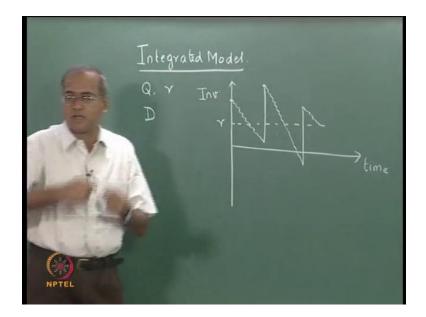
## Operations and Supply Chain Management Prof. G. Srinivisan Department of Management Studies Indian Institute of Technology, Madras

## Module No - 1 Lecture No - 22 Integrated Model, ROL for Normal Distribution of LTD and Given Mean

(Refer Slide Time: 00:16)



In today's lecture, we will develop an integrated model where, in a single integrated model we try to find both Q and r where Q is the order quantity and r is the reorder level. In the earlier models that, we saw in the previous lectures we had computed Q based on the economic order quantity equation and then, r was computed based on the lead time demand. Now, we will try and build an integrated model to see whether, there is any slight change in the value of Q that is computed using an integrated model or is there a slight difference in the cost computations, if we build an integrated model involving Q and r.

Now, let us go back to the basics of inventory where, we draw the usual time versus inventory graph, now let us say we start at a certain level; which right now we are not defining it either as Q or as I m and then, we start consuming. Now, we are going to

assume that the demand D, annual demand is D the aggregate annual demand is known but, at every instance the demand is probabilistic; it is not deterministic as in the earlier examples. So, we start consuming this inventory and let us say we use a curved line to indicate that, the demand is nondeterministic, so the inventory drops like this.

Now, let us assume that there is a reorder level r which we will determine this is our reorder level r. So when the inventory reaches this r we place an order, now the lead time is the time between placing the order and getting it and let us assume that we get the quantity, when we get the fresh consignment the stock is here. So the stock gets updated by a quantity Q and let us say it comes here, then once again we start consuming based on this and again when we reach the reorder level let us say we place another order.

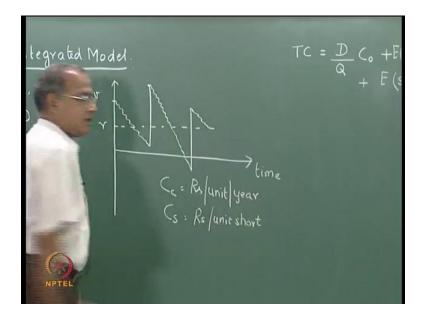
But then, let us assume in this cycle by the time the order comes the inventory has come down to this position. Now once again the quantity arrives there is a backorder here quantity arrives so, this stock is updated by a certain quantity Q. So, we can assume that, this is the Q and this is roughly the same as this and once again the cycle continues and so on. So, the only change that we have made is because of the probabilistic nature of the demand, there can be some cycles where the ending inventory is positive when the stock arrives and there can be some cycles; where the ending inventory is negative when the stock arrives.

This is different from the earlier deterministic models where, we consistently assumed that the stock will be 0 or the inventory will be 0 when the stock arrives or in the case with backordering; there will be backordering every time the stock arrives. So we could have cycles like this now we have two variables: which is which are Q and r. Q is the quantity which means this quantity that, we are going to order which we will get after a lead time and r which is the reorder level, which is the point at which we place the order. (Refer Slide Time: 04:31)

So, we will try and get both using an integrated model. So now, we write the total cost expression for this, so the total cost is made up of three parts: the annual demand is D with the aggregate annual demand is known. So, the number of orders per year will be D by Q into C naught which is the order cost, this is a familiar term D by Q C naught represents the annual order cost that we will have. Now, there is an inventory component there is a shortage or backorder component so, the inventory component will be given by average inventory into C c plus average shortage into C s. C c is always defined as rupees per unit per year and therefore, average inventory is in units.

Now, it again depends on how we are going to define C s we need to either multiply with the D by Q or not. If C s is defined as rupees per unit per year then, we do not multiply by D by Q but, if C s is defined as rupees per unit short or rupees per unit then, we have to do that because expected value of the shortage. Instead of saying average shortage let us say let us use expected ending inventory or expected average inventory plus expected shortage into C s.

