Lecture No - 16
Economic Lot Scheduling Problem, Supply Chain Inventory

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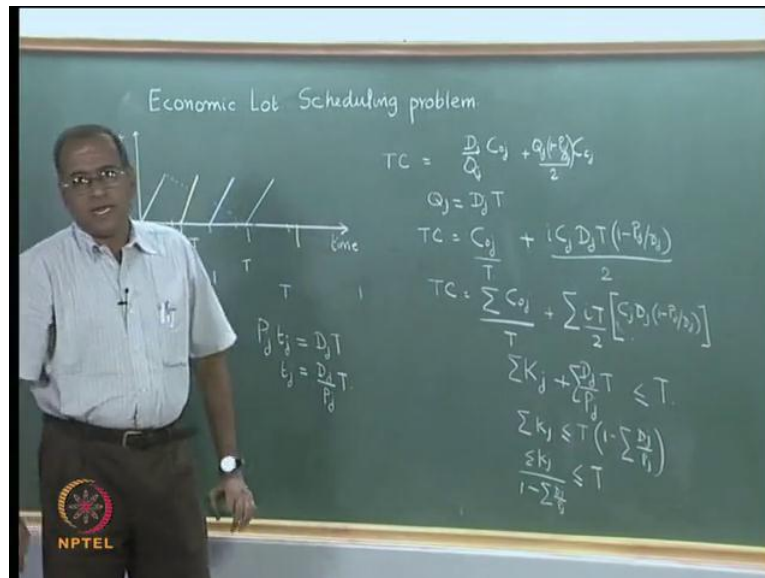
In the previous lecture, we were looking at the economic lot scheduling problem; we were looking at this graph, where we said that we are considering say 3 items. This is for the equipment or the facility that is producing these 3 items. So, the first item is produced for a period say t 1 and then inventory is built up and this inventory is used to meet the demand for the rest of the period. Then we produce the second item which is shown in yellow and we consume while we produce and then inventory is built up and then this inventory is used to meet the demand for the rest of the cycle. Then we produce the 3rd item which is shown in blue up to a certain point, it is not necessary that these peaks have to be the same these are only indicative.

So, we produce this 3rd item then stop production consume this inventory, then we come back to the 1st item. Now, this is called the cycle; this cycle is capital T and in the economic lot scheduling problem, we assume that the cycle is the same for all the 3 items. Then, we wrote an expression for the total cost which runs like this. So, total cost is the sum of the costs associated with all the 3 items. In general - n items. For each item

there is an ordering cost and there is a setup cost and there is a carrying cost. As mentioned earlier, setup cost is the equivalent of ordering cost; ordering cost is when we buy the item from outside and order, while setup cost is incurred when we produce.

Now, we also realize that in this figure, I have given a small gap between the ending of production of this item and the beginning of production of the next item and this small gap that much amount of time is used in the changeover or the setup to make this item. And there is a cost associated with that setup, which is our setup cost. So, if we take a particular item, the number of times we would make that item is D by Q. So, the usual expression is D by Q into C naught. Where, C naught for item j is the changeover cost. D j is the annual demand Q j is the order quantity or production quantity for that item plus we also have; we have already derived Q j - I m for item j by 2 into C c of j.

Now, we also know that this I m is equal to Q into 1 minus P by D, so this is written as plus Q j into 1 minus P j by D j into C c j divided by 2. Now, one important assumption in the economic lot sizing problem is that the cycle times T match. So, the cycle time for the 2nd item would also be the same T and the cycle time for the 3rd item would also be let us say, it goes through somewhere here would also be the same T. So, since the cycle times are the same, the production quantity Q j for item j, Q j will be D j into T, D j is the annual demand T is the cycle time.

So, in a cycle time of T, say T years if D j is the annual demand the production quantity is Q j is D j into T. Another way of deriving is that the number of times we place an order is D by Q which is equal to 1 by T and therefore, Q is equal to D j into T. So, we go back and substitute here, to get T C is equal to C naught j into T plus Q is D j into T the D j and D j gets canceled. C naught j divided by T which is here plus this C c is written as i C j, Q j is D j T into 1 minus P j by D j divided by 2. So, this is the cost for a particular item under the assumption that the cycle time is T, which is the variable that we want to find out. Now, total cost will be for all the items. So, we sum it up and say sigma C naught j by T because we assume that this T is going to be the same for all the items by t plus sigma i T by 2 into C j D j into 1 minus P j by D j.

