Now, from combinatorics and we can or we have this conclusion, we have this result. We have borrowed it from combinatorics F to the power N is equal to this expression, that is the red box which is further equal to the expression in the blue box. So, this is an a very important equation which we are going to use for further development.

The second part is quite straightforward and you simply expand E to the power ik R in terms of the cosines. And, you in this expression E to the power ik E, 1 plus E to the power minus ik E, 1 upon 2 gives you cosine k a k E, 1 and similarly you have cosine plus cosine k E, 2 plus cosine. So, that part is quite straightforward, the rest is from the in combinatorics.

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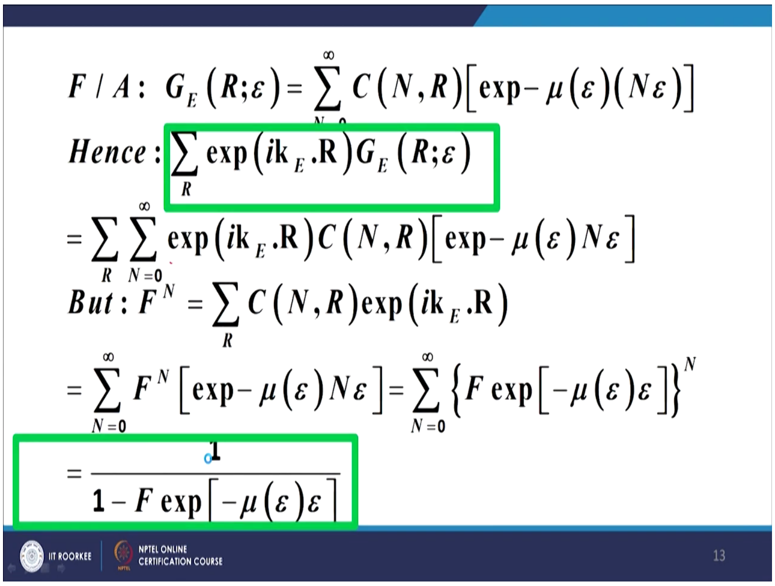

Now, comes the important part. Now, from above we have got this expression that is the path integral is given by this summation C N, R exponential minus mu epsilon into N epsilon. Remember just to recapitulate N epsilon is the length of the path joining from the origin to the point R, C N, R is the number of such paths which have length N epsilon we join the origin to the point R. Each of them is the and therefore, the action becomes minus mu epsilon N epsilon and the exponential of that is my weight factor.

Weight factor multiplied by number of paths, summed over all possible values of N gives me the total value of the paths integral. Now, we multiply both sides of this expression by exponential ik E dot R, multiplying both sides with this expression we get the expression that is here on the right hand side. We introduced this exponential ik R summation N 0 to infinity, this is what we are this expression we have multiplied both sides. So, we get this expression on the right hand side.

Now, this first part exponential ik E R C N, R; this expression gives you seen this expression earlier in the previous slide. This is nothing but this expression seen in the blue box, summation C N, R exponential ik R. And, this is equal to F to the power N. So, let us use this result here and then using this result I get F N exponential and the rest of the term I retain as it is.

What is the rest of the term? Exponential minus mu epsilon into N epsilon which is retained as it is. This can obviously, be written as F exponential minus mu epsilon into epsilon to the power N. Now, this is a geometric series clearly a geometric series and we can write it in the form of a summation of a geometric series which is given in the green box. So, you see this expression is summation of a geometric series. So, we can write this as the summation of geometric series in the form a upon 1 minus common ratio.

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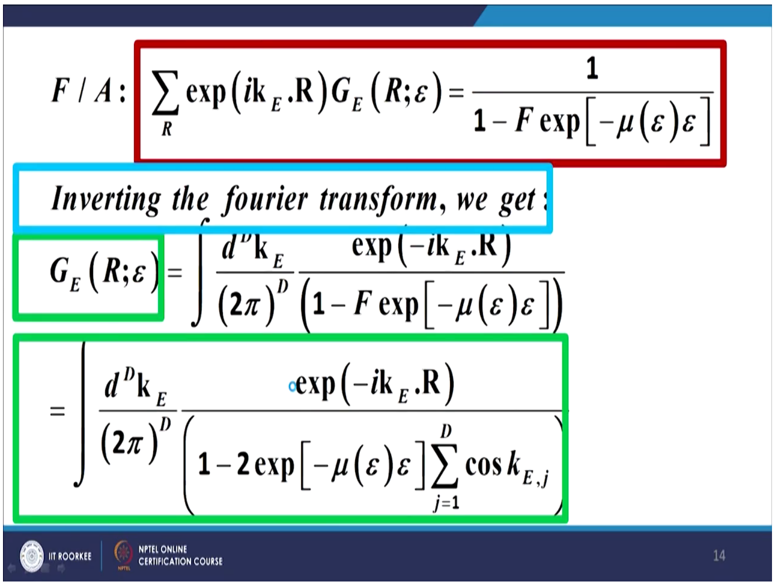

So, that is what we have here. Now, if we invert the Fourier transform, the result that we get is and the result that we will get on the left hand side. We get the expression for the path integral, but please note it is a function, still are in the discretized framework. Therefore, we have written it to as a function of R and epsilon.

