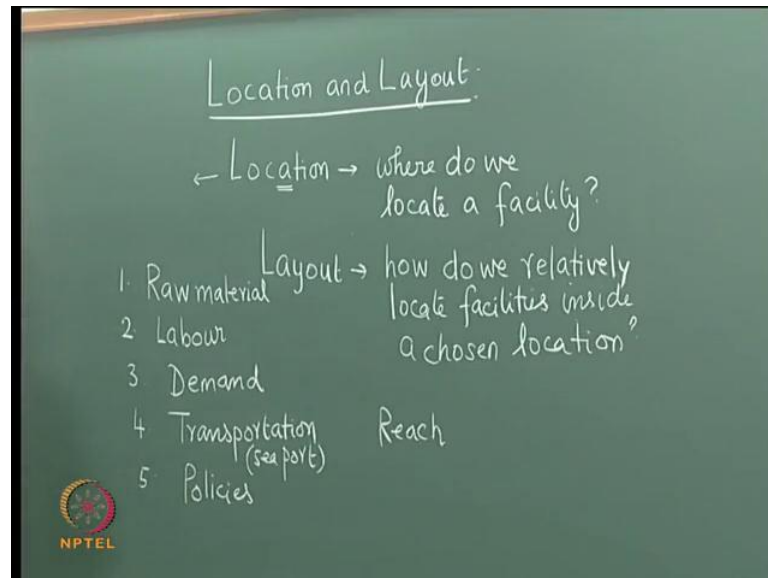


Operations and Supply Chain Management
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Lecture - 32

Location Problems - P Median Problem, Fixed Charge Problem

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In today's lecture, we address location and layout. Location essentially deals with, where we want to locate the facilities and layout deals with, how we locate the other facilities inside, once the location is chosen. So, we are going to see two things, one is called location, which answers the larger question of, where do we locate a facility and layout, how do we relatively locate facilities inside a chosen location. So we ask two questions, one is on location and the other is on layout.

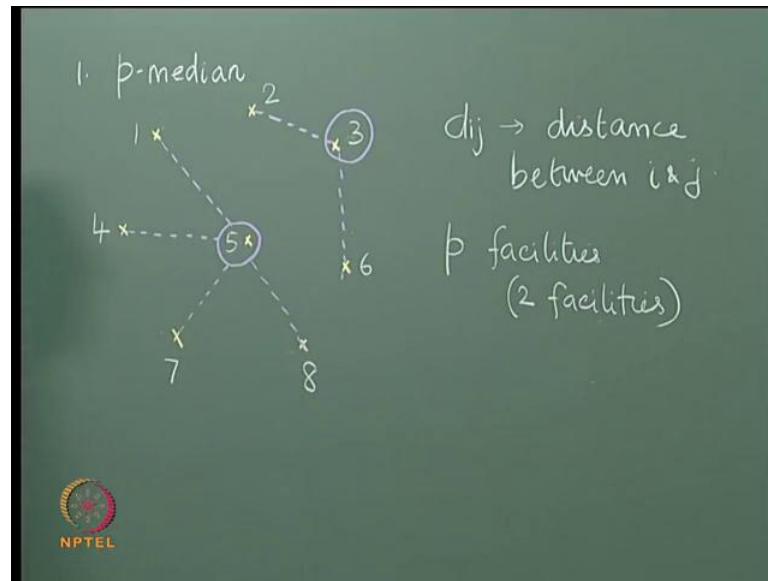
Now, location decisions are important as well as the layout decisions are. Location decisions are strategic in nature and location decisions mostly are qualitative in nature, even though several quantitative models are available for location. The decision on location would also depend on many qualitative factors, some of which we will see now. The first important decision is raw material, if we are looking at locating a manufacturing plant or a facility, which is raw material intensive, then it is customary to locate these near the places, where we have raw material is available.