So, i T by 2 into C j D j into 1 minus P j by D j. Now, this is the function that we want to minimize and find out T that is common for all the items. Now, in addition this is where we actually stopped in the previous lecture; in addition, we also have a constraint.

Constraint is like this. If I take the beginning of this, let us call this item 1 if we take the beginning of the 1st cycle of item 1 and the beginning of the 2nd cycle of item 1, the time duration is T, which is the same as beginning of item 2 in the 1st cycle and item 2nd cycle for item 2.

Now, in between this time t. What we have to do is, we have to produce this item plus this item plus this item plus we have to account for the change over from white to yellow, which is item 1 to item 2 and from yellow to blue which is item 2 to item 3 and blue to white which is item 1 from item 3. So, within this time period T we need to have the production of the 3 items made as well as, the setup time for all the 3 items to be accommodated. So let us call K j as the setup time to make item j.

Now, this is the setup time required to make the yellow item, this is the setup time required to make the blue item and somewhere here this is a setup time that is required to make; this is setup time to make the yellow item, this is the setup time to make the blue item, this is the setup time to make the white item. Note that these three t times need not be the same and secondly right now, we assume that this setup time, is time required to make this item and does not dependent on the other item. For example, if we move from white to yellow or if we have another situation, where we move from blue to yellow. The time taken to set for the yellow item which is item number 2 will be the same.

So, let us call the K j as the setup time to produce or to setup for the production of item j. So, K j is the setup time. Now, what is the production time for this item? The time we produce is T is the cycle time here, so we produce up to a certain period which is if P j is the production rate of this cycle. So, P j into some t 1 is what we produce in this cycle. P 1 is the production rate for item 1. So, we call this as P j into let us say some t j is the time in which we produce item j in the cycle. Now, this should be equal to the demand of the entire cycle because with what we produce and consume here. We produce P j t j we also consume the demand during this period, build up this inventory and this inventory is used to meet till the next cycle begins.

So, what we produce in 1 cycle should be equal to what we consume in that cycle. So what we produce is P j t j and what we consume is D j into T D j is the demand annual demand and T is the cycle. So, from which the production time t j is equal to D j by P j into T. So, this is the production time in a cycle. This is the setup time in cycle for item j

so K j plus D j by P j into T is the time required for 1 item; this is the setup time required for the item, this is the production time required for the item. So, for 1 item this is the time required, for all the items summation K j plus summation should be less than or equal to T which is the cycle time that we have. So, this is the constraint.

Now, this constraint also has to be rewritten in such a manner because, there is a T time that comes here, there is also a T term that comes here. So, this can be rewritten as sigma K j plus T into 1 minus sigma D j by P j, should be less than or equal to 0 or in other words this is written as sigma K j, I am sorry. This is less than or equal to this, which is rewritten as sigma K j by 1 minus sigma D j by P j is less than or equal to T. So, this is how the constraint; this is the objective function and this is how the constraint is written, this is the constraint. So we now, have to solve this constrained optimization problem to get the value of T.

Now, this objective function is non-linear because the term T appears here in the denominator and the term T appears here in the numerator and it is subject to a linear constraint of T greater than or equal to constant. Now, there are two ways of doing it: one is to try and use the method of lagrangian multipliers and take this constraint into the objective function. Setup the Lagrangian function partially differentiate and try to get the best value of T. The other which is a simpler method is, to actually solve the unconstrained version of this problem that is; to temporarily ignore this constraint, just solve the unconstraint version of this problem get the value of T.

And if it, happens that the T that is obtained by solving the unconstrained version satisfies this constraint, then it is optimal to the constrained optimization problem. This is a very simple result, you can always relax a problem by removing a certain set of constraints and then solve a relaxed version of the problem, if it is easier to do so. And then, the solution to the relaxed version. If it satisfies the constraint, then it is optimum to the constraint problem.

The issue comes only when it is not optimal, then we will look at it as an when it happens. So, we first solve the unconstrained version of this problem which is to straight away differentiate this with respect to T.

