**Path Integral Methods in Physics & Finance**
**Prof. J. P. Singh**
**Department of Management Studies**
**Indian Institute of Technology, Roorkee**

**Lecture – 21**
**Harmonic Oscillator Path Integral**

Welcome back. So, today I propose to take up some examples to illustrate the application of the Quantum Mechanical Path Integral. But before I do that let us recap the various steps that enabled us to write down the Feynman's path integral for the quantum mechanical system for a general quantum mechanical system and then we will apply them to certain specific cases.

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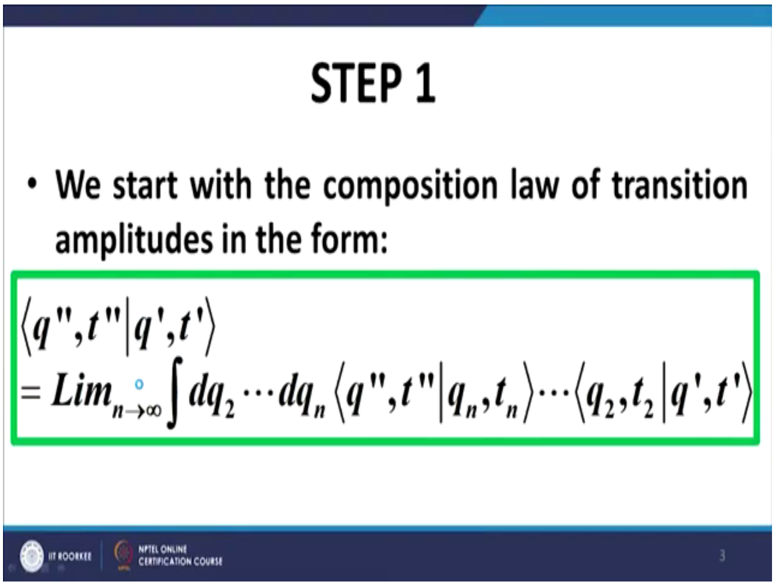


So, let us start. The first step of course, we started with the composition law of the transition amplitudes in the form that is given in the green box. The composite the propagators or the transition amplitudes follow the propagation law which is given in the form in this particular box.

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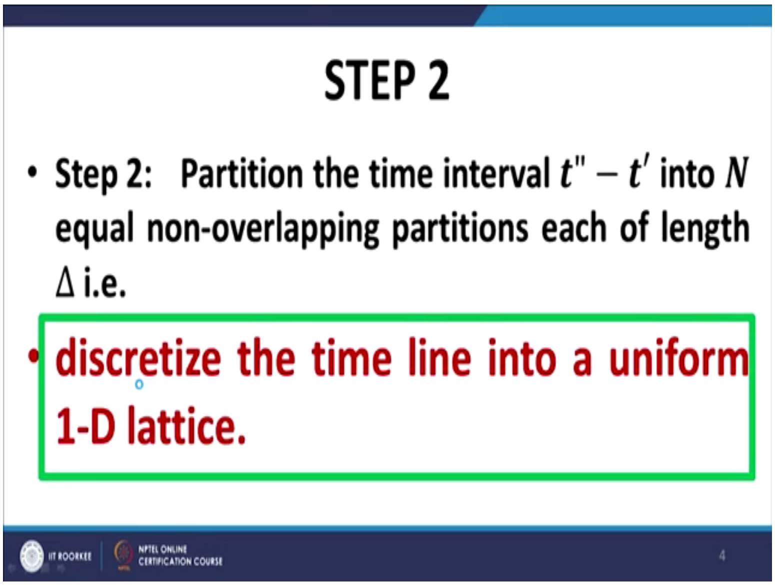


The second step involve, partitioning the time slice t double dash minus t dash into small infinitesimal partitions small blocks. In other words we are simply discretizing the timeline into small into a lattice, a lattice that is having lattice points or a lattice constant of size delta and there are N such points on the lattice starting from t dash which is the initial state of the system. Or at which point the system is being observed initially to the final state of the system which is being observed at time t double dash.

The entire timespan is split up into small fragments discrete fragments of time. So, that is the second step.

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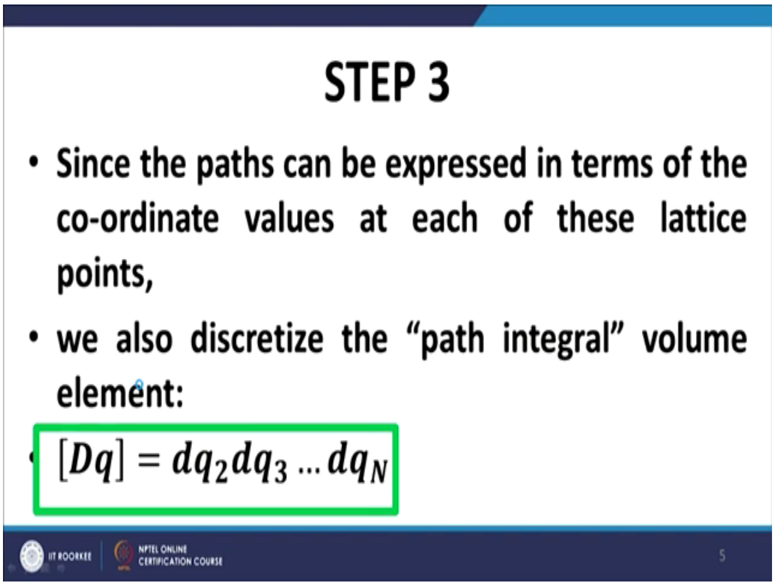


Then, now because this this discretization is so fine the partitions are so, fine that we believe that this is the paths that is followed by the system any particular part that is followed by the system; can be described by the coordinate values at each of these lattice points.

The discretization is sufficiently small, sufficiently fine to enable us to make the approximation that by describing the coordinates or by capturing the coordinates at the various time slices time points; we are able to extract complete information about the path of the system that is the next step in. And that enables us and that enables us to write the path integral volume element in the form of this in the form of the respective coordinates and these are the infinitesimal integration volumes with respect to these variables.

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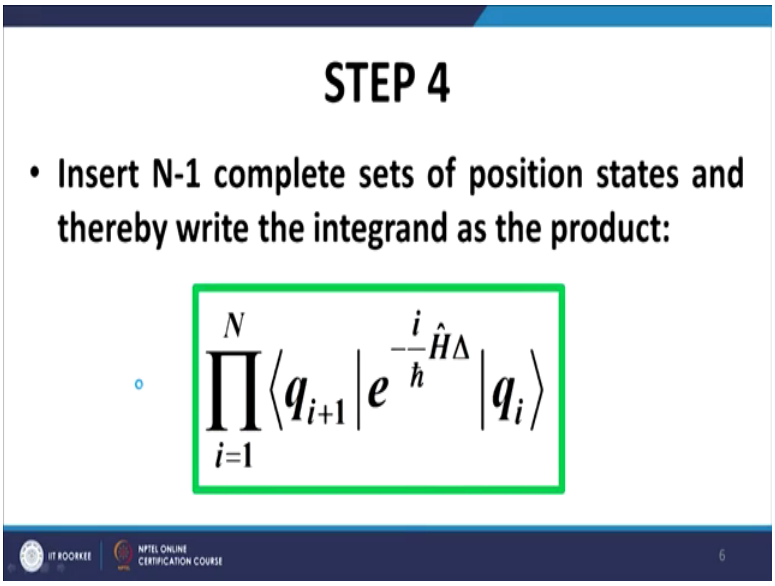


And the next step we insert N minus 1 complete set of position states in the in our expression for the transition amplitude. And that enables us to write us write it in transition amplitude in the form of this set of products a sequence of products of this form which is given in the green box at the bottom of your slide.