Sometimes, location decisions are also based on labor availability. Areas where labor is abundantly available and labor is not very costly. Sometimes, location of facilities also happens near the demand areas, particularly retail stores, banks, etcetera are all located near the demand area. Sometimes, locations are also made based on transportation decisions for example, near a sea port, depending on the amount of material that come in and have to be transported.

Sometimes, the location decisions also depend on the policies of the government. Governments can give subsidy to locating certain types of industry in certain areas. Certain areas in the state are declared as areas that are suitable or that are allocated for location of manufacturing plants and people who choose to locate plant there, would get certain benefits and subsidies on tax, etcetera. So, these are other things that force the manufacturing and other organizations to locate plants in certain areas, depending on the policies of the government.

Now, the location decisions also become slightly different when we want to locate public facilities such as a police station or a fire station, etcetera. In such cases, the reach becomes a dominant factor and we try to locate them in such a way that, the time or distance between the place, where these central facilities are located and the farthest customer is minimized. So, it will be like a mini max problem, where we try to minimize the maximum distance between the location and any one of the customers. Now, in addition to these qualitative factors, there are other quantitative models that are available for location. And in this lecture series, we would concentrate a little more on the quantitative models that are available for location as well as for layout.

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So, the first model that we will see is called the p median model and this model originated from location and layout, though this has several useful applications. Now, the p median model is like this, and suppose we have 8 points, say we call them 1 to 8. Now, from the location point of view, what we are interested in is, there are 8 points and these 8 points represent demand areas or 8 customer demand areas, where we do locate the facilities such that, we are able to meet the demand with minimum distance of transportation.

Now, as far as the p median problem is concerned, to begin with we may say that, there are 8 points and the coordinates of these 8 points are known. So, we can also have a distance matrix d_{ij} between points i and j , so d_{ij} will represent the distance between i and j . Now, let us assume that, these 8 points represent 8 demand points or areas where the products are required. And we wish to locate, let us say p facilities, let us say 2 facilities such that, we produce in these two facilities and we meet the demand in all the 8 facilities with minimum transportation distance.

Example, if we decide to have say 5 and 3 as the two facilities, if we decide to do that then we say we are going to locate facilities in these two points. And we will say that, we will find out, to which facility the remaining six points are to be attached such that, there is minimum distance. So, if we fix 3 and 5 then we will attach 1 to 5, because 1 to 5 is assumed to be lesser than 1 to 3, 2 will go to 3, because 2 to 3 is lesser than 2 to 5. Let us

say 4 will go to 5, 7 will go to 5, 8 will also go to 5 and let us say, 6 goes to 3, because 3 is lesser than 6 5.

So, if we fix 3 and 5 as the two places, where we are going to locate the facilities then based on minimum distance we may say that, 1 4 7 and 8 will go to 5 and 2 and 6 will go to 3. Now, at the moment, we are not looking at, what are the demands in these eight places or we may assume that, the demands are equal. We are not looking at, when we locate a facility in 5 and attach 1 4 7 8 as well as 5 to itself, we assume that, this facility will have enough capacity to meets the demands of all the points that are attached to it.

So, this being a first model, we are not looking explicitly at what is the demand and what is the capacity or alternatively we assume that, capacity is enough to meet the demand of all the points that are assigned to it and therefore, the decision is based only on minimum distance. Now, these two are called the median points and since we are talking of p medians, this problem is called the p median problem. Now, this problem can be formulated as a binary integer programming problem, which we will see right now.

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$X_{jj} = 1$ if point j is a median
 $= 0$ otherwise
 $X_{ij} = 1$ if a non median i is attached to a median j
 $= 0$ otherwise

$\sum_{j=1}^n X_{jj} = p$

$\sum_{j=1}^n \sum_{i=1}^n X_{ij} = 1 \quad \forall i$

$X_{ij} \leq X_{jj} \quad \forall i, j$

Minimize $\sum \sum d_{ij} X_{ij}$

$X_{ij} = 0, 1$
 n^2 dec. var.
 $1 + n + n^2$ constraints

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So, we will assume now that, we will have X_{jj} equal to 1 if point j is a median and X_{ij} equal to 1 if a non median i is attached to a median j , so X_{jj} equal to 0, otherwise automatically follows. Now, since we have defined X_{jj} equal to 1 if point j is a median, for this feasible solution, we would have X_{33} equal to 1 and X_{55} equal to 1. Now,

there are 8 possible values of X_{jj} and if there are m general points, there are m possible values for X_{jj} .