So, when we differentiate this with respect to T and set it to 0 we get something like minus sigma C naught j by T square this T to the power minus 1, so minus 1 by t square plus sigma i by 2 into C j D j into 1 minus P j by D j equal to 0. From which, t is equal to root over 2 C naught root over i by 2 sigma T square equal to 0, from which. So, this goes to the other side, so we will have let me just write it down. So, this would give us, sigma C naught j by T square is equal to sigma i by 2 into C j D j into 1 minus D j by P j, sorry this should be D j by P j 1 minus. So, t square goes to the other side this will come here. So, T is equal to root over sigma C naught j by sigma i by 2 into C j D j into 1 minus D j by P j. So, we get the value of T this way.

So, we compute this value of T and then we check whether the value of T here satisfies this particular inequality or this particular constraint. If it satisfies this constraint, then it is optimum. If it is does not satisfies this constraint, we try and realize once again from the very nature of the function that, it will be made as an equation at the optimum and then we will see how we handle that through a particular numerical example.

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So, let us look at an example here, now let us look at this example where we consider three items. So, let us consider: C naught equal to 1000, D 1 is 3000, D 2 is 5000, D 3 is 20,000. Let us say: i equal to 20 percent; C 1 is equal to 100, C 2 is equal to 200, C 3 is equal to 400; P 1 10,000, P 2 20,000 and P 3 50,000. So, we also observe that D by P is less than 1 so 3000 and 10,000 5 and 20 and 20 and 50. So, we would get T is, first we need to find out this portion for each one of them and then we sum it up for each one of the items and then we can get T.

Now, suppose we call this as some H 1 for the 1st item, so H 1 is equal to i by 2 into C 1 D 1 into 1 minus D 1 by P 1 which is 0.2 divided by 2 which is 0.1, into C 1 into D 1 is 3000 into 100 into 1 minus D 1 by P 1. D 1 by P 1 is 0.3. So, 1 minus D 1 by P 1 is 0.7 which on computation would give us, this is 3000 into 0.1 is 300. 100 into 0.7 is 70. So, 300 into 70 will give us 21,000. Similarly, we can calculate H 2 which will be 0.2 divided by 2 which is 0.1 into C 2 d 2 which is 5000 into 200 into 1 minus D 2 by P 2 which is 0.75.
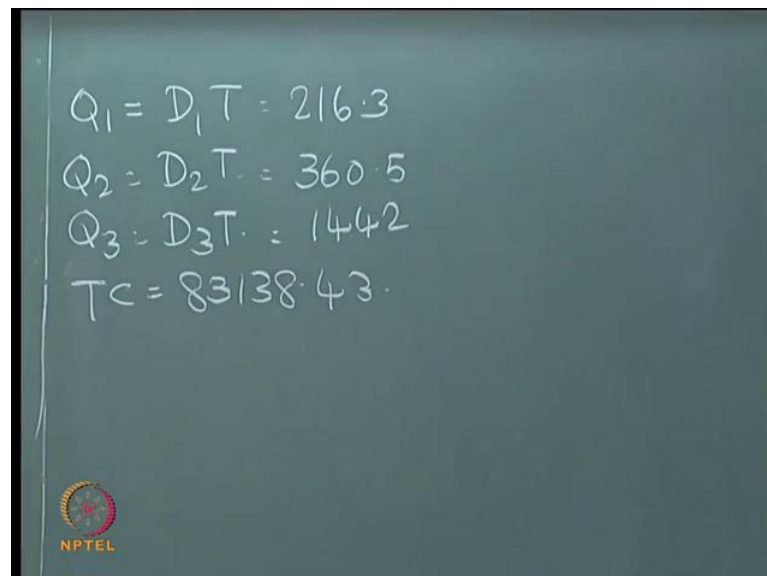
So, we can calculate that and H 2 becomes 75,000. H 3 is equal to 480,000 so we have computed: H 1, H 2 and H 3 so H 3 is 4 80,000. Now, we have to substitute all this here to get the value of T. Now, T is equal to root of sigma C naught j by sigma H j. Now, this is equal to root of C naught is the setup cost per item. Now, we have assumed that 3 items and there is a total setup cost of 3 into 1000; 3000 divided by sigma H j. We have

individually calculated the 3 values: 21,000 plus 75,000 plus 480,000 which would give us 576,000. This would give us a value of 0.0721 years.