And, on the right hand side we get, if we introduce the Fourier transform variable k in Euclidean space and we write the entire thing as the integral with respect to D-dimensional integral with respect to k E, this is the integrand Fourier space, transform space. So, a bit of simplification here we have simply replaced, if you look at it carefully we have replaced this expression F with 2 summation cos k E, j.

Whereas, this come from let us have a look at it, it has come from here this expression C N, R exponential minus i k R. And, if you look at this first expression F to the power N is equal to 2 cos, this expression to the power N and that gives me F is equal to 2 cos summation cos F is equal to summation.

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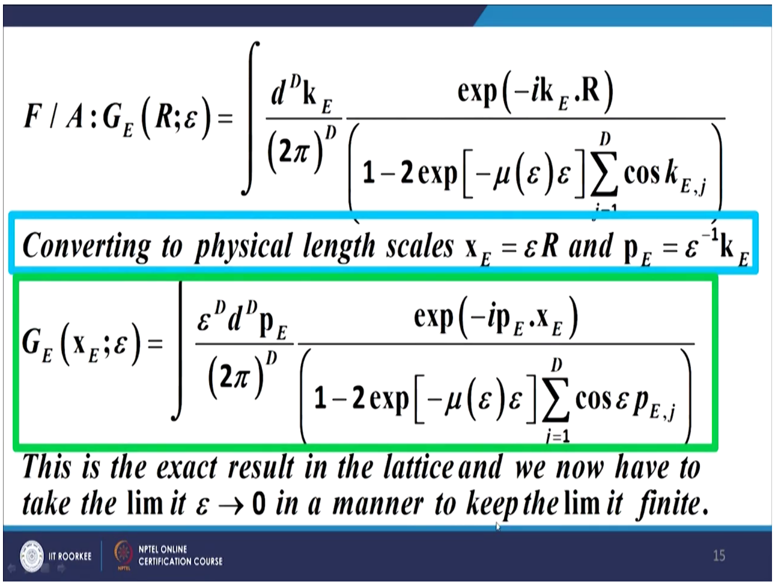

I am sorry, F is equal to 2 summation cos k j. So, we have got this in this result, we have got this result here, we have got this expression here up to where in the green box. Let us move forward. Now, from here we convert back to physical scales, we write that is the X E we write as epsilon R and p E we write as k E upon epsilon; p E we write as k E epsilon (Refer Time: 13:04).

We are transforming from this wave vector to the momentum and that is precisely what is happening here on shifting the variables, on transferring the variables. We get the expression that is given in the green box here, simply transforming the variables nothing else. And so, far this is an exact result on the lattice. This is a discretized version of the path integral; we still have not taken the limit epsilon tending to 0.

But, we now have to do that, we now have to transform this problem from a discretized lattice framework to a continuum framework to a continuum continuous underlying space time. And, for that purpose we need to take the limit epsilon tending to 0. But, at the same time we need to keep the limit finite, otherwise it becomes a redundant exercise.

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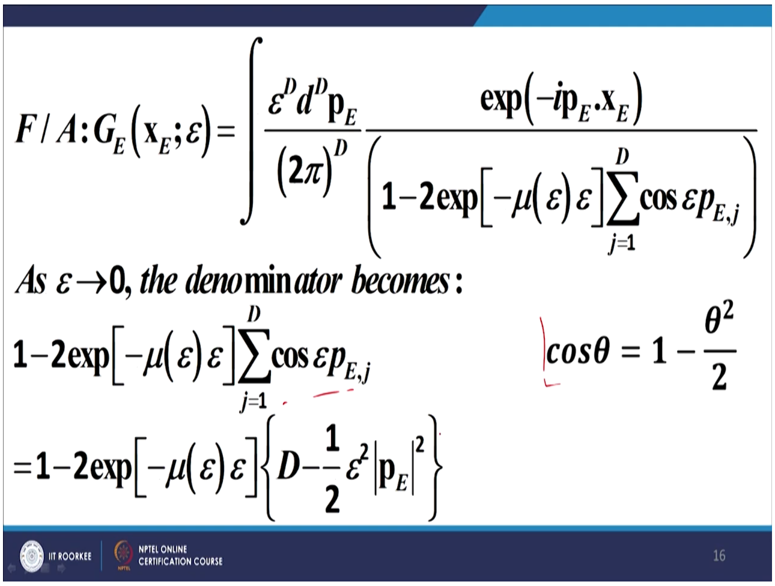

So, how do we do it? And, that is our next step, we introduce certain expressions. If you look at the denominator, if you look at the denominator, the denominator can be simplified as follows: 1 minus 2 exponential, this thing the cos as the cos term here, the cos term here can be simplified using this expansion up to second order in theta.