(Refer Slide Time: 06:20)



Now, in this case we are going to define C s as rupees per unit short so C c will be defined as, rupees per unit per year and C s will be defined as rupees per unit short. So the E shortage is the expected shortage in a cycle that, into rupees per unit short into D by Q will give us the expected cost per year because, there will be D by Q cycles in a year. Now, E of s or E of shortage is the expected shortage in a cycle D by Q is the number of cycles per year and C s is rupees per unit short.

So, all these when multiplied would give us expected shortage cost per year. Now we have to start defining this as well as this, now in order to do that, now let us call the expected ending inventory as E of ending inventory, let us call it as E of ending inventory. Now, expected ending inventory or expected beginning inventory will be expected value of ending inventory plus Q because, the any cycle if there is an expected value of the ending inventory for example, if there is something here then the cycle begins with an addition of Q.

So, expected average inventory will be beginning inventory plus ending inventory divided by 2 which will be this is the ending, this is the beginning inventory plus ending inventory divided by 2 which will be expected value of ending inventory plus Q by 2. So, when we substitute we should now start writing this, now we call this expected shortage

as S bar of x as the expected shortage. And then, the expected ending inventory will be; what is the expected ending inventory? The expected ending inventory is the difference between the reorder level and the lead time demand.

Because, when we reach the reorder level we place an order and when the there is a lead time and there is a lead time demand. So, we expect that we will meet the lead time demand using this reorder level so, r minus expected value of lead time demand, will become the expected ending inventory, because there could be some cycles where there is positive inventory, there could be some cycles where there is negative inventory.

That would depend on the actual lead time demand but, we are not modelling the actual lead time demand we are modelling the expected ending inventory. Therefore, expected ending inventory will be r minus expected value of lead time demand. So, expected ending inventory will be r minus f of x, where, f of x represents the expected value of lead time demand or we could use slightly a different notation we could not call f of x here we would say r minus expected value of lead time demand.

(Refer Slide Time: 10:41)

$$TC : \frac{D}{Q} C_{0} + \left(\frac{Q}{2} + T - f(x)\right) i \zeta + \frac{\overline{S}(x)C_{s}D}{Q}$$

$$\frac{\partial TC}{\partial Q} : 0 \rightarrow \frac{-D}{Q^{2}} C_{0} + \frac{i \zeta}{2} - \frac{\overline{S}(x)C_{s}D}{Q^{2}} : 0$$

$$Q: \int \frac{2D(C_{s} + \overline{S}(x)C_{s})}{i \zeta} = 0$$

$$\frac{\partial TC}{\partial r} : 0 \rightarrow i \zeta + \frac{\partial \overline{S}(x)}{i \zeta} \leq \frac{D}{2} : 0$$

$$\frac{\overline{S}(x):}{S} (x - Y) f(x) dx = 0$$

$$\int \frac{f(x)}{D \zeta_{s}} - \frac{D}{D \zeta_{s}} = 0$$

So now, substituting in all of them we would get T C is equal to D by Q C naught plus Q by 2 plus r minus, let us also call it f of x so, r minus f of x into i C plus S bar of x C s

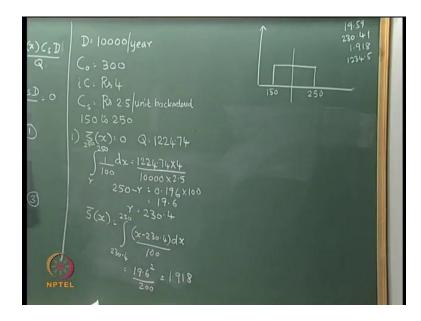
into D by Q. Now, partially differentiating this with respect to Q and r we can get the optimum values of Q and r which are the 2 variants. So, partially differentiating with respect to Q so dou T C by dou Q equal to 0 would give us minus D by Q square C naught plus i C by 2. Because, the expected value of the lead time demand does not depend on the order quantity, so it does not have any effect. So, plus i C by 2 minus S bar of x C s D by Q square equal to 0, so this on simplification will give us Q equal to root of 2 D C naught plus S bar of x C s divided by i C. So, this is the equation for the order quantity in the integrated model, so this is our first equation.