Now, we compute that the cycle time is 0.0721 years. Now, we have to find out whether this value of T satisfies the constraint. And the constraint that we have to satisfy is sigma K j by 1 sigma K j by 1 minus sigma D j by P j is less than or equal to T. We need to check whether this constraint is satisfied by the T. Now, in order to do that we need to first find out this sigma K j, which is the sum of the setup times. So, let us assume that the setup time is 10 hours per setup. So, this would give us 10 hours per setup will give us 10 by 24 by 365 so these many years. This on calculation would give us 0.00114.

So, 0.3 into 0.00114 divided by the 3 comes because of 3 items, each item has a setup time of 10 hours. So, 3 into 0.00114 divided by 1 minus sigma D j by P j which is 1 minus D 1 by P 1 is point 3 minus D 2 by P 2 is 0.4 and D 3 by 0.25 minus 0.4. So 1 minus 0.95 is 0.05 and this would give us 0.0685 years. Now, the constraint sigma K j by 1 minus sigma D j by P j is 0.0685 which, is less than the computed value of T of 0.0721. And therefore, if the setup time is 10 hours per setup, we can conveniently take this T to be 0.0721.

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Now, for this T equal 0.0721 we can now, calculate Q 1 is equal to D 1 into T, Q 2 equal to D 2 into T, Q 3 is equal to D 3 into T. So, Q 1 will be 216.3 units, this will be 360.5 and this will be 1442 giving us a total cost of 83138.43. Now, if we have a situation

where the setup time is 10 hours per setup (Refer Slide Time: 17:07) then we would accept the 0.0721 as the cycle time T. But, on the other hand if K j is 50 hours instead of 10 hours, then we again have to find this out, sigma K j by 1 minus sigma D j by P j.

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Now, if K j is equal to 50 then sigma K j by 1 minus sigma D j by P j will become 0.0685 into 5. The reason is the denominator remains the same the numerator instead of 10 is now, 50 therefore; it is multiplied by 5 which gives us a value of 0.3425. Now, 0.3425 is greater than 0.0721. And therefore, in this case we would now, take T equal to 0.3425 and compute the rest of the numbers, from which Q 1 will be 1027.5, Q 2 will become 1712.5 and Q 3 will become 6,850 with total cost is equal to 206039.1. We also observe that the total cost increases when the setup time becomes 50 hours instead of 10 hours.
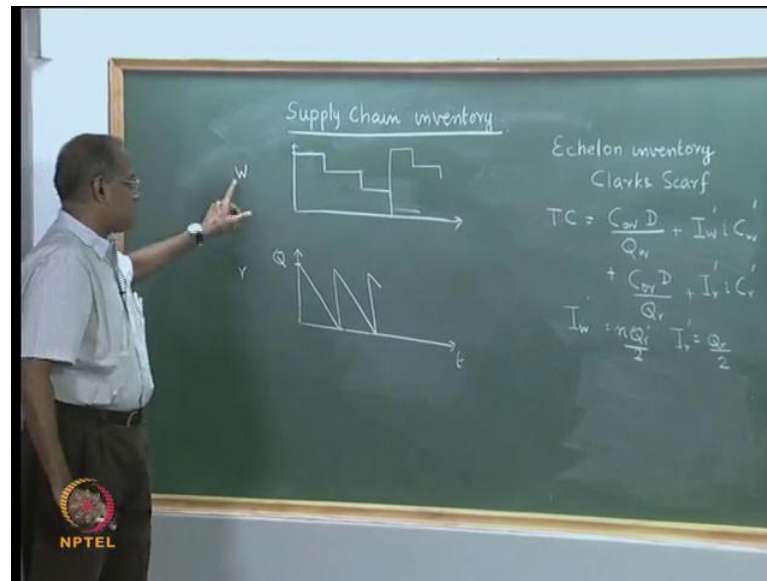
We look, at one more aspect of inventory, which let us call as supply chain; we call it supply chain inventory. Now, the very basic inventory model, where we had this graph for time and quantity. We assumed that there is instantaneous replenishment which means there is 0 lead time. And then, there is continuous demand for the item. So, we could start with a certain Q here and then we consume till we reach 0, at that instance place an order it gets replenished. Once again we do this and then we place an order and the cycle goes. Now, let us assume that we or this person is the retailer. There is a demand; there is a continuous demand. So, someone is going to pick up demand every moment. So this will be the inventory position for the retailer; so the inventory will begin at Q. It will be consumed at the rate of D to 0. The retailer places an order, gets items instantly it shoots up to Q once again it is consumed.