If you look at this expression cos theta is equal to 1 minus theta square upon 2 plus higher or higher powers of theta, but if you retain assuming theta to be reasonably small, if you retain powers up to theta square. And then this summation over various values of theta will give me over D values of theta, because the summation is over D terms.

So, it will be a product of it will be a summation of D terms. And therefore, I will get D here minus 1 by 2 epsilon square into p 1 p E 1 square plus p E 2 square plus p E 2 square which is nothing, but p E square modulus. So, that is the simplification that we have done here right .

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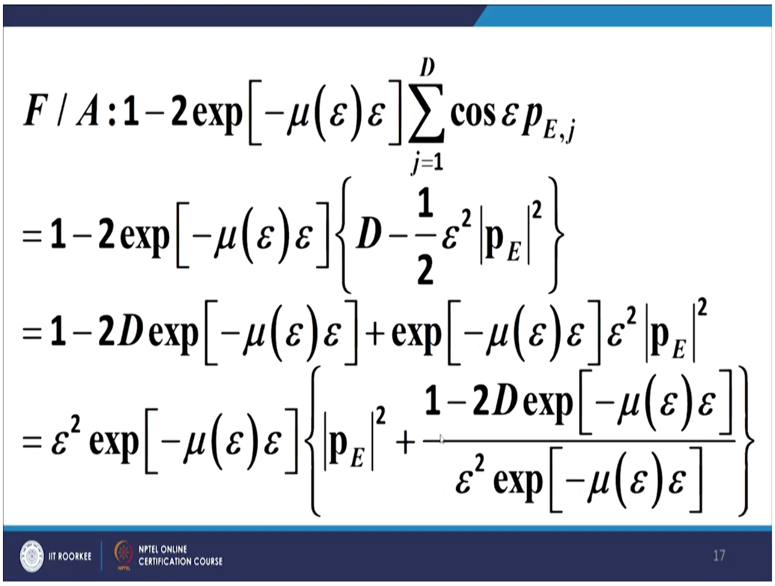

Now, further simplification can be done by taking this expression outside the brackets epsilon square exponential minus mu epsilon into epsilon, you take outside the bracket. And, you have the rest of the term within the bracket and that is what we have done here and we carry it forward.

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$$F/A: G_E\left(x_E;\varepsilon\right) = \int \frac{\varepsilon^D d^D p_E}{(2\pi)^D} \frac{\exp\left(-i p_E . x_E\right)}{\left(1-2\exp\left[-\mu(\varepsilon)\varepsilon\right]\sum_{j=1}^{D} \cos \varepsilon p_{E,j}\right)} \quad -(4)$$

$$Denominator := \varepsilon^2 \exp\left[-\mu(\varepsilon)\varepsilon\right]\left\{\left|p_E\right|^2 + \frac{1-2D\exp\left[-\mu(\varepsilon)\varepsilon\right]}{\varepsilon^2 \exp\left[-\mu(\varepsilon)\varepsilon\right]}\right\}$$

$$so\ that\ G_E\left(x_E;\varepsilon\right) = \int \frac{A(\varepsilon) d^D p_E}{(2\pi)^D} \frac{\exp\left(-i p_E . x_E\right)}{\left\{\left|p_E\right|^2 + B(\varepsilon)\right\}} \quad -(1)$$

$$where\ A(\varepsilon) = \varepsilon^{D-2}\exp\left[\mu(\varepsilon)\varepsilon\right]\ (3);\quad B(\varepsilon) = \varepsilon^{-2}\left[\exp\left[\mu(\varepsilon)\varepsilon\right] - 2D\right]\ (2)$$

Now, we so, the denominator works out to this expression, substituting this expression for the denominator in our Fourier transformed path integral, in our expression for G E X E epsilon, now we are back in x coordinate space. So, G E X E epsilon what we get is and this expression where this expression means, let us call it expression 1.