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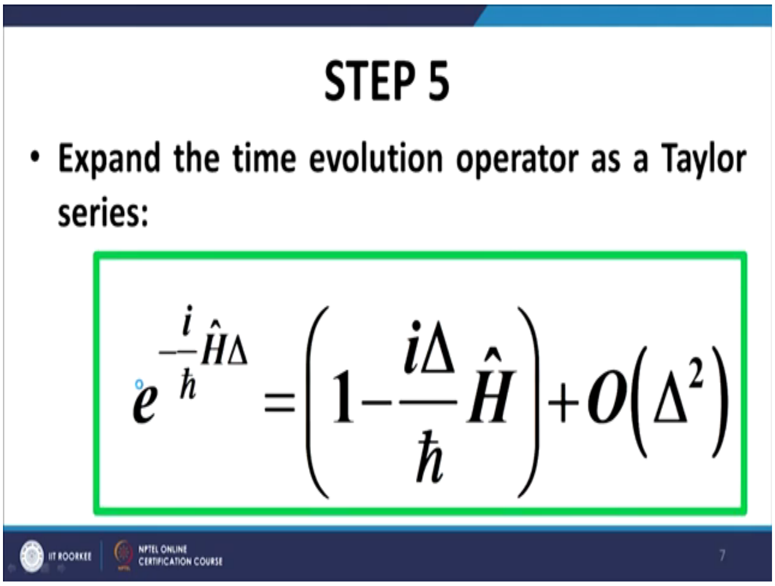


And, the next step is to view the exponential that contains the Hamiltonian operator can now be exponential; can now be expanded as a Taylor series as an exponential series up to first order in the Hamiltonian. And then the higher order terms in delta are collected together and represent by O of delta square which as we shall see in in a subsequent step does not contribute significantly to the overall dynamics. And the dynamics can be sufficiently accurately represented by a first order expansion of this exponential given on the left hand side of your equation in the box.

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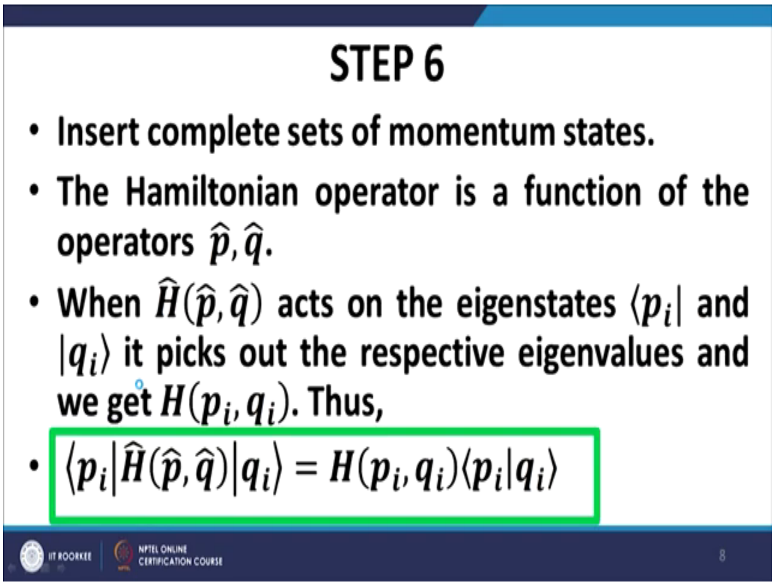


Now, we insert do another insertion in this you see in the last case we inserted N minus 1 complete sets of position states, now we insert complete sets of momentum states. Now why we insert momentum states will become very clear when you look at the equation in the green box.

Now, the Hamiltonian operator is a function of the position and momentum operators with q and p. When we sandwich the this Hamiltonian operator between the momentum states and the position states in the form that is given in the left hand side of your box. What happens is, the position states operate on the or the position operator also the momentum operators in the Hamiltonian operate on the momentum states eigen states and the position components or the position operators in the Hamiltonian operate on the position states.

As a result of which they extract the Hamiltonian these operators q and p extract the eigenstates corresponding to these states that are given with that sandwich operator. And thereafter return the eigenvalues corresponding to these respective states the effect is what is shown on the right hand side to reiterate. This is an important step technical step the Hamiltonian operator consists of the the position and the momentum operators in a certain combination for example, for a free particle we have the Hamiltonian p at the operator p square upon 2 m.

Similarly, we could have a momentum we could have a potential term as well in the Hamiltonian for a system which is not free which is subject to a potential. And in any case this Hamiltonian when it is sandwich on the (Refer Time: 06:49) side by the by the momentum states. And on the case side by the position state the momentum the momentum operator in the Hamiltonian will operate on the momentum states and will pick out the corresponding eigen values.

Similarly, the position operators in the Hamiltonian will operate on the position states and pick out the corresponding eigenvalues and as a result of which what we get is the Hamiltonian which is now not a function of operators, but is a function of variables. So, it is the standard function rather than being a function of variable the operators are replaced by the corresponding eigen values.

This is a very important step, but in this particular step a care has to be taken or a note has to be made of the fact that we have assumed implicitly that the momentum operators in the Hamiltonian are to the left and the position operators in the Hamiltonian are to the right.

So, that the momentum states operate are operated upon by the momentum operator in the Hamiltonian. And the position states are operated upon by the position operator in the Hamiltonian, but that need not necessarily be that is more of a convention and that need not necessarily be the assumption invariably made.