Now, let us look at the other aspect who does the retailer place the order and how does the retailer get it? So, the first assumption is there is instantaneous replenishment. Now, let us assume that the retailer places an order with either a distributer or a manufacturer. So, let us call a distributor and let us try and plot the inventory position for the distributor. So, this is again the position this is again time. Now, let us assume that the distributor has a certain amount of stock, which is here and for the sake of our argument we assume that there is only 1 distributor, who has warehouse. Let us say so 1 distributor and this distributor has a certain stock. So, the stock for that person is somewhere here it keeps flat like this.

So, the stock is going to come down, only when there is a demand from the retailer. So, this person has a stock right here, at this point there is a demand for Q from the retailer. So, this person is going to supply the Q let us say, a person supplies Q here. Now, the retailer gets this Q and then starts consuming somewhere here. Now, as far as this person is concerned there is no other order so this stock will remain the same up to this point. Once again this person is going to make an order so this person is going to replenish a Q and so on. Note that these two are not to scale this is Q; same Q is shown here as Q so they are not to scale but essentially this quantity Q becomes this quantity in the other graph.

So, the point that we wish to make is while the shape of the inventory position curve is triangle this way or Sawtooth as it is called. Now, this is more like a step function it goes on, then somewhere this person also would reach 0 stock or let us say the person reaches 0 stock here. And then, this person orders which means a person may get it somewhere here up to this and then his cycle may continue. Now, if we start looking at inventory planning of this person. Let us say this person is not a manufacturer whereas, this person is a warehouse or a distributor then this person has to place an order to the manufacturer and to get it. So, this is how the inventory moves along the supply chain now, modeling this person is reasonably easy because the order cost is D by Q into c naught carrying cost is Q by 2 into C c.

Now, the Q by 2 into C c comes because of the nature of this thing which is a triangle. So, we could get total inventory divided by and so on. Now, the moment we get into this we cannot apply the same economic order quantity formula for this person; the reason being the inventory curve is not the Sawtooth curve it is a different kind of a curve. So, the inventory average inventory calculation will be very different here and therefore, we will not be able to apply the same economic order quantity formula here, for this person. Now, how do we handle such a situation? Now to handle such a situation an important term called Echelon inventory; was used and introduced by Clark and Scarf in their 1960 paper where, they spoke about Echelon inventory between this and this.

So, let us explain this Echelon inventory and see what it is. Now, when we say that inventory here shows a certain behavior. Whereas, inventory here shows a different kind of behavior, when we compute the Echelon inventory as the total inventory in the system, which would include the inventory of this person, as well as the inventory of this

person. Then it was shown that the Echelon inventory more or less shows the same kind of behavior here. So, for the Echelon inventory we could write the total cost function and then we could differentiate it with respect to Q to get the order quantities for the retailer as well as for the distributor.

Now, the total cost quantity will be like this. So, the total cost for both of them put together T C will be T C will be C o w into D by Q w plus I w dash i C w dash plus C o r D by Q r plus I r dash into i into C r dash. Where, I w dash is the Echelon inventory given by n Q r dash by 2 and I r dash is equal to Q r by 2. Now, I have written a fairly complicated expression here but it is quite easy to explain each of these. Now, what we are assuming is that we call this person as r which is the retailer and we call this person as w which could represent a warehouse or a distributor. Since, w stands for warehouse let us call this person as a warehouse.

Now, this term is for the retailer, this term for the warehouse. Now, let us look at only this term, which is the retailer term; now there are 2 terms in this for the retailer: one is the order cost for the retailer other is the carrying cost for the retailer. So, the retailer's order cost is C o r per order D by Q into C naught is the standard thing that we know which represents the annual ordering cost. So, D by Q into C naught is for the retailer so D is the annual demand which is the same for both the retailer as well as for the warehouse. So, D by Q r Q r is the order quantity for the retailer D by Q r into C naught r order cost for the retailer plus Q by 2 into C c which is nothing but the average inventory of the retailer into C c.