We get expression 1, but where we have introduced 2 quantities A and B, A epsilon and B epsilon both are functions of epsilon naturally, that is the lattice constant which we will be later taken to 0. And, where B epsilon turns out to be this quantity which is let us say this is expression number 2 and this quantity is A epsilon, this is expression number 3. So, to reiterate we are simply manipulating the algebra, the expressions in terms of the algebra; rewriting it in a format which is compatible for taking limits.

We write it in the form of this expression number 1, where this now p square plus this whole expression is now taken at the B epsilon term. And, we have an A epsilon correspondingly appearing in the numerator, when we simplify this; when we substitute B epsilon according to 2 in equation number in equation number let us call it 4.

When I substitute B epsilon in equation number 4, B epsilon as per equation number 2 in equation number 4, I get a certain expression on the in the numerator which I termed as term as A epsilon, that is what I have done. So, a bit of simplification, bit of notational convenience for the purpose of taking the limit that is what term gives me equation number 1.

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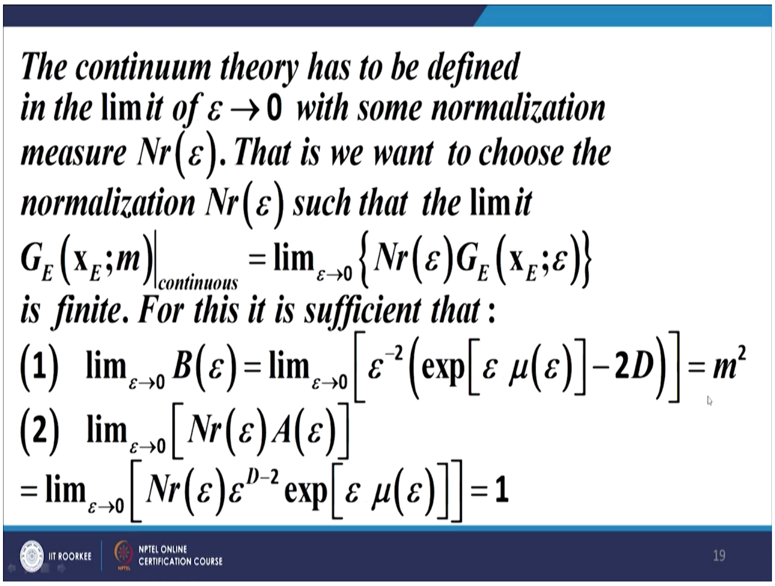

Now, in for the purpose of taking the continuum limit, we need to take the limit epsilon tends to 0 with some kind of a normalization which we already identified as N r epsilon. How we proceed to take the limits? The limits should be taken in such a way that we are able to recover the path integral in the continuous framework as a limit of epsilon tending to 0 N r epsilon into the expression, that we have got.

In other words, if I add on let me go back, if I add on a factor and normalization factor here and then take the limit as epsilon tending to 0. I should be able to recover the continuum version of the path integral and more importantly, that should be finite. So, that is where we are. And how we do it? Well, we do it by taking these two limits identified as 1 and 2. We take limit epsilon tending to 0 B epsilon as equal to m square and we take the limit epsilon tending to 0 Nr epsilon, Nr is the normalization; Nr epsilon A epsilon equal to 1.

So, these are two limits which we identify as or we impose so, that we get precise or a workable expression for the propagator or for the path integral.

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$$\textit{If } \lim_{\varepsilon \to 0} \left[\varepsilon^{-2} \left(\exp\left[\varepsilon \, \mu(\varepsilon) \right] - 2D \right) \right] = m^2$$

then for small ε, *we have*:

$$\mu(\varepsilon) = \frac{1}{\varepsilon} \ln\left(m^2 \varepsilon^2 + 2D \right)$$

$$= \frac{1}{\varepsilon} \ln 2D + \frac{1}{\varepsilon} \ln\left(1 + \frac{1}{2D} m^2 \varepsilon^2 \right)$$

$$\approx \frac{1}{\varepsilon} \ln 2D + \frac{1}{2D} m^2 \varepsilon \approx \frac{1}{\varepsilon} \ln 2D$$

So, when we take over these limits, these are some results that arise on the basis of these two and these two assumptions or these two parameters being introduced, we get mu epsilon is equal to 1 upon epsilon approximately equal to 1 upon epsilon log of 2 D on making certain first order approximations.

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Whence from 2nd condition

$$\lim_{\varepsilon\to 0}\left[Nr(\varepsilon)\varepsilon^{D-2}\exp\left[\varepsilon\,\mu(\varepsilon)\right]\right]=1 \text{ we get}$$

$$Nr(\varepsilon)=\frac{1}{\varepsilon^{D-2}}\exp\left[-\varepsilon\,\mu(\varepsilon)\right]$$

$$=\frac{1}{\varepsilon^{D-2}}\exp\left[-\varepsilon\left(\frac{1}{\varepsilon}\ln 2D\right)\right]=\frac{1}{2D\varepsilon^{D-2}}$$

And, similarly from the 2nd condition we get Nr epsilon is equal to 1 upon 2 D epsilon to the power D upon 2; again by using the previous expression for mu and introducing certain approximations.