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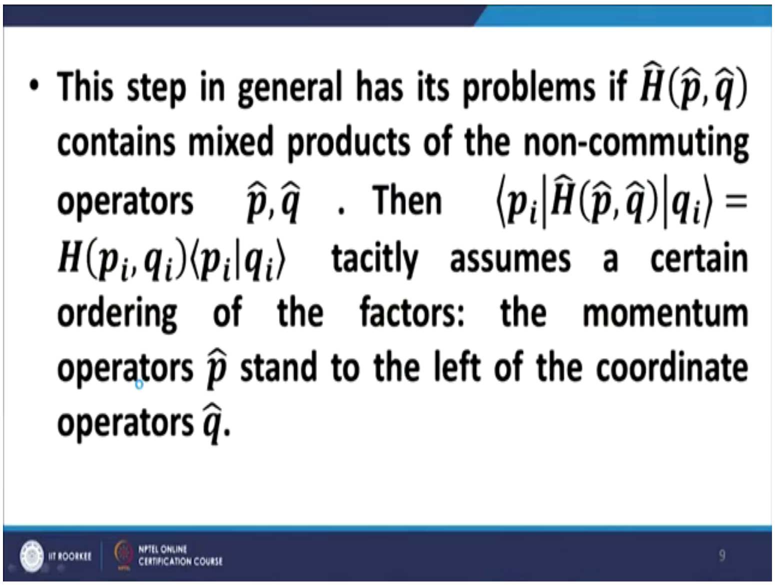

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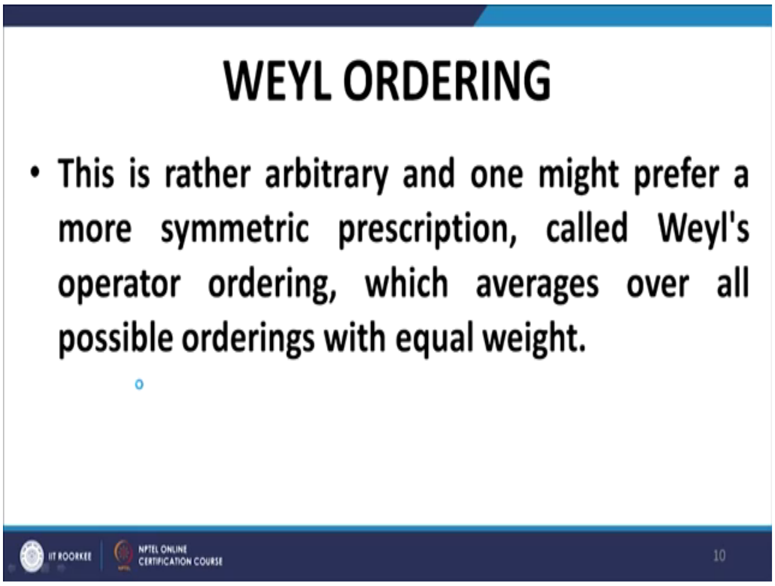


In contrast to this you can have other assumptions for example, the while ordering of operators which is which involves taking, which involves taking an average over various combinations of various pairing of these operators.

So, this specific ordering of operators where the momentum operators at are placed to the left and the position operators are placed to the right. In the Hamiltonian is a more a convention rather than dictated by the any physics of the system and it can be replaced by other orderings like the while ordering.

The next step that step was the fundamental step the next step once you have extracted the Hamiltonian and Hamiltonian has become a function of the eigenvalues. It can be taken outside the outside the dot products and we are left with the dot product of the position and momentum states which is given by this expression given in the green box.

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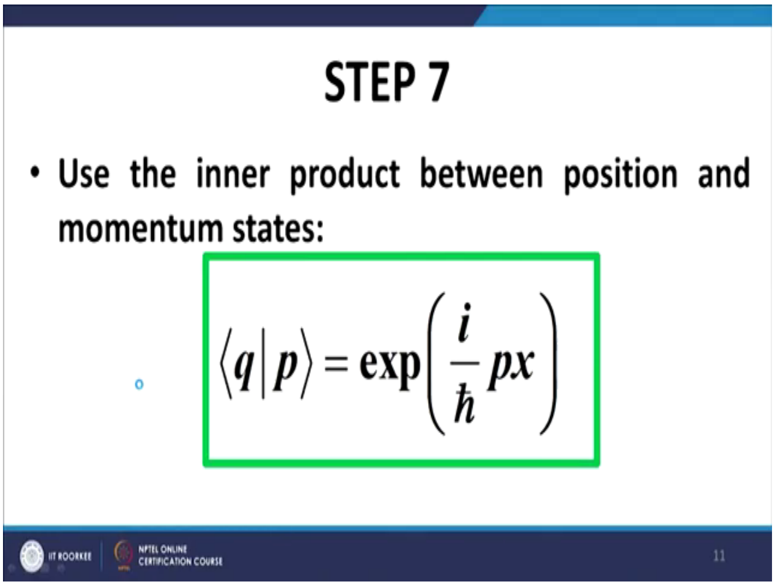


So, we can use this expression wherever required and we can substitute this product this dot product with this particular exponential.

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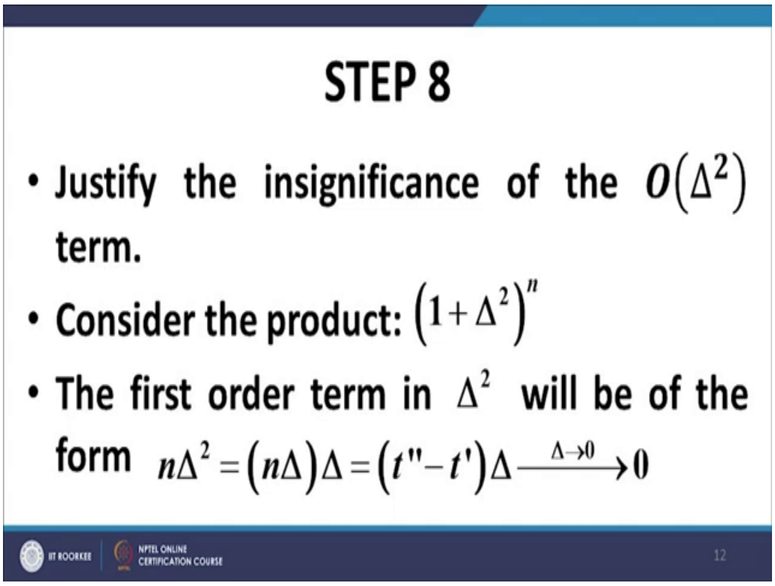


Now, we come to a point that I made earlier when during the course of this discussion just now that the second order and higher order contributions. When you expand the Hamiltonian or the evolution operator as the exponential of the Hamiltonian the second order and higher order terms do not contribute; first order expansion is sufficient. And that is that is justified on the fact that if you look at the delta squared terms there will be n such terms and they will be multiplied.

If you can these terms would be multiplied n times and so, the first the first order terms in that product would contain n delta square, now n delta square can be written as n delta n of delta square n of delta square can be written as n delta into delta. Now n delta because we made n partitions of t double dash minus t we can write it as t double dash minus t t dash I am sorry t double dash minus t dash into delta.

Clearly t double dash minus t dash is finite, therefore when this expression trans is translated to the continuum limit delta will be tending to 0. And therefore, this whole term tends to 0, the net effect of what I said is that, the higher order terms other than let us go back a bit let me go back. The higher order term this order delta square terms these terms do not contribute to the diagonals and this expansion to the first order is good enough for our purpose right.