So, the first constraint would be that $\sum_{j=1}^n X_{jj}$ is equal to p , so in our example, $\sum_{j=1}^8 X_{jj}$ equal to 2 and this feasible solution has X_{33} equal to 1, X_{55} equal to 1, so for p equal to 2, this will be satisfied. The next constraint is the non medians 1 2 4 7 6 and 8 will have to attached to some median. So, we write this as a constraint, which is like this, $\sum_{i=1}^n X_{ij}$ equal to 1, this is j equal to 1 to n for every i .

Now, there are 8 points here, now each point is attached to only one other point, which could be itself or it could be some other point. So, as far as this solution is concerned, X_{15} is equal to 1, X_{23} is equal to 1, X_{33} is equal to 1, X_{45} is equal to 1, X_{63} is equal to 1, X_{75} equal to 1 and X_{85} equal to 1. So, if I take a particular i then for that i , it has to be attached to only one j , which could be itself or it could be some other point.

Then we have another constraint, which is called X_{ij} , should be less than or equal to X_{jj} for every i, j . This means, that a non median i can be attached only to a median, this tells that, every point whether it is a median or a non median, is attached to only one point. If it is a median then it is attached to itself, which comes out of this. If the point is a non median then it can be attached only to another median, but it is attached only to one point comes from here. And from here it says that, if it is a non median, it has to be attached only to a median for example, if we take this solution, 1 is a non median.

If 3 and 5 are declared as medians, we cannot have X_{12} equal to 1, because X_{ij} as less than or equal to X_{jj} . So, since 3 and 5 are medians, X_{33} equal to 1, X_{55} equal to 1, but X_{22} is equal to 0. Since X_{22} equal to 0, we cannot have X_{12} equal to 1, X_{12} has to be 0, because X_{ij} is less than or equal to X_{jj} . Therefore, a non median point 1 can only be attached to either 3 or to 5, so it will be attach, so either X_{13} will be equal to 1 or X_{15} equal to 1, so that way for a non median point 1, this will be satisfied, this would also be satisfied.

This will ensure that, a non median point 1 does not get attached to more than one point or more than one median. This will ensure that, it will get attached to only one median and cannot get attach to a non median point, so all these are taken care of, by this set of constraints. So, the objective function now is to minimize $\sum_{i,j} d_{ij} X_{ij}$, where the moment

there is a link X_{ij} , there is a distance that is going to be travelled. So, we want to minimize the sum of the distances, so obviously d_{jj} is 0, distance between a point and itself is 0, so we want to minimize the sum of $d_{ij} X_{ij}$.

So, in this example with 8 points and 2 medians, there will be 6 distances whose, sum we try to minimize. Or essentially, what we try to minimize is, if there are, these are the two median points and if we take the remaining 6 non median points and assign them to the median with minimum distance, we are automatically minimizing the $d_{ij} X_{ij}$. So, the objective function $\sum d_{ij} X_{ij}$ would take care of, minimizing the distance between the non median points and the median points under the assumption that, every non median point is attached to only one median point and that distance is taken.