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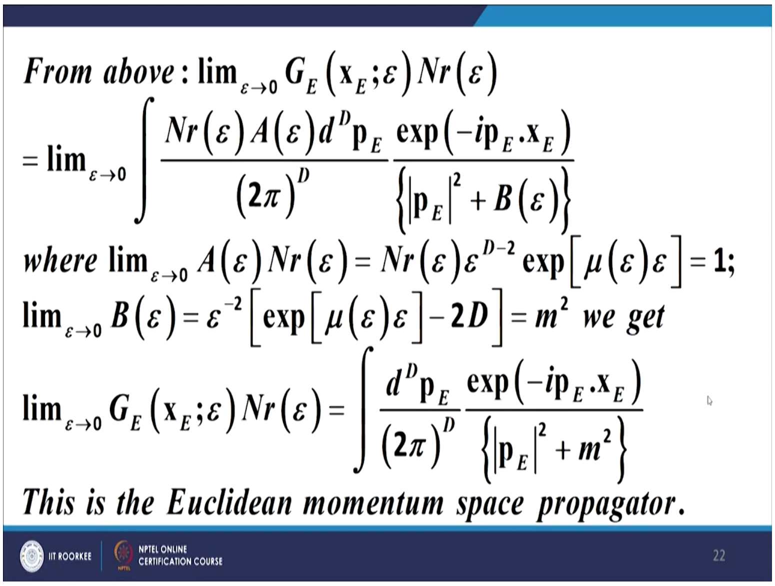

So, having put these having got these expressions, we can write the path integral in the given path integral, they continue in the continuous framework as limit epsilon tending to 0; the path integral in the Euclidean framework together with the normalization. We you have N r epsilon E epsilon equal to 1; we have B epsilon equal to m square. We make these substitutions and we get the result that is in the bottom equation of your slide which is the Euclidean momentum space propagator, Euclidean momentum space path integral.

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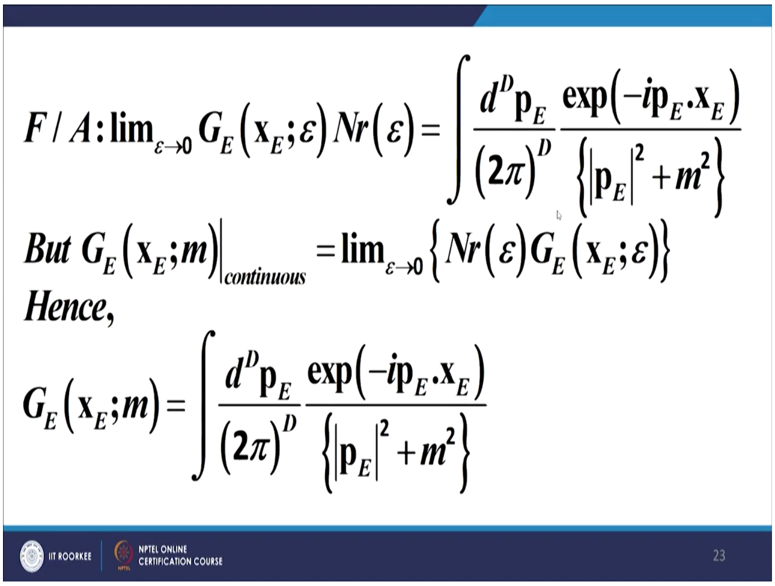

So, having done this now, we have already identified this as the continuous space or the continuum path integral by definition. And therefore, or continuum path integral in Euclidean space, in Euclidean space; please note this one. We will not transform to Minkowski space here, in Euclidean space works out to the expression which is given in the right bottom of your slide.