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**STEP 9**

Use $\lim_{N\to\infty}\prod_{i=1}^{N}\left(1+\frac{x_i}{N}\right)=\exp\left(\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}x_i\right)$

Set $\frac{x_i}{N}=-\frac{i\Delta}{\hbar}H(p_i,q_i)$ so that $\frac{1}{N}\sum_{i=1}^{N}x_i=-\frac{i\Delta}{\hbar}\sum_i H(p_i,q_i)$

write $\lim_{N\to\infty}\prod_{i=1}^{N}\left[1-\frac{i\Delta}{\hbar}H(p_i,q_i)\right]$

$=\exp\left[\lim_{N\to\infty}-\frac{i\Delta}{\hbar}\sum_i H(p_i,q_i)\right]$

Then we use this formula this is the standard formula that we have it is a generalization N in fact of the binomial and this formula enables us to write a product in the form of an exponential of a sum. We use this with the settings that is given here xi upon capital N is equal to minus i delta upon H bar into H p i q i. When I use this on the left hand side on the right hand side what I get is, what is given in the expression the expression at the bottom of your slide exponential limit n tending to infinity in this expression.

Now, look at this the product of the N terms involving the Hamiltonian has been translated to an exponential involving summation of the Hamiltonians.

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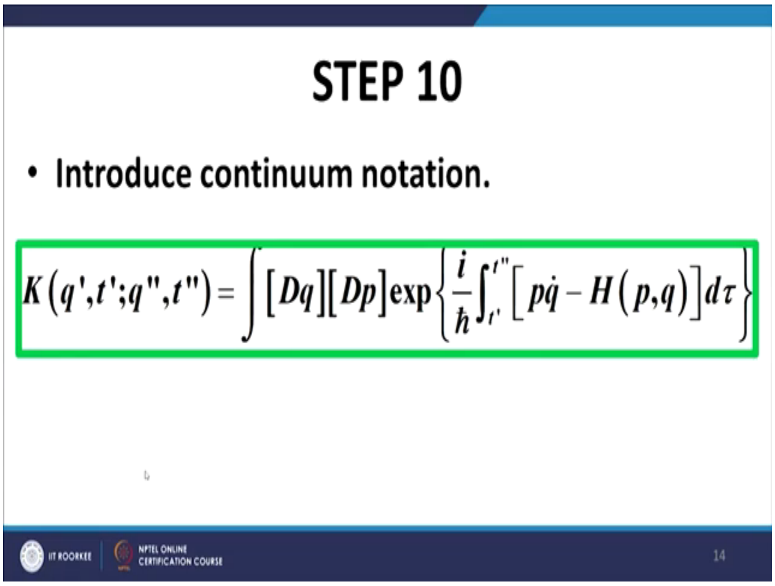


Now, the we are literally through we simply introduce continuous notations wherever required. And on introducing the continuous notations we get the expression for the transition amplitude in the form that is given in the green box that is here.

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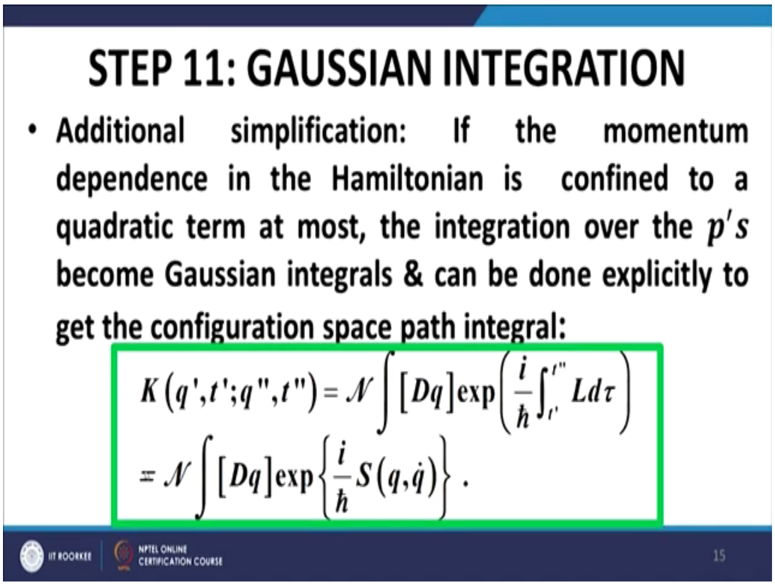


And in the special case; in this special case of course, when the Hamiltonian is no more than quadratic in the momentum operators. For example as in the case of a free particle, we can do the momentum integrations can be done because they turn out to be Gaussian and they can be done explicitly and we get a simplified form of the of the transition amplitude in the form of the equation given at the right at the bottom of your slide in the green box. And this clearly invokes the action of the system which is given by S or which is denoted by S which is a function of course, the coordinate and its derivatives.

So, these were the various steps that we followed in order to arrive at the Feynman integral in configurations this is an integral in configuration space now. So, this is the Feynman integral in configuration space and this is the cornerstone it the backbone or the backdrop of a lot of field theory that we shall be doing in the rest of this course and in fact, we shall also be exploring the role of this integral in pricing a financial products.

So, let us get along with that let us take some examples.

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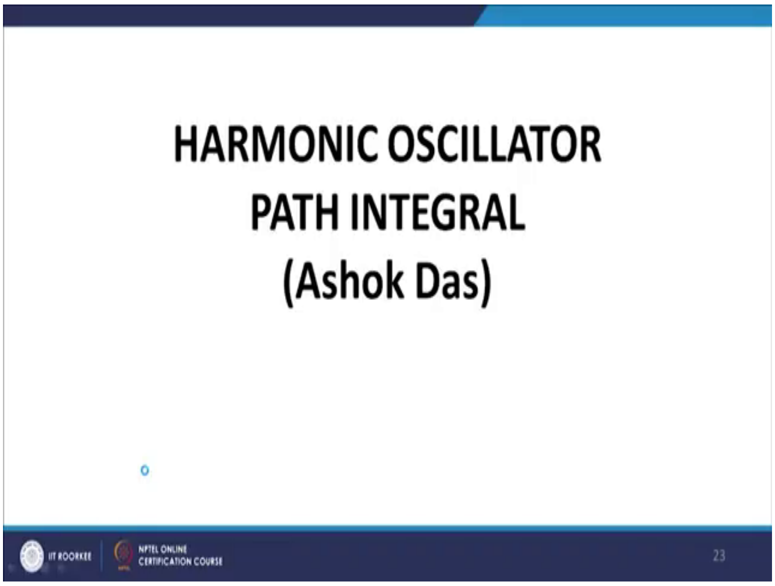


Now, the first example that I propose to take up is that of the harmonic oscillator, the harmonic oscillator forms the fundamental building block of field theory. So, let us start with the harmonic oscillator we work out the harmonic oscillator path integral.