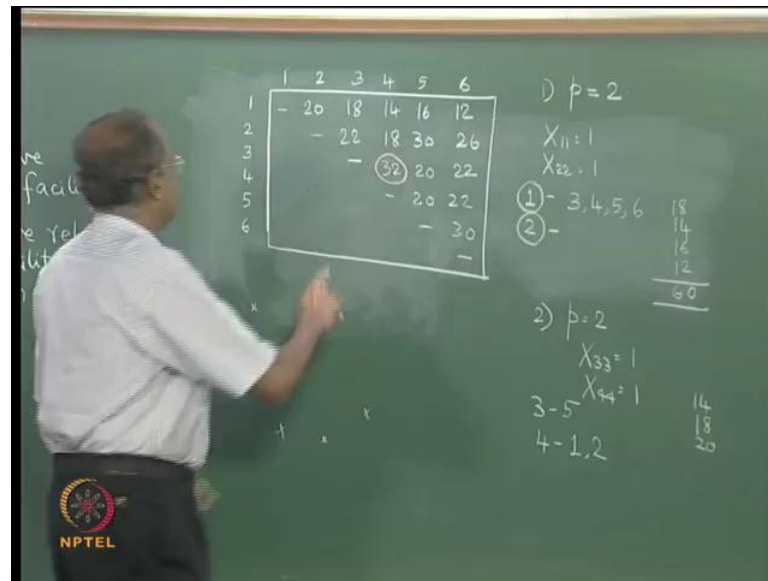
So, as a binary integer programming, so X_{ij} equal to 0 1, so if there are n median points, there are n square variable, n square decision variables. Now, there are n constraints here, one for each point, there is one constraint and there are n square constraints that come from here for every X_{ij} . So, this will be 1 plus n plus n square constraints in this formulation. This formulation can also be seen as, given 2 seed points, the problem is to allocate the rest of the non medians or non seed points.

The medians acts as seed points, so given 2 medians or p medians, the problem of it is to find out, how the non medians are allocated to the medians such that, the distance is minimized. So, the thumb rule is, allocate or assign each non median to the closest median. So, given the median points, getting this other rest of the solution is very easy and it is not very difficult at all. The real issue in the p median is, how do I find the correct median points?

Now, if there are 8 points and if we want 2 medians, we are looking at $8 C 2$ ways of actually creating the medians. And for each median set that contains p medians, the remaining n minus p non median points are allocated to closest median. So, the problem is about, essentially enumerating if necessary, all the $n C p$ ways of getting p medians out of n points.

Now, this problem is also a difficult problem to solve from a binary integer programming point of view and many times, heuristic solutions are given to solve this problem. So, we will look at one heuristic solution to this problem and also we will compare that heuristic solution with an optimum solution to this problem.

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So, let us take 6 points, instead of 8 and the distance matrix among these six points is given here. Now, we can assume symmetry, so distance between 1 and 3 is 18 distance between 3 and 1 is also 18. So, let us try and solve this problem for p equal to 2, now if we take for example, 1 and 2 as the two median points then the non median points 3 4 5 and 6 will have to be allocated to the two median points 1 and 2. So, for 3, the minimum distance 3 to 1 is 18, 3 to 2 is 22, so the two median points are here.

The non median point 3, 3 to 1 is 18, 3 to 2 is 22, so 3 goes to this with value 18. 4 to 1 is 14, 4 to 2 is 18, so 4 goes to 1 with 14. 5 to 1 is 16, 5 to 2 is 30. So, 5 also goes here with 16. 6 to 1 is 12, 6 to 2 is 26, so 6 also goes here with 12. So, we get the solution with facility located in 1, will meet the requirements of 1 3 4 5 and 6, facility created at 2 will meet the requirements of 2 only and the total distance will be 8 plus 4, 12, 18 20 2 6, so total distance will be 60, if we look at this solution.