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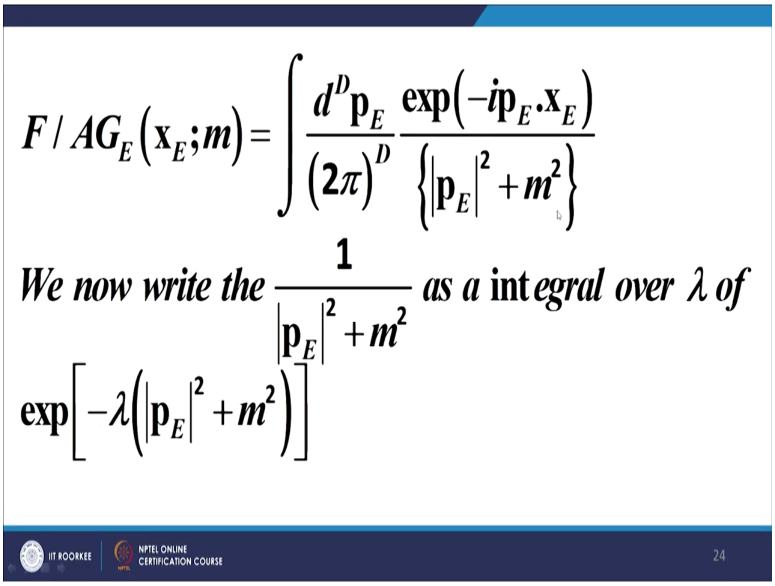

So, this is where we are, now we simplify this expression further. We carry out the integral or part of the integral at least. We write 1 upon p square plus m square as an integral over lambda n in this expression.

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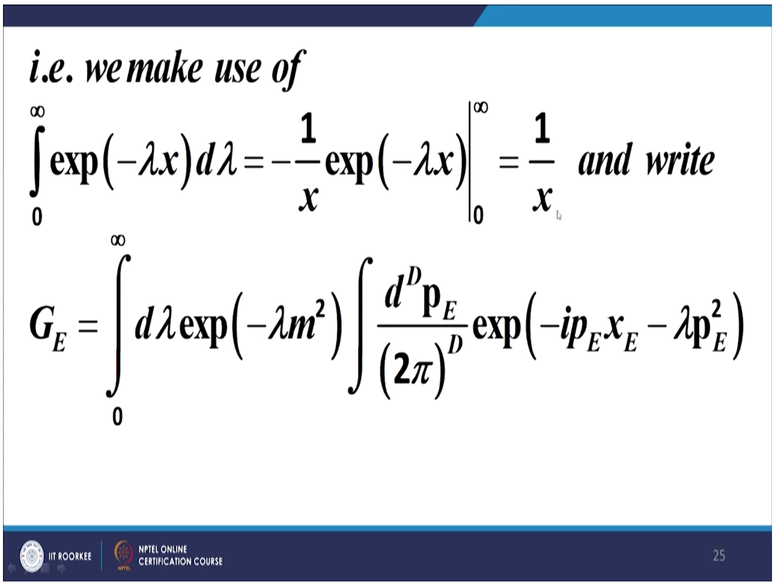

In other words let me move to the next slide, where I will explain this. You see if I work out this integral exponential minus lambda x d lambda integrated between 0 and infinity what I get is 1 upon x. So, using this expression, using this methodology or using this trick I write G E in the form or I write this expression 1 upon p square plus m square in the form of an integral using exponential of introducing a parameter lambda. And, writing it as exponential minus lambda into p square plus lambda square.

So, that is precisely what we would have done here, we have introduced exponential minus lambda m square is here, the p square factor is here in the second integral exponential minus lambda square into m square plus p square. And, this m square and p square have been split up. So, the m square has been retained with the lambda integral, the p square has been retained with the p integral. The second integral is clearly Gaussian; it can be integrated explicitly.

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$$F/A: G_E = \int_0^\infty d\lambda \exp\left(-\lambda m^2\right)\int \frac{d^D p_E}{(2\pi)^D}\exp\left(-ip_E x_E - \lambda p_E^2\right)$$

Doing the p_E − integration, *we get*

$$G_E = \int_0^\infty d\lambda \exp\left(-\lambda m^2\right)\int \frac{d^D p_E}{(2\pi)^D}\exp\left(-ip_E x_E - \lambda p_E^2\right)$$

$$= \int_0^\infty d\lambda \exp\left(-\lambda m^2\right)\left(\frac{1}{4\lambda\ \pi}\right)^{D/2}\exp\left(-\frac{|x_E|^2}{4\lambda}\right)$$

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You do the explicit integration and you get the expression which is here in the bottom equation of your slide, of this slide; the bottom equation is obtained after doing this Gaussian integral. And, the lambda integral still remains now what we you see.