Now, I r is the generic term so I r is the echelon inventory or average inventory at the retailer which is Q r by 2. So, Q by 2 into C c. C c is i into C i which we have already seen i is the interest rate and C is the unit price of the item at the retailer. So, this part is also there for the retailer. Now, for the warehouse we need to look at D by Q into C naught plus Q by 2 into C c for the warehouse. So, D by Q again D is the annual demand Q w is the order quantity that this person orders so Q w into C naught w order cost for this person so this term is fine. This term is Q by 2 into C c, C c is i into C i into cost of the unit price of the item at the warehouse. So, i C into Q by 2. Now, this is Echelon inventory at the warehouse this will be n into Q r by 2. We assume that the order quantity for this Q w is an integral multiple of the order quantity of this Q r. So, C dash w is the

worth of the Echelon inventory; unit Echelon inventory at the warehouse so this will become n into Q r by 2.

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$$TC = \frac{C_{ow} D}{n Q_r} + \frac{C_{or} D}{Q_r} + \frac{n Q_r i C_w'}{2}$$

$$+ \frac{Q_r i C_r'}{2}$$

$$- \frac{C_{ow} D}{n Q_r^2} - \frac{C_{or} D}{Q_r^2} + \frac{n i C_w'}{2} + \frac{i C_r'}{2} = 0$$

$$Q_r = \sqrt{\frac{2D\left(\frac{C_{ow}}{n} + C_{or}\right)}{i\left(n C_w' + C_r'\right)}}$$

So, what we can do now, is we can substitute and try and get the expressions for, so now we can substitute here to get T C is equal to C naught w D. Now, (Refer Slide Time: 26:51) Q w is equal to n Q r by n Q r plus C naught r d by Q r plus (Refer Slide Time: 26:51) I w dash is n Q r by 2, n Q r by 2 i c w dash plus Q r by 2 i C r dash. So now, differentiating with respect to Q r which is the variable here. So we get and setting it to 0 minus C 0 w D by n Q r square minus C o r d by Q r square plus n i C w dash by 2 plus i C r dash by 2 equal to 0. From which, Q r will be equal to root over 2 times D into D is common this 2 is also a common.

So, 2 times D into C 0 w by n plus C 0 r divided by i into n C w, n C w dash plus C r dash. So, 2 D n C w dash plus C r dash is the economic order quantity for Q r. So, if we know all these values ordering cost at the warehouse, ordering cost at the retailer, annual demand, unit price at the warehouse for the Echelon inventory and unit price at the retailer for the inventory i, we can calculate it. The only variable that we do not know is n. Now, there are more involved ways of computing this n but the best and the easiest thing to do is to substitute n is equal to 1 get Q r. Because, n is equal to 1 the total expression is somewhere here; this is the total expression substitute for n equal to 1 get Q

r for n equal to 1 substitute here. Do for n equal to 2 get Q r again substitute here and somewhere, you will see the minimum that is happening and accept it.

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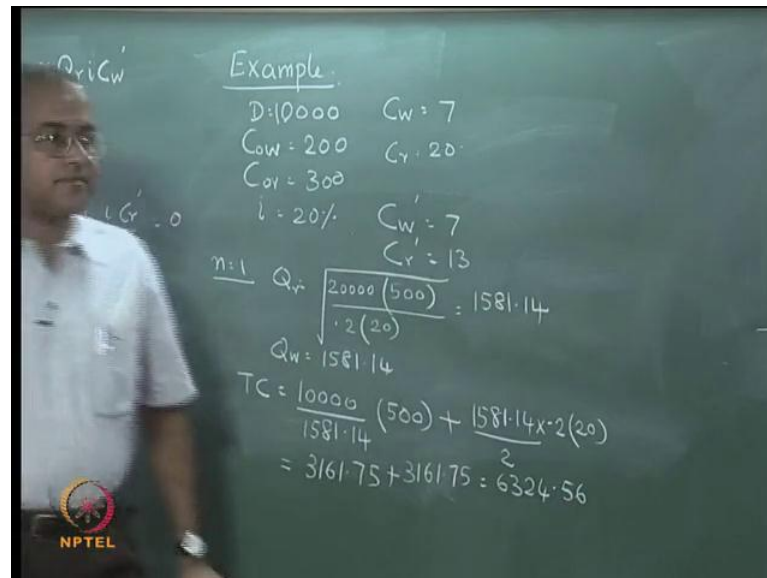
So, we will take a numerical example to explain this. So, let us take: D equal to 10,000, C naught w is at the warehouse the order cost is 200, at the retailer the order cost is 300, rate of return is 20 percent for both of them. And we assume that the unit price at the warehouse is 7 and the unit price at the retailer is 20. So now, this would give us the Echelon inventory's cost is 7 and C r is at the retailer which also absorbs this 7. So, you get C r dash the effective value at the retailer is 13; additional unit price at the retailer is 13, which is what we will substitute here. So now, on substitution we would get; we know this value of (Refer Slide Time: 36:53) Q so we use n equal to 1 to get.. For n equal to 1, Q is equal to root of 20,000 into (Refer Slide Time: 36:53) C 0 w by n plus this so 500. This is 200 divided by 1 plus 300 is 500 plus 0.2 into C w dash plus C r dash which is 20.