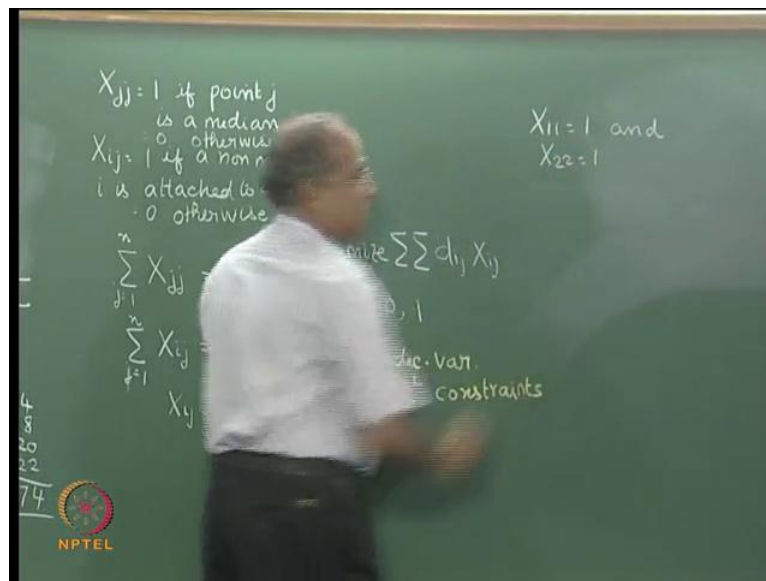
Now, let us look at another solution for p equal 2, now we have not given the coordinates of the 6 points, but we are given the distance matrix among the six points, but let if we assume that, there are some six points, which are like this. And we are interested in two medians, if we choose the two medians here then we realize that, we may not get a very good solution. Whereas, if we choose one of these as median and one of the other three as another median then we automatically get a solution, which is slightly better than choosing two medians that are near each other.

So, a simple thumb rule is to choose medians that are far away from each other so that, they can attract other points, which are nearer to them. So, using this, we can look at the largest number in this matrix, which happens to be 32 for 3 and 4, so choose 3 and 4 as the two medians. So, the two medians based on a thumb rule will be X_{33} equal to 1 and X_{44} equal to 1. So, thumb rule is to choose two points, which are farthest away as two medians.

Now, when we do this, we have the two medians 3 and 4, the non medians are 1 2 5 and 6, so 1 to 3 1 to 4, so 1 will go with 4 with distance equal to 14. 2, 2 to 3 and 2 to 4, so 2 will also go to 4 with distance equal to 18. 5, 5 to 3 is 20, 5 to 4 is 20, so let us put 5 here with 5 to 3 as 20 and 6, 6 to 3 is 22, 6 to 4 is 22. So, let us put 6 here, so 6 is 22. So 18 plus 14 is 42 62 64, distance equal to 74. So, we would now end up locating two medians, which are at 3 and 4, the location at 3 will meet the requirements of 3 5 and 6, the location at 4, would meet the requirements of 1 2 and 4.

Now, if we compare this arbitrary solution here with this, we realize that, this seems to be better than this. So, the thumb rule base solution did not work very well for this particular type of data. Now, if we try and solve this problem as a binary integer programming problem with, as I said n equal to 6, so there are 36 decision variables.

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And then we have 36 plus 6, 42 plus 1, 43 constraints we would get the optimum solution X_{11} equal to 1 and X_{22} equal to 1. So, the optimum solution has 1 and 2 as

median points and we have already calculated the assignment of the non median points with 1 and 2 as median points and the solution is shown here. So, with 1 and 2 as median points, the optimum solution is 3 4 5 and 6, all of them get attached to 1 and 2 is by itself. So, point 2 will meet the requirement of customer number 2, which is itself and point 1 will meet the requirements of 1 3 4 5 and 6 with distance traveled equal to 60.

So, the p median problem also can be solved, let us say for 3 groups, in which case we would put X_j equal to 3. So, 3 out of the 6 will become medians and the remaining non medians will then be attached to the chosen medians. Now, in that case, there are can be $6 C 3$ ways of trying to fix the medians, after which fixing the non medians becomes easy, because each non median will get attached to that median point, with which it has the least distance.

But then if we want this thumb rule based algorithm to find out based on a thumb rule, three different median. It is customary to take the largest distance and those two points as the first two medians. Now, going by what we saw here, all the three medians have to be away from each other, for them to attract points that are nearby. So, the third median that we should choose should be such that, it is far away from the first two chosen medians. So, generally another thumb rule is used, where if points i and j have distance greater than the average distance of this then they qualified to be medians.