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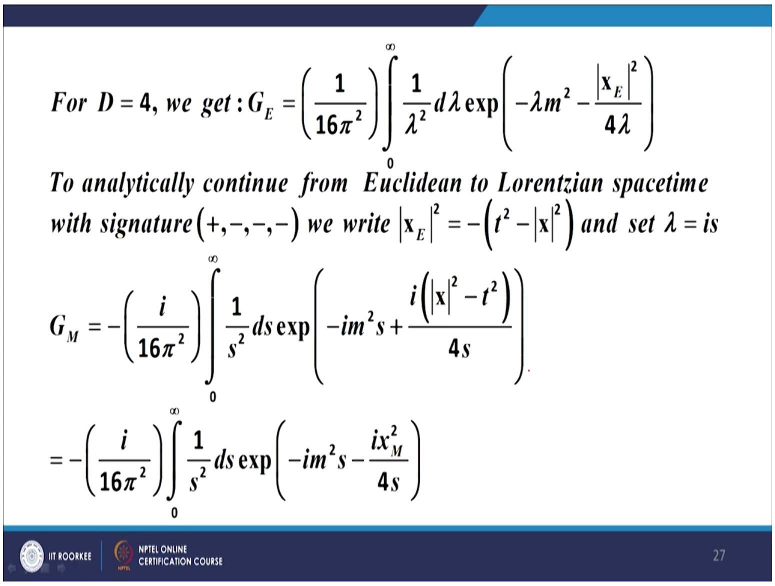

Now, we are literally through with the with the obtaining of the expression for the path integral in Euclidean space. Of course, if you take D equal to 4 for the 4 dimensional space which is relevant to us, we get this normalization factor of 1 upon 16 pi squared and the rest of it is also its what it is and. Now, to move from the Minkowski's to move from the Euclidean space to the Minkowski space or the Lorentzian space with this signature, that we have been working in ah; we write X E square as minus of t e square minus x square.

In other words, we write X E square as minus of t e square; please note this is this is modulus of X E square. So, modulus of X E square we write as minus of t e square minus x square and set lambda is equal to iota into s. On making these substitutions; so, what I get is X E square is equal to x square because of this minus sign I get x square minus t square. And, the lambda becomes is, I have substituted that here the lambda square d lambda 1 upon lambda square d lambda becomes 1 upon x squared ds with the with a negative sign. And, we get this expression for the path integral in Minkowski space.

So, this is this is the result that we obtained for a point particle, a relativistic path integral in Minkowski space. that is very interesting, but it has certain very interesting implications as well. Let us look at that for a look at this for a minute.

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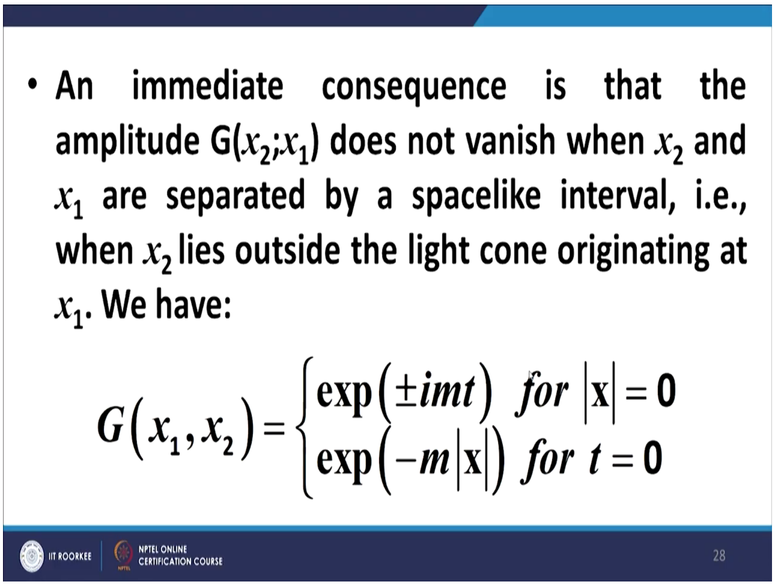

Now, let us consider two points where x 1 and x 2 are separated by a space like interval; that means, x 2 lies outside the light cone originating from x 1, x 2 lies outside the light cone originating from x 1. In other words, that distance x 1 x 2 is spaced like.

In that you see in that with this expression that we have obtained here, this expression that we have obtained here. In this slide for the path integral of a point particle that we have obtained in the bottom equation of this slide can be simplified to the expression, that we have at the bottom of this slide by using this saddle point approximation.

The approximation tells us in the second, the second case will apply and the exponential and the approximation shows that the path integral or the transition amplitude is gradually exponentially terminating or exponentially going to 0. It is not 0 right away, that is the important part; its exponentially falls to 0.