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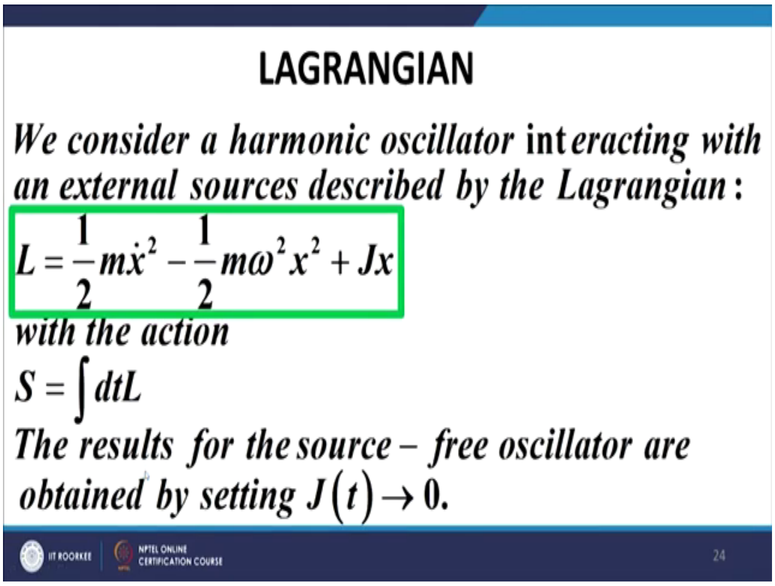


We start with the Lagrangian of this harmonic oscillator which is given in the green box which contains this. In fact, there is a source term here we have included a source term here in this Lagrangian which enables us to model at the oscillator under act which is under the influence of which is interacting with a source term which is represented by this term J x, this J x is these source term. And of course, the action is given by the integral of this Lagrangian with respect to time. In the case of free oscillator and the source term would vanish and you can get the dynamics of the free oscillator simply by substituting J equal to 0.

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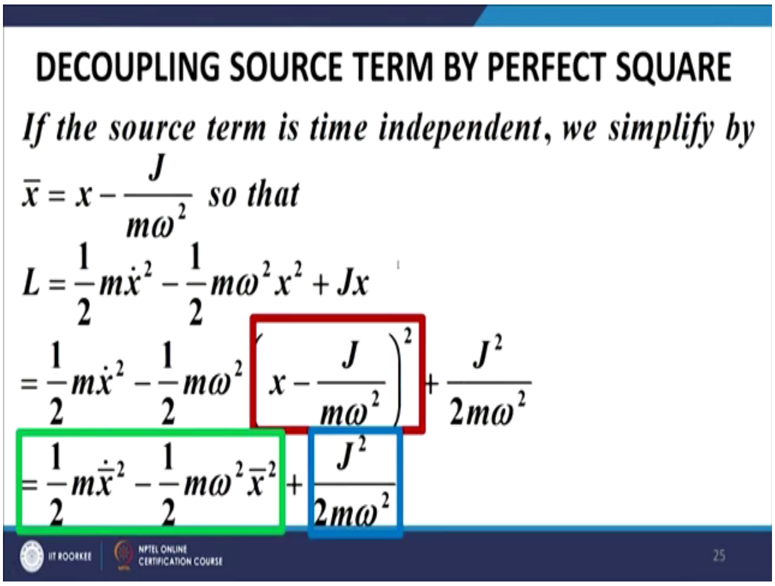


Now, in this special case when the source term is time independent; in the special case when the source term is time independent an immediate simplification of the Lagrangian can be achieved by shifting the origin. And that enables us to write the Lagrangian in a simplified form in the expression that is given in the green box plus this additional term which captures the effect of the source term in the blue box.

So, the source term impact is captured in the blue box and the impact and the two effects are in a sense decoupled. If, but this is possible only if the source term is time independent otherwise it the simplification cannot be done and this simplification is simply done by shifting the origin of coordinates.

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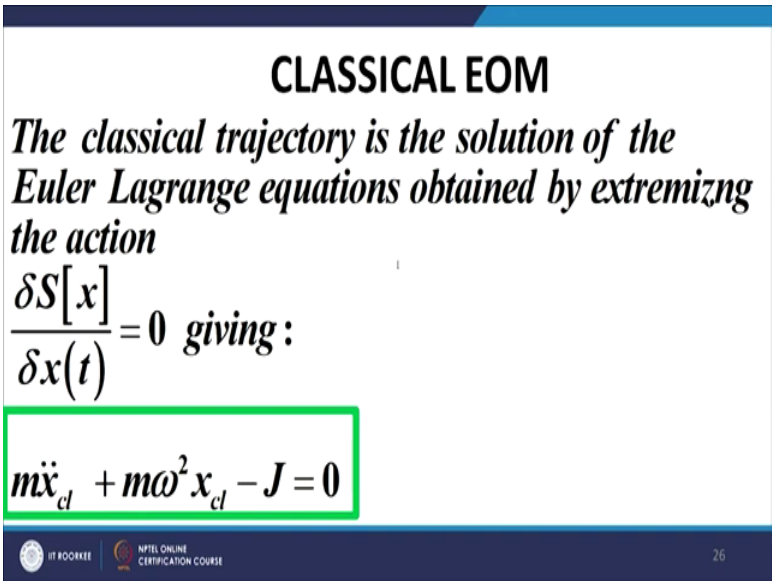


The classical equations of motions can be obtained straight away by solve by extremizing the action within they give us the Euler Lagrange equations. And the Euler Lagrange equations for this particular system if you work them out in detail you get the expression that is given in the green box at the bottom of your slide, mx double dot plus cl the subscripts cl denotes that these are classical this is these are equations for the classical path. So, mx cl plus m omega square x cl minus J is equal to 0 x cl represents the classical variable or the classical trajectory.

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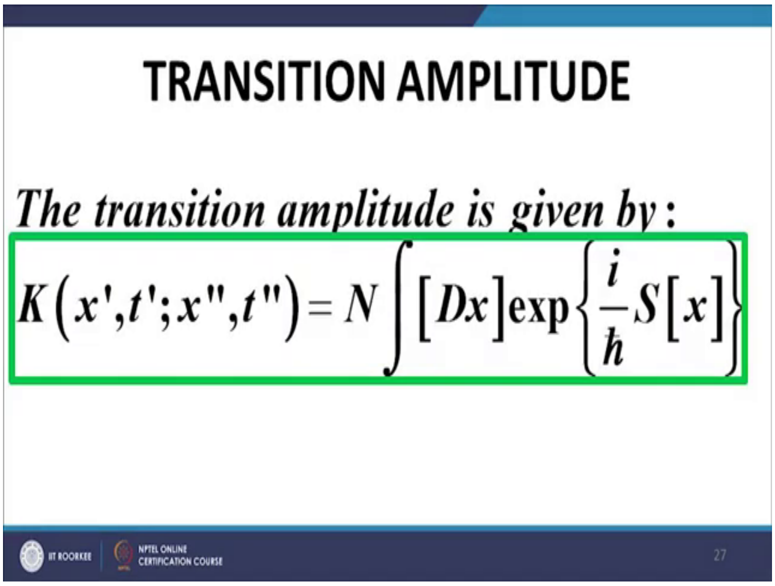


The transition amplitude the expression for the transition amplitude that we talked about just a few minutes back is given by the normalization factor into integral D x .The path integral Dx exponential i upon h into the action of the of the harmonic oscillator N is the normalization constant. We shall be working out the value of N, but for the moment we write it in this form.