So, using this thumb rule, we can chose three points such that, their mutual distances are all greater than the average, that way three median points are chosen and the non medians can go and get attached to the medians. In any case, solving this problem optimally for p equal to 3, would give us the best allocation for a three median problem out of this given data. Now, there are two more aspects that have to be looked at in the p median problem.

Now, one of which is, the p median problem is also seen as a way to group points or to group entities, where distances among these points or similarities among these entities are given. Another way of looking at it is, I have 6 points here, I want to make them into two groups or two clusters as they are called such that, the distances are minimized. Now, we would say after solving a p median problem that, points 1 3 4 5 6 will become one cluster and point 2 will become another cluster.

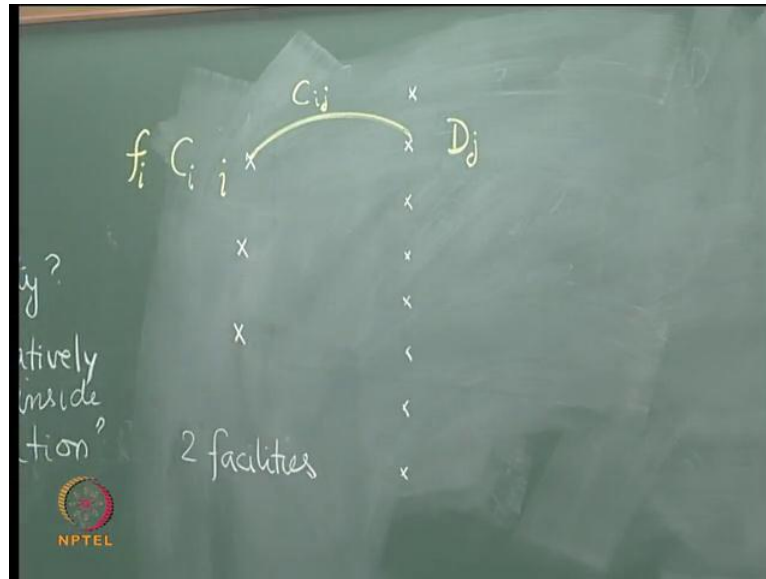
Now, if distances are not the criteria for clustering and some similarities are the criteria for clustering then the p median problem now will become one of maximizing $\sum_{i,j} s_{ij} X_{ij}$, where s_{ij} would be the similarity and the rest of the constraints will remain the same. So, p median can be seen as some kind of an optimization problem to cluster points or entities, that is another application in use of the p median problem. Second is, as I mentioned in the beginning, when we looked at say these six points, assuming that these are the six points, we are not looking explicitly as what are the demands.

We are also assuming that, if we have these two points as medians, say and locate facilities, the cost of locating the facility at any median point is the same. And therefore, it is not going to influence the choice of the location, the choice of the location is merely influenced only by the distance between the median and the non medians, which also means that, the cost of location in any of the six points is the same. The demands are not explicitly taken, which is also can be assumed that, the demands are the same in all the six points.

Capacities are not explicitly taken which means that, we have assumed that, capacity is a either infinite or each median when a plant is located there, has the capacity to handle the requirement of all the six points. So, p median can be seen as a location model with certain assumptions that, the cost of locating in every facility is the same, demands are equal and when a facility is located, it will have the capacity to meet the requirements of all the customers or all the points.

Each point here acts like a customer, so obviously the next location model will be like, if I have some demand points and if I have some potential location points, where do I locate, with some restrictions on capacity, with some different requirements at different points and with different costs of locating the facility. Now, such a problem is called the fixed charge problem or it is called a location allocation problem that we will see now.