Now, differentiating with respect to r would give us - this has no effect this also has no effect, so there is a i C which is here; from here i C plus dou of S bar of x by dou r into C s D by Q equal to 0. This is what we get because S bar of x represents the expected shortage, so this will depend on the reorder level because, shortage happens when lead time demand exceeds the reorder level. So, S bar of x can be defined as integral r to infinity x minus r f x dx. Now, when we differentiate this with respect to r we would get a minus term here, which comes from the x minus r

So differentiating and substituting this we would finally, get the expression integral r to infinity f x dx will be equal to i C Q by D C s. The minus term comes from here from the partial derivative we get a minus, so minus goes to the other side i C Q by D C s would give this. So, this is our second equation which is this. And our 3rd of course, is the expression for s bar of x, so we have three expressions which come here. This derivation is quite similar to the earlier one, if we see very carefully it is very similar because, we still have that same term i C Q by D C s, which we wrote in the previous lecture as Q C c by D C s, Which represented 1 minus integral which is kind of represented area under the curve, so in the some sense in the previous expression we got the value of r directly from this.

This is the well known equation for economic order quantity; with the only change that s bar of x comes explicitly into this function. Earlier we did not have this portion so we had Q equal to 2 D C naught by i C when we separated Q and r we got Q equal to 2 D c naught by i C. Now because, we brought r into the picture we are getting another expression plus S bar of x C s. So, in some sense this derivation is very similar to the earlier one and then, when we want to use it how do we compute the values of Q and r; which is a purpose of this derivation. So we explain the computation of Q and r through a numerical example, so let us go to the numerical example here.

(Refer Slide Time: 17:28)



So we try and use the same example, for the purpose of comparison: so we assume D equal to 10,000 per year, so we are going to assume c naught equal to 300, i C equal to rupees 4, C s is equal to rupees 2.5 per unit backordered. Then, we want to find out Q and we also assume, that the lead time demand follows, a uniform distribution between; 150 to 250 with 200 as the expected value. In the earlier discrete case we took from 100 to 300 and we followed a discrete distribution. Now, we are going to assume that it follows a uniform distribution between; 150 to 250 with 200 as the expected value. In the earlier discrete case we took from 100 to a uniform distribution between; 150 to 250 with 200 as the expected value. We had already seen how the uniform distribution will look like; so it is like saying this is how the lead time demand will look like. So, this is from 150 to 250 with equal probability with the mean at 200.

So we now want to find out Q and r as well as T C. So, first in order to find (Refer Slide Time: 10:41) Q we know 2 D C naught plus S bar of x C s by i C, we do not know the term S bar of x. So, if we know the term s bar of x we can find out Q. Now, let us go back

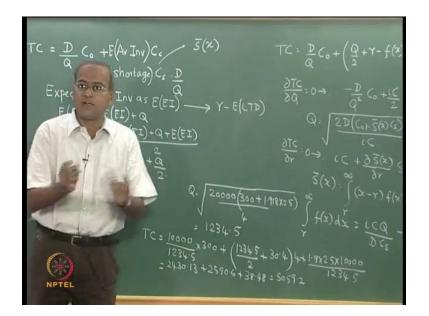
and see what we require to find out S bar of x. To find out S bar of x we need integral of r infinity x minus r f x d x. So if we know r we can find S bar of x because, f x is known so, integral x minus r f x d x. So, in order to find S bar of x we need r and now you want to find out; how do we find out r? So, in order to find out r you have once again r to infinity f x d x is i c Q by D C s, which means you need Q. So, essentially to find Q you need S bar of x; to find S bar of x you need r and in order to find r you need Q. So what we do is we start with the simple assumption that S bar of x is 0 and then get Q.

Now, use this Q to get r use this r to get S bar of x come back and recompute Q and use the kind of an iterative algorithm, which will affectively get Q and r to converge into their ultimate optimum values. So, first begin with S bar of x equal to 0, so iteration 1 s bar of x equal to 0; would mean Q equal to (Refer Slide Time: 10:41) 2 into D into C naught by i c. So, Q is equal to 1224.74 we have computed the value of Q, Q equal to 1224.74. We now, use this value of Q in order to get r so the corresponding equation is integral r to infinity so integral r to 250 in this case it is a uniform distribution; it does not go up to infinity. It has an upper limit of 250 or to infinity f x d x, so f x is 1 by 100 so 1 by 100 d x is equal to Q C c by D C s, so 1224.74 into 4 divided by 10,000 into 2.5.