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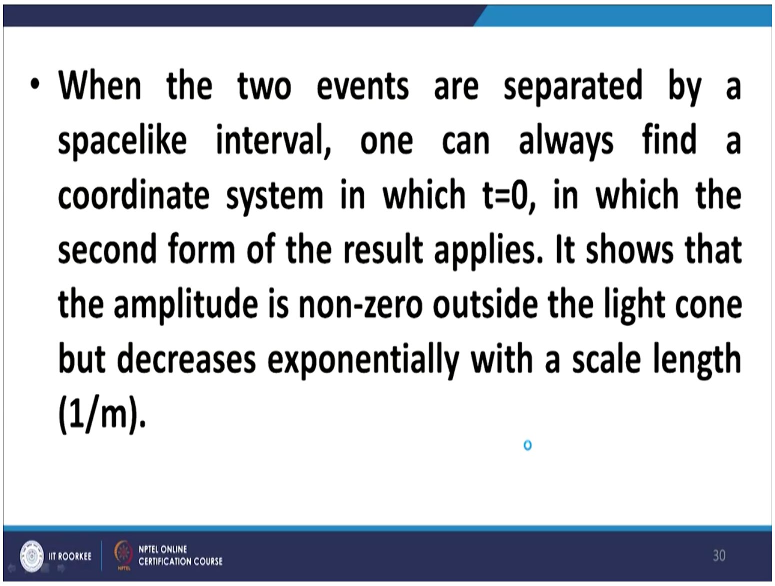

The its it is non-zero for some point in time and then it gradually decreases exponentially to with a scale length of 1 upon m. And, the important thing is that it is not strictly 0 outside the light cone.

This is a very interesting result which will have a lot of implications when we talk about the need for quantum field theory or multi-particle system, multiple multi particle interpretations in view of the single particle interpretation of quantum mechanics.

The need for quantum field theory, why do we need quantum field theory in view of quantum mechanics, it is a very interesting area and we shall be covering it in the next lecture probably.

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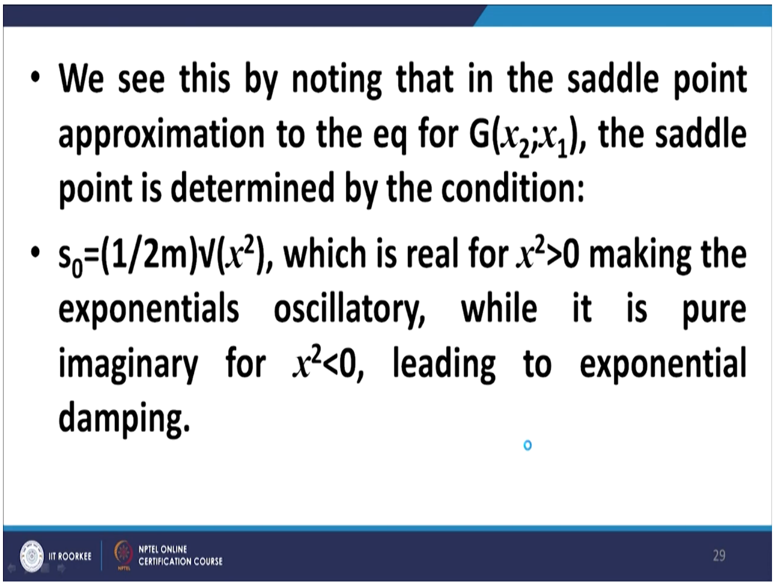

Now, so, that is a very very important fallout of the result that we have obtained here.

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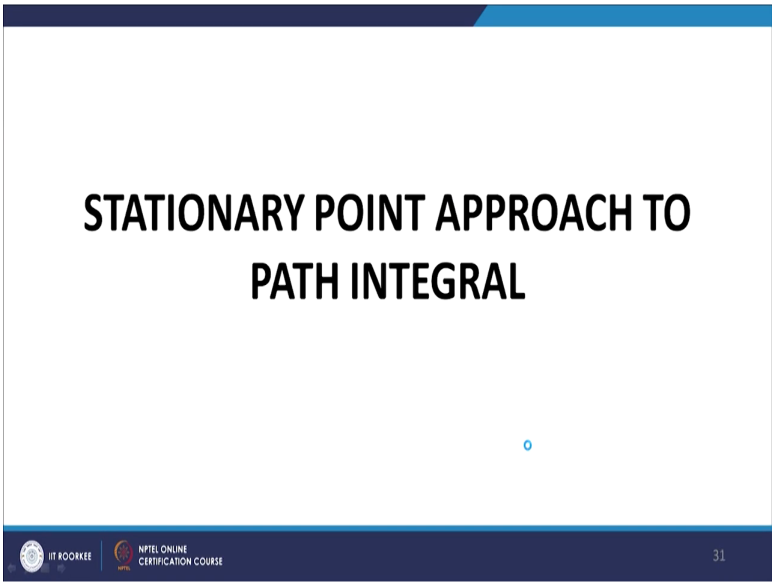

This is the result and let us take a break and then we will continue with the a new interpretation or a new approach to working out the path integral which is far more suggestive, far more instructive and far more explicit. So, let us continue with that after the break.

Thank you.