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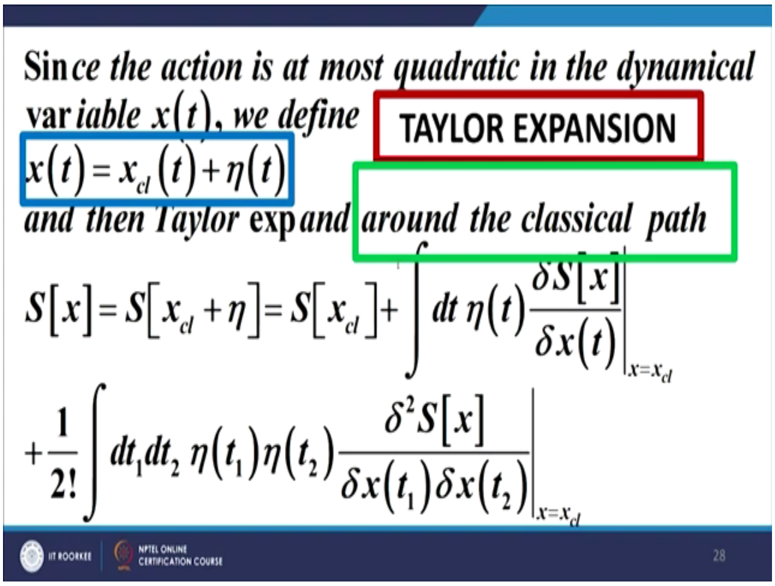


Now, we what we do is we assume that the paths that the has the quantum mechanical harmonic oscillator follows can be represented as deviations from the classical paths by introducing another variable eta t. And we can write the variable x t the original variable x t in terms of the classical paths classical trajectory and the deviations from the classical trajectories.

So, in effect eta t is the expression that captures the fluctuation the quantum fluctuations, the fluctuations due to the quantum nature of the oscillator and x cl will represent the classical state of the or the classical flow or the classical trajectory of the system. And we what we do now is, we expand the action; we expand the action again as a Taylor series around the classical action. That is precisely what is given here we are expanding the action the action of the quantum mechanical system as a Taylor series around the classical action.

This x this S of x cl is the classical variable. So, S x cl is the classical action and these are the Taylor expansions. Now the important thing to note is that we shall see later that this expression because we were talking about the Euler - Lagrange equation gives us that the functional derivative of the action with respect to x will be 0 at the for the classical path. In other words what we are saying is that the oscillator will follow that trajectory classically for which the action is extremized that enables this term to go to 0 and with this term going to 0, we can write the action as and we can write the action as we will come back to it right.

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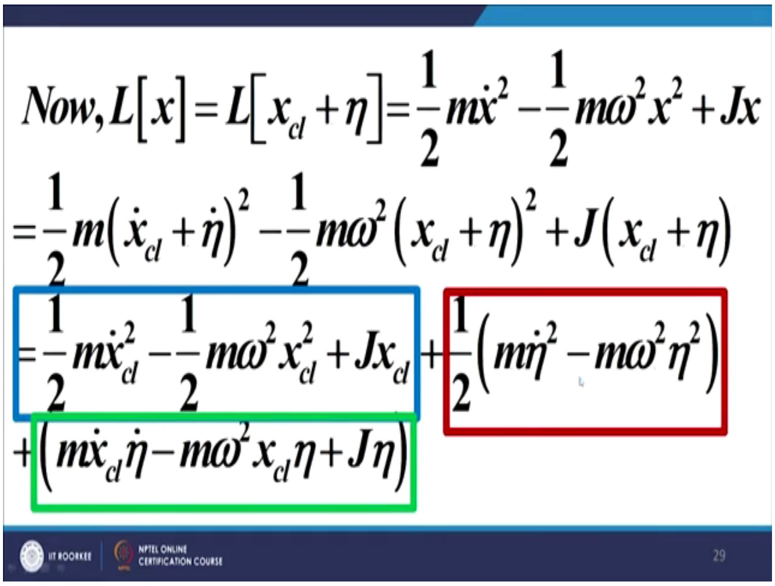


So, now let us expand now let us expand the Lagrangian let us also expand the Lagrangian in terms of the classical trajectory or the classical variables and the quantum fluctuations represented by eta. The expression that I get is the equation that is shown at the bottom of your slide.

The first term is the expression in the blue box, the second green red box and the third in the green box. Now the first term that is the expression in the blue box represents the classical Lagrangian that we started with; this is the classical Lagrangian. The second term represents the effect of the quantum fluctuations. And the green box the green box see now the important thing is when you how will I, how will I get the action from this? I will integrate the Lagrangian with respect to time and over the limits of time being the initial time and the final time.

But the initial time and the final, at the initial time and the final time the paths are fixed the path points are fixed, in other words at these two points eta has to be 0. And therefore, when I integrate this this term the term that is in the green box, when I integrate it between the limits of the initial time and the final time these this term goes to 0.

In other words the term in the green box does not contribute anything to the action and the action can therefore, be written as a combination of the classical action plus this particular term or the contribution due to the term that is there in the red box.

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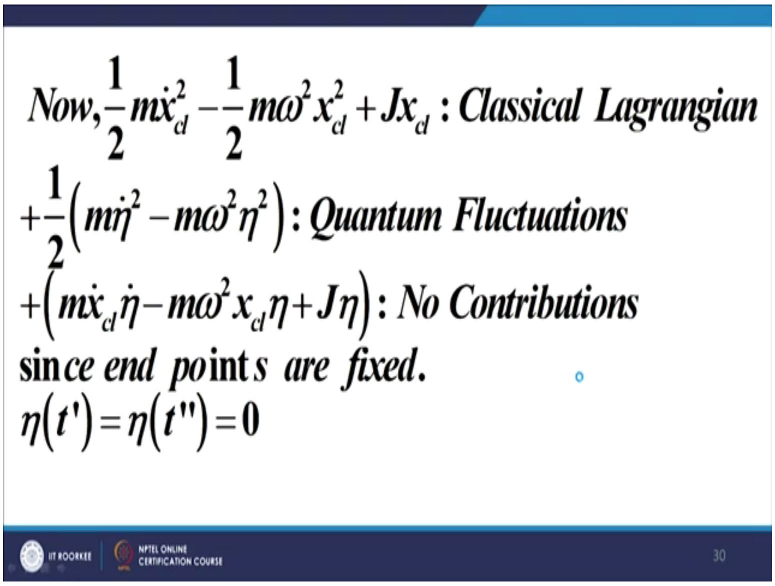


So, that is what I have explained just now is what it is there on the slide the first term the term that was there in the blue box is the classical Lagrangian. The second term represents a quantum fluctuation the component of Lagrangian due to the quantum fluctuations. And the third term will not make any contribution to the action the third term will not contribute make any contribution to the action since the end points are fixed which implies that this condition will hold. Because the classical and the quantum trajectories will necessarily coincide at the fixed points at the points at t equal to t dash and t equal to t and double dash the initial point and the final point.

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$\eta(t)$ represents the quantum fluctuations around the classical path.