So, this 1 by 100 will come out so this is 250 minus r is equal to, so we would get 250 minus r is equal to this quantity multiplied by 100. So, this is 1224. So this on simplification would give us point 1959. 1224.74 into, gives us 0.1959 or 0.196. Now, this is 0.196 so 250 minus r will be into 100 which would give us 19.6 from which r is equal to 250 minus 19.6 which would give us 230.4.

Now, that we have the value of r we can now use that r here, to get S bar of x, so S bar of x will become integral r to infinity x minus r f x d x. So integral r is 230.4 to 250 or to infinity x minus 230.4 by 100 d x, so this would give us x minus 230.4 the whole square divided by 2 into 100. So this is on the upper limit it is 250 on the lower limit it is 230.4; so the contribution will be 0 from the lower limit from the upper limit it will be 19.6 the whole square by 200, which would give us 1.918 as the expected value of s.

(Refer Slide Time: 23:59)



Now, we have to use this 1.918 into this equation to go back and compute Q again we show that somewhere here, so the new computation of Q will be root over 20,000 which is 2 D, C naught plus s bar of x into C s which is 1.918 into 2.5 divided by 4 which would give us 1234.5. Now we can use this 1234.5 into this equation and compute r again and once again use this r to compute S bar of x and keep iterating till the values of Q r and S bar of x converge.

So let us assume that Q has converged to 1234.5 let us assume that r has converged to 230 (Refer Slide Time: 17:28) and S bar of x is converged 1.9. So, finally, we can compute the total cost from this equation, from this equation and for these chosen values the total cost T C will be D by Q into C naught which is 10,000 by 1234.5 into 300 plus Q by 2; 1234.5 divided by 2 plus r minus expected value of f of x. let us say r is 230.4 (Refer Slide Time: 17:28) expected value is 200, so plus 30.4 into C c 4 plus S bar of x lets say is 1.918 which is say 1.9 into C s into D which is 1.9 into 2.5 into 10,000 divided by 1234.5.

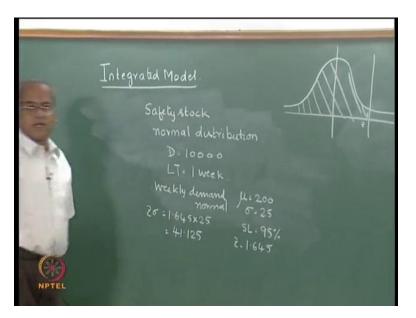
So, this on computation would give us this is 2430.13, which is the order cost component, the carrying cost component will be 2590.6 plus shortage cost component will be 38.48 and this would give us a total cost of 5059.2. For the same illustration when we had

deterministic demand our total cost was about 4,900. Now, when there is the demand is nondeterministic and follows a uniform distribution (Refer Slide Time: 17:28) we get about 30 extra units as the safety stock or the buffer and then, we have an expected shortage of about 2 units in every cycle.

So, this is the effect of having the integrated model, now integrated model has given us values which are very close to what we had independently found out. For example, when we use the normal E O Q model we had 1224.74; here we have 1234.5. We also know that in practice the slight difference of about 10 units is not going to make a big impact many times vendors may not be able to give accurate quantities such as 1234.5 etcetera, it is also uneconomical to do that.

So, we would choose a convenient number like 1250 which we used for earlier illustrations; so there is not much of a change here. Similarly, there was not much of a change here 230.4 would gives us a service level of about 80 percent and so on. So, this is also a comparable and the total cost is also comparable. So, either one could use the integrated model or one could simply separate the order quantity computation Q and the reorder computation r separately. And then, depending on the distribution of the lead time demand calculate the reorder level and use it so that the overall cost is minimized.





Now, let us come back to one more aspect of probabilistic inventory models - something that we had already seen and then, something that is going to have a large implication in the way we order. In an earlier illustration, we had calculated the safety stock assuming a normal distribution. S o, let us go back to that example, so we first said that, the annual demand is 10,000 but, we also said that, the lead time is 1 week and we also assume that weekly demand follows a normal distribution with mean equal to 200 and sigma equal to 25.

Now, we worked out a certain computation and said that, if the lead time is exactly 1week and if the weekly demand follows a normal distribution like this. If we want a service level of 95 percent what is our safety stock? This calculation we did in the earlier lecture so, what we did was we went to the normal distribution and find out the z value at which the area was 0.95 and the z value at which the area is 0.95 happened to be for z equal to 1.645. And then, we said z sigma is safety stock so, 1.645 into 25 which would give us about 40 units as the safety stock, there was 41.125 units as the safety stock.