So, this gives us a Q value of 20,000 into 500 divided by 20 1581.14 so this Q r is equal to 1581.14 Q w is also equal to 1581.14. Because, if here we have defined Q w as n times (Refer Slide Time: 26:51) Q r. So now, for this the total cost will be, we will have to go back and (Refer Slide Time: 36:53) substitute here for the total cost. So, this is D by Q r is the same, so d by Q r is 10,000 divided by 1581.14 into (Refer Slide Time: 36:53) C 0 w by 1 plus C o r which is 500 plus (Refer Slide Time: 36:53) Q r by 2 into i

Q r by 2 into i n into C w dash plus C r dash. So, Q r by 2, 1581.14 by 2 into point 2 into C w dash plus C r dash which is 20. So, this would give us 10,000 divided by 1581.4 into 500, 3161.75 plus 1581.14 into point 2 into 20 divided by 2; which is another 3161.75, this is 6324.56.

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 Now, we could look at n equal to 2 (Refer Slide Time: 36:53) to get Q r is equal to root over 20,000 which is 2 into D (Refer Slide Time: 36:53) C 0 w by 2. So, which is 100 plus 300; 400 divided by n into C w dash (Refer Slide Time: 36:53) plus C r dash so 14 plus 13; 27 into 0.2, 0.2 into 27. This would give us 20,000 into 400 divided by 0.2 divided by 27 root over 1217.16 is what we would get and for this Q w, will be 2 times this quantity 2434.32. See what we have done here is, we have assumed that the order quantity here (Refer Slide Time: 26:51) is 2 times the order quantity here.

So, we have substituted for n equal to 2 in this to get Q r so Q r is 1217 and Q w is 2434.32. Now, for this value of Q r and Q w are essentially (Refer Slide Time: 36:53) for this value of Q r we will now try and find out T C. So, T C will become C 0 w (Refer Slide Time: 36:53) into D by n into Q r so 200 into 10,000 divided by 2 into 1217.16 n into (Refer Slide Time: 36:53) Q r plus 10,000 into C o r by Q r 300 divided by 1217.16 plus n Q r into (Refer Slide Time: 36:53) i C w dash by 2. So, 2 by 2 into 1217.16 n Q r i Q r into 0. 2 into C w dash which is 7 plus (Refer Slide Time: 36:53) Q r by 2 into i C r dash 1217.16 by 2 into 0.2 into 13.

So, we have 4 terms here. So, let me quickly calculate all these 4 terms. So, 200 into 10,000 divided by 2 divided by 1217.16 this is 821.58 plus 2464.75 plus this will get canceled 1704.024 plus 1582.308 so this 4 will add up to 6572.76.

So now, we have got this formula and then we have tried (Refer Slide Time: 36:53) to find Q r for n equal to 1 and n equal to 2. When we do n equal to 1 we get Q r is 1581, Q w is also 1581. Total cost is of the 2 systems is 6324. When we put n equal to 2, Q r is 1217 it comes down but Q w gets doubled of this 2434.32. Therefore, this cost comes down, this cost goes up. This inventory cost will go up, this inventory cost will come down, because of the unit cost and the total is higher 6572.76.

Therefore, in this case we would accept this solution with n equal to 1. So, this is how we handle, if we have to look at inventories (Refer Slide Time: 26:51) between retailer as well as the warehouse or what is called Echelon inventory. Now, we will go back to more traditional inventory models. So, far we have seen in all the inventory models that the demand does not change with time it is the same 10,000 per year and so on. Now, the next question is what happens if the demand for an item changes with respect to time and is different in different time periods and is known. So, we will address such types of inventory problems in the next lecture.