Since the end points of the classical trajectory are fixed,

$$\eta(t') = \eta(t'') = 0.$$

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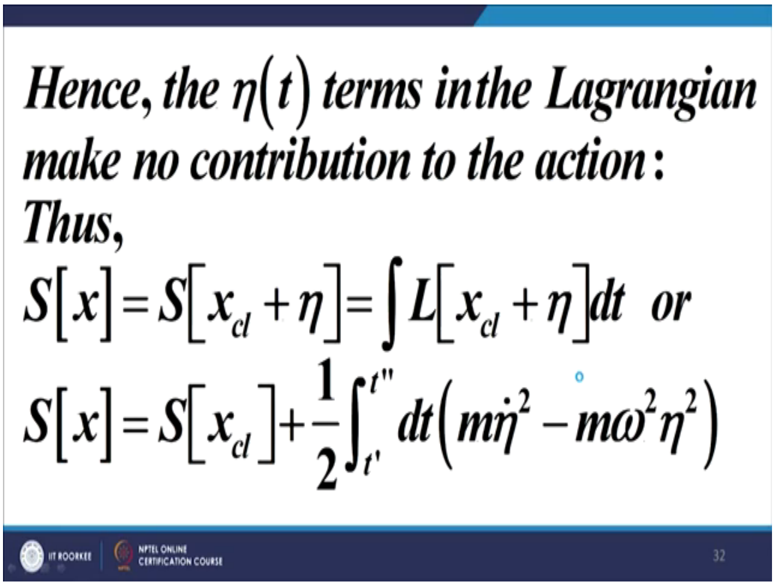


So, therefore, we can ignore the term that was there on the green box you can have a look at this. Again if you like this term we ignore and the action will now comprise or shall be developed or shall be determined by the Lagrangian comprising of the term that is there in the blue box and the term that is there in the red box.

So, that is precisely what is written here, the term that is there in the blue box is nothing, but S the classical action and the third term in the red box is written out explicitly.

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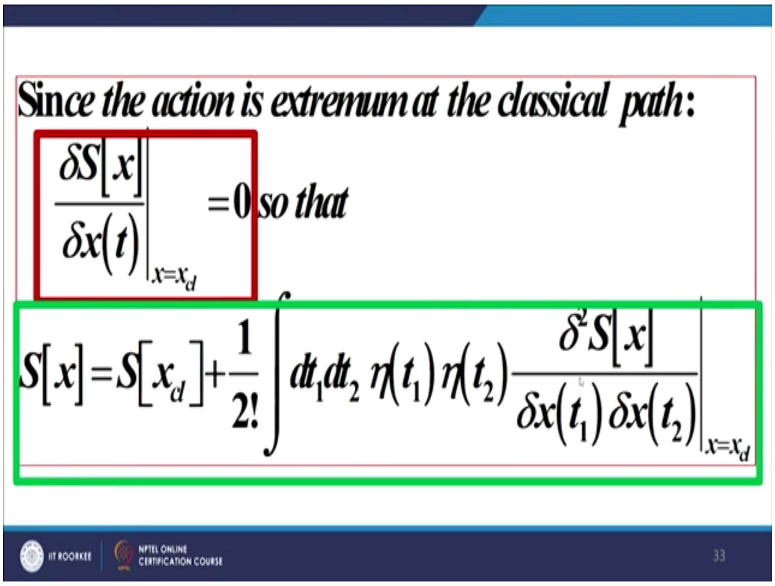


So, as I mentioned as I mentioned earlier because in the context of the classical path. How do we determine the classical path, we determine the classical path by extremizing the action. So, when we extremizing the action the required condition is represented by the expression in the red box.

So, when I use this expression in the red box the Taylor series expansion of the action gets simplified to just the two terms. Of course, because the action is quadratic therefore, there will be only a terms up to second order. And out of these two terms up to the second order the first order derivative vanishes; because of the optimality condition because of the extremizing condition and we are left with this expansion which is there in the green box for the for the Taylor expansion of the action around the classical action.

So, now we have in the sense we have two expressions for the action this is one expression which has been obtained by the Taylor expansion and the other expression which has been obtained through development of the Lagrangian and directly and this is that expression. So, we have got two independent expressions for the action of the quantum harmonic oscillator.

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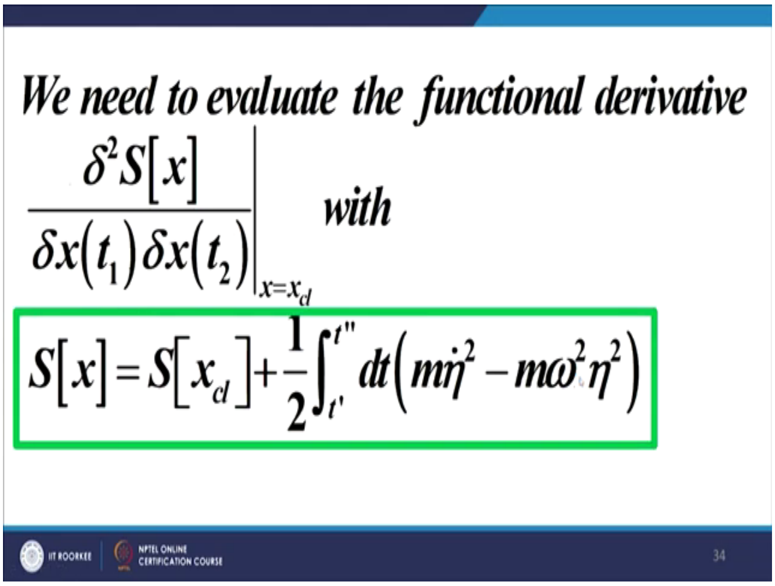


Our problem is to evaluate this particular derivative the second functional derivative with respect to x t 1 and x t 2; this is precisely, what is the agenda that we are going to follow now using the expression for the action that is given here.

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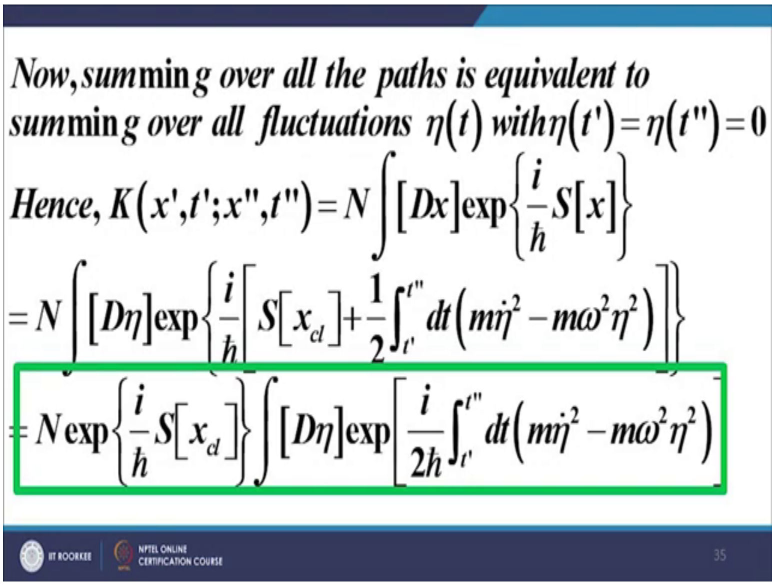


Now, so, this is substituting the value of the action here we have the in the path integral, in the expression for the transition amplitude what do we have, we have this expression for the transition amplitude our object is simply to simplify it. Now if you look at this carefully the D x or the path integral volume has been replaced by the volume around eta. Because you see classical action is in a sense the fixed action and it is the fluctuations of the summing of paths the various paths are captured or the dynamics of the path are captured by eta, they represent the various fluctuations around the classical path.