(Refer Slide Time: 31:48)

0000

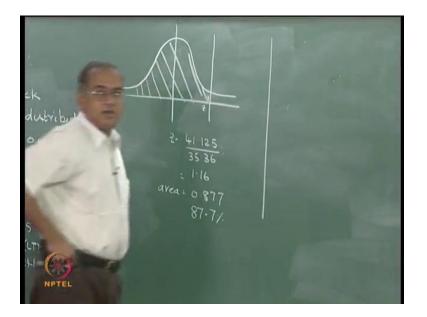
Now, let us look at the situation to what happens when lead time is equal to 2 weeks. So now, when lead time is equal to 2 weeks of course, in this case safety stock is 41.25, so reorder level in the case of 1 week is reorder level is equal to expected value of lead time

demand plus safety stock which is 200 plus 41.25 which is 241.125 so, this is the reorder level here. Now, when lead time is equal to 2 weeks expected value of lead time demand is 400 because, weekly demand is has an average of 200. Now we need to calculate the standard deviation of the demand (Refer Slide Time: 29:25) during the lead time, now standard deviations are not additive but, variances are additive.

So, what we do is variance of the lead time for the (Refer Slide Time: 29:25) first week is 25 square, variance of lead time for the 2nd week is 25 square so, total variance is 2 into 25 square. So, total variance, variance of lead time demand will be 2 into 25 square from which standard deviation of lead time demand will be root 2 into 25. So, if the weekly demand is D and the lead time is L weeks then, the standard deviation of the lead time demand is root over L into D. So, this on computation would give us 25 into root 2 which is 35.36, so standard deviation of lead time demand is 35.36 and now once again for a 95 percent service level we would have z is equal to 1.645.

So, for a service level of 95 percent, we would have z is equal to 1.645 which means safety stock is 1 point 645 into 35.36, this would give us 58.16 and reorder level is equal to 458.16. Now, let us try and understand what is the effect or what has happened when the lead time move from 1 week to 2 weeks. Now this is 241.125, this is 458.16, now the 400 comes as the contribution of the expected value of lead time demand and the safety stock which was 41.125 here, has now increased to 58.16 here. So, as lead time increases we end up carrying more safety stock, so it is necessary to keep the lead time small or alternately, if we had not adjusted the safety stock and brought it up to 58 but, in the 2 week case if we had still kept the safety stock at 41.125.

(Refer Slide Time: 36:00)



If we had done that, then what would have happened if in the 2 week case if we had retained at 41.125 then, the corresponding z will be 41.125 divided by 35.36 which is 1.16. And for z equal to 1.16 the area from minus infinity to z in the normal distribution would now become 0.877, so area equal to 0.877, telling as that our service level will only be 87.7 percent, if we had kept 41.125 and the lead time increased from 1 week to 2 weeks.

So, if there is an increase in the lead time then, there has to be an increase in the safety stock to maintain the same service level. If we maintain the same safety stock then, service level will come down. So that is the first lesson that we learnt. In both the cases it happened that while the demand followed the normal distribution the lead time was still deterministic. The lead time was deterministic with 1 week and the lead time was deterministic in this case with 2 weeks.

(Refer Slide Time: 37:55)

400

Now, if the lead time also follows a certain distribution then what happens, now let us call this as now demand during a week; so demand follows, let us call this as mu demand as the mean and sigma demand as the standard deviation of the demand. And now, lead time also follows a distribution. So, let us call mu L as the expected value of the lead time distribution and sigma L as the standard deviation of the distribution of lead time. So, in this particular example both demand as well as lead time are following certain distribution.

Now, in such cases the expected value of the lead time demand; expected value of the lead time demand is equal to expected value of demand into expected value of lead time which is mu D into mu L. The standard deviation of lead time demand is now, given by root over mu L sigma D square plus mu D square sigma L square mu D square sigma L square. Now, let us understand the effect of this and when we apply it to the same numerical illustration.

Now, we have seen a case here where, (Refer Slide Time: 31:48) demand follows a normal distribution with mean 200 per week (Refer Slide Time: 29:25) and standard deviation 25 per week. So, let us say mu demand is 200 per week and standard deviation of demand is 25 per week when we looked at this case (Refer Slide Time: 31:48) mu lead

time is 2 weeks but, standard deviation of lead time was 0. Now, let us look at a case where mu lead time is 2 weeks and standard deviation of lead time is also 2 weeks.