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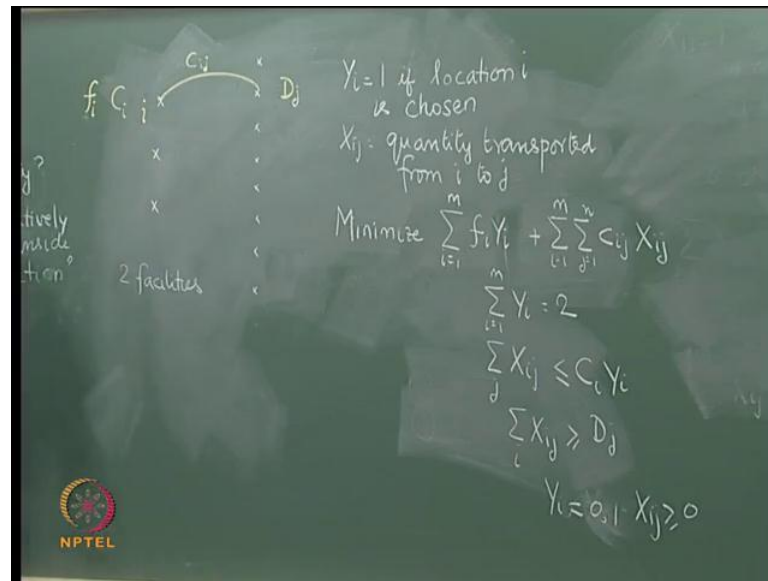


Now, in a fixed charge problem or a location allocation problem, we may say that, say we have three potential locations, where we want to locate a facility and let us say, there are 8 customer points that are there. Now, let us say that, if we choose to locate a facility in point i , we choose to locate a facility in point i , i equal to 1 to 3, there is a fixed cost of locating a facility, which is given by f_i . We could also say, that there is a capacity which is C_i , which is the maximum that this facility can produce.

Now, there are say 8 demand points and there is a demand D_j in point j , which has to be met. Now, the products are made manufactured in these facilities and if let us assume a single product for the sake of convenience then there is a unit cost of transportation between i and j , which is C_{ij} . Let us assume that, this is upper case C , which represents the capacity and this is the lower case C_{ij} , which represents the unit cost of transportation from i to j .

Now, the question is, if I want to locate say 2 facilities such that, I have enough capacity to meet the demands and I am able to meet the demands at minimum cost of locating the facility and distributing it or transporting it from the chosen facilities to the demand points.

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Now, for this problem, the formulation is as follows, Y_i equal to 1 if location i is chosen and X_{ij} is the quantity transported from i to j . So, the objective function has two costs, costs of location and costs of distribution. Now, this distribution is also called allocation, where in the located facilities, how much of this capacity that is available in the located facility, is allocated to meet the demands of each of these points.

So, this problem is also called location allocation problem. It is also called a fixed charge problem, because this problem explicitly assumes that, there is a fixed cost or a fixed charge of locating facilities. So, it is called fixed charge problem, it is also a called location allocation problem. So, the two components of the objective function are the fixed cost, which are the location cost and the transportation cost, which are the allocation costs.

So, the location costs are $\sum_{i=1}^m f_i Y_i$, i equal to 1 to m , there are m possible places, Y_i equal to 1 if it is chosen and therefore, there is a fixed cost, plus there is an allocation cost $\sum_{i=1}^m \sum_{j=1}^n c_{ij} X_{ij}$, i equal to 1 to m , j equal to 1 to n . So, here, n is equal to 8 and m is equal to 3, so whatever is allocated from i to j , there is a transportation cost. Now, we can have a constraint, if we say that, we want to allocate a exactly 2 facilities then we would have a constraint $\sum_{i=1}^m Y_i = 2$, i equal to 1 to m .

Now, with every facility, there is a capacity C_i , so whatever goes out of the facility, should be less than or equal to it is capacity. So, if I take this facility then X_{ij}

summation over j , whatever goes out of this to the various destination points should be less than or equal to its capacity. So, this should be less than or equal to $C_i Y_i$, the C_i is the capacity and Y_i comes, because only when Y_i is equal to 1, this capacity is available when the facility is located there.