In other words, as we have done explicitly we have split up the path integral into two paths, one is the classical path and then we have captured the deviations around the classical path to represent all the other paths. So, because the classical path is in that sense a fixed path we take can take it outside the integral and the rest the path integral can equally well be represented by the fluctuation variable eta instead of the original variable x because x classical represents in a sense a fixed path a unique path which is which is obtained by extremizing the action right.

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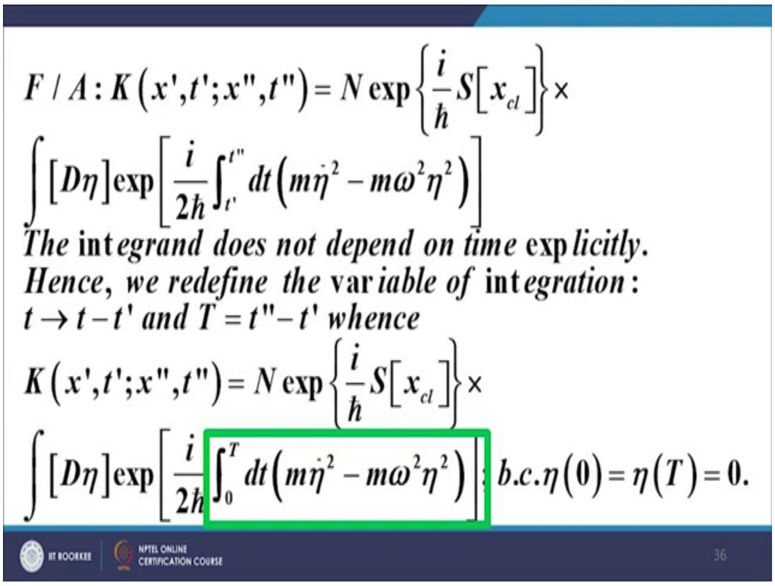


So, the classical term go goes out of the integral the integration variable the path integration variable changes from x to eta. And this is what we are now supposed to evaluate the expression that is there in the in the square brackets or the expression of the path integral with respect to eta of the expression, which is there in the square bracket, exponential of the expression which is there in the square bracket.

Now, because the integrand does not depend on time explicitly we can simplify further we can simplify further we can redefine the origin of time in such a way that we have t dash we define t as t minus t dash or we shift the origin of time and we take capital T as the as the time length between t double dash and t dash. In other words, we are doing nothing, but simply shifting the origin of time we are shifting t 2 t minus t dash right. So, in a sense our system was starts at t equal to 0 in terms of the new time variable our system starts at t equal to 0 and our system final observation of our system is at capital T instead of t double dash right.

So, we can write the path the action as integral 0 to 2 t dt into this expression of course, the classical part of the action is already out of the integration out of reckoning in a sense and it is already gone outside the in path integral as a together with the normalization constant. And the boundary conditions, the boundary conditions also change the boundary. What are the boundary conditions? The boundary conditions where eta t dash is equal to eta t double dash equal to 0 in terms of the new time variables they become eta 0 is equal to eta capital T is equal to 0.

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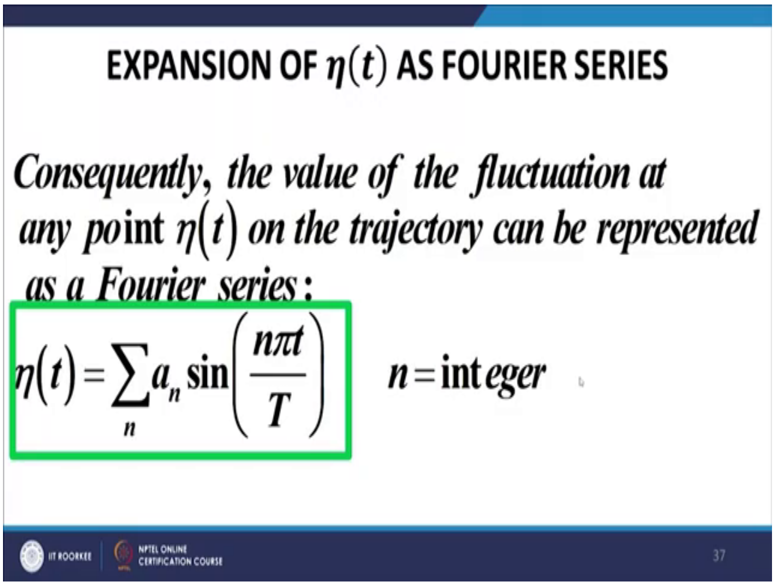

A next step would be to expand this expression as a Fourier series expand eta t as a Fourier series that is our next step. What we do is, we expand eta t as a Fourier series in the form which is given in the green in the green box summation over n a n sin n pi t upon capital T.

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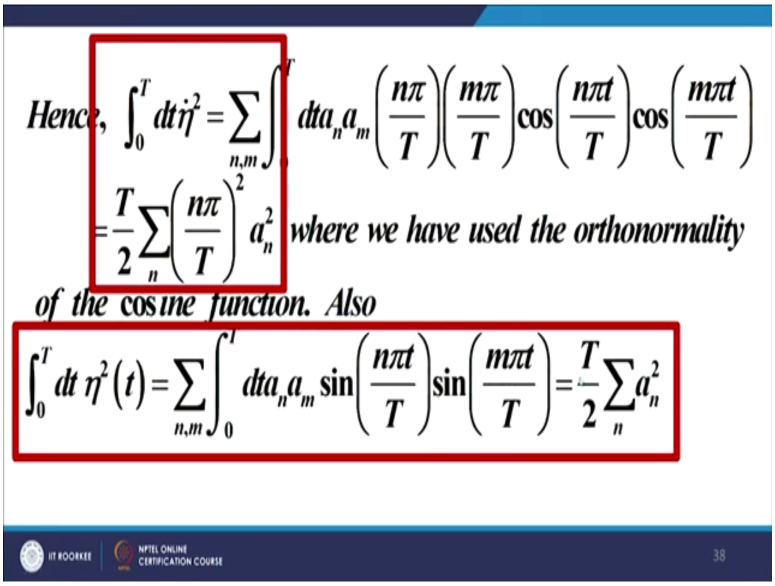


Now, there is a very important point here when we do the expansion which will come to in a minute, but before I do that these are some simple computations and the values of integral of eta dot square between 0 and t. Because we shall be needing that for working out there or for working out the action and we work out the value of eta square as well for working out the action you will recall that we need them. Why do we need them? We need them for working out the expression in the green box here.

So, we need the value of eta dot square integrate it between 0 to capital T and we need the value of eta square integrated between 0 to capital T. As I mentioned we expand eta t as a Fourier series in this form from here I will continue after the break.

Thank you.