Now what happens to our expected values, now the expected value of lead time demand is simply the product of mu D and mu L, so it is a product of 200 and 2 which is 400 which was exactly the same here (Refer Slide Time: 31:48) expected value of lead time demand was 400. Now the only difference that happens is in the computation of the standard deviation of the lead time demand. So, this becomes root over mu L which is 2 into sigma D square which is 25 square plus mu D square which is 200 square into sigma L square which is 4.

So this becomes root over 625 into 2 1250 plus 200 square is 40000 160000 plus 160000. So, this will become 160000 plus 1250 root over 401.6, so standard deviation of this would become 401.6. So, standard deviation is given by the formula, mu L into sigma D square plus mu D square into sigma 1 square. So, we do have this D square term that keeps coming in, so this is roughly of the order of; this is roughly of the order of mu D into sigma L.

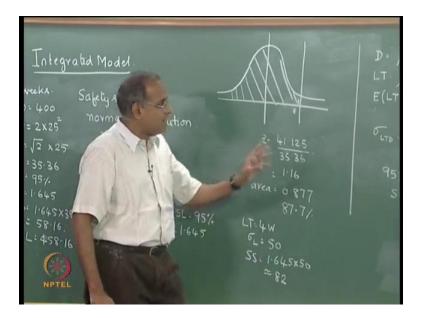
So, our mu D is of the order of 200 sigma L is of the order of 2, so this is of the order of 2 into 200, which is 400. So the standard deviation of lead time demand becomes 401.6 in this case. Now, what happens is your safety stock should be z sigma now for a 95 percent service level we have calculated z equal to 1.645. So, z sigma which is the safety stock is 401.6 into 1.645 which is 660.56 units.

So now, the same scenario here (Refer Slide Time: 31:48) that we saw when the lead time did not show variation which means sigma lead time was 0, gave us a safety stock of 58.16. The moment we have the lead time also showing variation with the sigma and sigma equal to 2 we realize that, the safety stock has moved up to 660.56 units. So, the safety stock itself is about three weeks of inventory is held as safety stock and we already know that, such a safety stock is never going to be consumed off. So, so much of worth of inventory is going to remain as safety stock at 95 percent service level.

So, the lesson is this, now if we have to choose between suppliers, the ideal situation is something like this (Refer Slide Time: 29:25) where the lead time is small, the lead time is deterministic, the lead time does not show any variation. So, once the lead time is small all we do is simply have a very small amount of safety stock. 2nd is the lead time is slightly (Refer Slide Time: 31:48) larger but, it is still deterministic and does not show any variation.

Now, in this case the safety moves up a little bit from 41 to 58 the 3rd scenario is when the lead time shows variation automatically, the safety stock shoots up. For example, if we went ahead here and computed the case where lead time is actually 4 weeks, (Refer Slide Time: 31:48) now, let us look at a case when lead time is 4 weeks but, then we are not looking at any variation

(Refer Slide Time: 45:26)



So lead time is 4 weeks; now sigma lead time is 50 root over 4 into 25 which is 50, safety stock is equal to 1.645 into 50 which is roughly 164 is 82 is roughly 82. So, if we have a situation where (Refer Slide Time: 31:48) lead time is 4 weeks but, does not show variation safety stock is eighty 2. But, if the lead time is 2 weeks (Refer Slide Time: 37:55) but, shows a variation of another 2 the safety stock is 401, safety stock is 660.56. So, it is always advisable to go for a supplier, who has less variation in lead time and lead

time may be higher but, then the supplier is very sure about delivering it in a certain amount of time.

So, the most critical thing is to understand the variation in lead time and the effect of such a variation on the safety stock. So, it is always good to chose, if you have to choose between 2 suppliers who even has the higher lead time but, definitely sticks to it, versus another person who has variation in lead time. So, variation in lead time is not desirable it is always good to go for a supplier who has no variation in lead time but, who can stick to the suggested lead time. So, a variation in lead time is going to play a major part in our supplying in our choice of suppliers and in the purchasing policy of the organizations. We will see some more aspects of this affect of lead time particularly, in the context of supply field and we will do that in the next lecture.