So, if this is not chosen then nothing can come out of it, because the right hand side will become 0, all the corresponding left hand sides will have to be 0. Now, as far as each demand point is concerned, whatever it receives should be greater than or equal to the demand there. So, $\sum_j X_{ij}$ summed over i is greater than or equal to D_j and then we will have Y_i and $X_{ij} - Y_i$ is binary and X_{ij} greater than or equal to 0, X_{ij} need not be declared as a binary variable.

So, this is the basic location allocation problem or a fixed charge problem. Many times it is not necessary to have this constraint, whereas this is an additional further binding on this. Sometimes, unless we are very sure about this constraint, this constraint can actually harm the optimization. For example, if sum of all these demands, let us say is 10000 and let us say the capacities are 4000 each, so automatically we need to locate in all the three places.

If we put a restriction that we need only 2 then it will be becoming infeasible, because the capacities will not be able to meet the demands. So, unless we are very sure about capacities as well as demands, or unless we have some very binding constraints that we do not want more than a sort of number then we do not use this. We use this, only when we have some other additional restrictions that we want to locate exactly a certain number or we do not want to locate more than a certain number, otherwise this can be left out and this is enough.

It will automatically ensure that, minimum number of facilities will be located, because locating an additional facility is going to increase this $f_i Y_i$. So, the problem will automatically find out the correct number of facilities to be opened such that, this cost is minimized. Now, there is another variation to this problem or variant to this problem, now if we look at this problem, the way this is formulated. Now, we can have a situation, where both these facilities are opened and both these facilities transport to this.

So, this does not assume that, every customer is attached to only one facility, which was the assumption in p -median. Every customer or every non median point was attached to

only one median point; here we can have situations, where both two facilities can actually supply to one particular customer. Many times this is desirable, but sometimes it is also necessary to have each customer being supplied from a dedicated facility or each customer is attached purely or entirely to only one facility. So, we would bring that also into the optimization.

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$Y_i = 1$ if facility i is chosen
 $X_{ij} = 1$ if demand at j is met entirely from i
 Minimize $\sum f_i Y_i + \sum \sum c_{ij} X_{ij} D_j$
 $\sum_j D_j X_{ij} \leq C_i Y_i$
 $\sum_i X_{ij} = 1 \quad \forall j$
 $Y_i, X_{ij} = 0, 1$

The model slightly changes, where Y_i equal to 1 if facility i is located are chosen, X_{ij} equal to 1 if demand j or demand at j is met entirely from i . So, the objective function will be to minimize $\sum f_i Y_i$ plus $\sum C_{ij} X_{ij} D_j$, because C_{ij} is the unit cost of transportation, the entire D_j is transported, so D_j into X_{ij} will be the total quantity that is transported. Now, these two constraints will be, for every facility that is chosen, summation over j $D_j X_{ij}$ is less than or equal to $C_i Y_i$.

So, if a facility is located here let us say, wherever it supplies, it should be able to meet the entire demand of the customers to which there is supply from this place. So, $C_i Y_i$ is the actually supply, because that supply exist only when Y_i equal to 1. Now, if this one is supplying to these three customers, when we should be able meet to D_1 plus D_2 plus D_3 , which is given by $\sum D_j X_{ij}$, it is less than or equal to $C_i Y_i$. Then you will have $\sum X_{ij}$ equal to 1, for every j summed over i , because every demand point will receive the entire thing only from one supplier.

So, $\sum_j X_{ij} = 1$ for every i and in this case, Y_i and X_{ij} both are binary, so this is the variation or variant of this model, where all the demand of a customer is met from only one supply point. And this is a more generalized version, where a supply point can meet the demands, part demands or full demands of more than one customer. So, here in this case, we can model a situation, where this will supply to this and this or more importantly, this can get from this and this. Whereas, in the second model, this can get only from one of the chosen points, so these are the two variations in the fixed charge problem or location allocation problem. We explain this further using a numerical illustration and we will see that in a next lecture.