An Introduction to Point-Set-Topology (Part II)
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Lecture No. 09
The retraction functor k(X)

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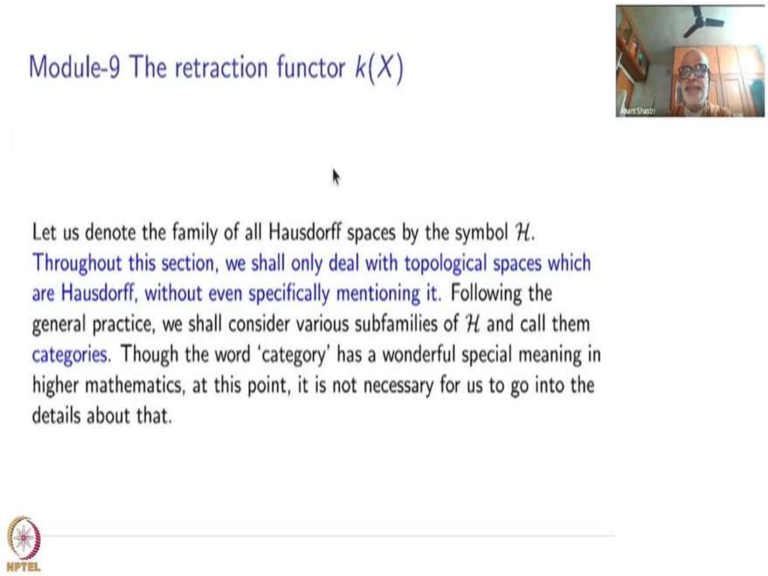

Hello, welcome to Module 9 of NPTEL NOC course on Point Set Topology part II. Today we shall study retraction functor $k(X)$ which is closely related to the topic we are studying namely local compact spaces. Let us denote the family of all Hausdorff spaces by the symbol $\mathcal{H}$.

Throughout these sections we shall only deal with topological spaces which are Hausdorff and perhaps we even do not mention it specifically. Following the general practice, we shall consider various sub families of $\mathcal{H}$ and call them categories. Though the word category has a very special and very wonderful meaning in higher mathematics, at this point it is not necessary for us to go into the details about that.

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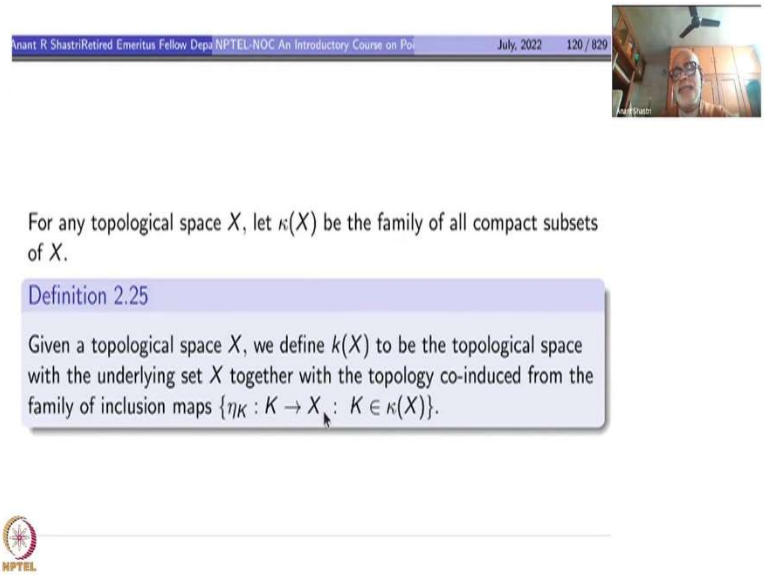

For any topological space X, let this $\kappa(X)$ denote the family of all compact subsets of X. So, this is a notation for the family of all compact subsets of a given topological space. Now, we define another topological space denoted by $k(X)$, please do not confuse this with $\kappa(X)$; they are quite distinct notations. This $k(X)$ is the topological space with its underlying set X itself, but the topology will be different, and what is the topology? It will be the co-induced topology from all the inclusion maps of all compact subsets of X. So, this co-induced topology.

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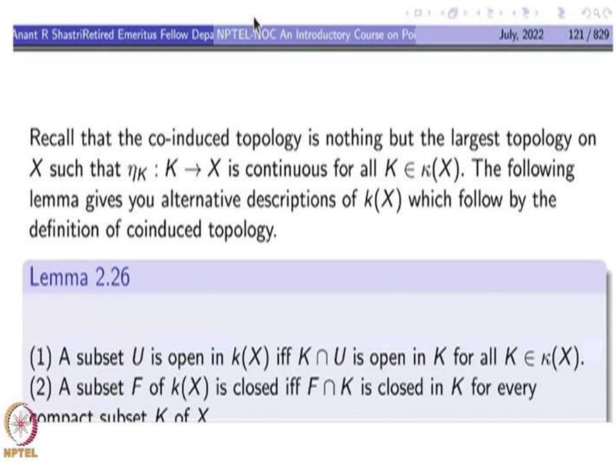

Recall that the co-induced topology is nothing but the largest topology on X, such that all these functions in this family are continuous. Once you declare a family of functions from arbitrary spaces into X, fix that family, take the largest topology on X such that all the members of this family are continuous. That is called the co-induced topology from this family. Since, I am talking about the largest family, there is only one such and that is called co-induced topology.

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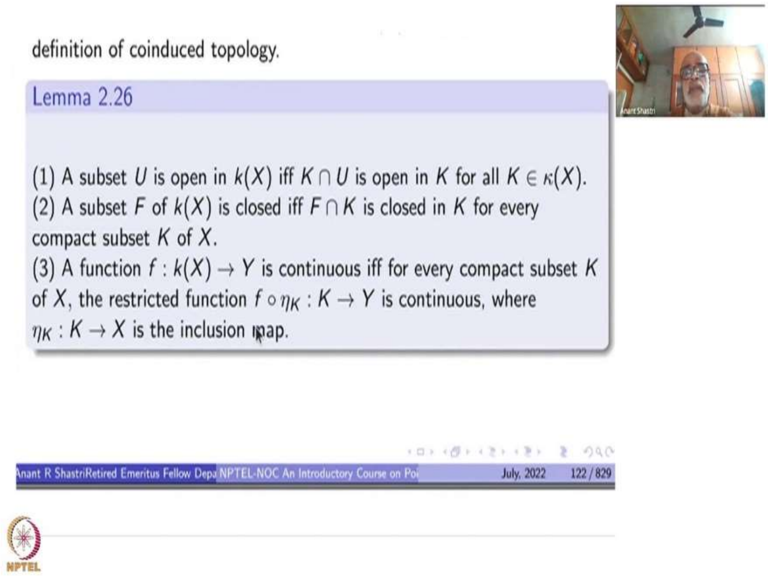

Just to recall what this topology is which we have introduced in part I, wherein, we have studied these things very thoroughly, I am summed it up here in this lemma. Each of these conditions is equivalent to the definition of co-induced topology. What are these three conditions?

(1) A subset U in the co-induced topology $k(X)$ is open, if and only if $K \cap U$ is open in K for all compact subsets of X.

(2) Next, instead of open, here similar condition by replacing the word `open' with the word `closed'. A subset $F(k(X))$ is closed if and only if $F \cap K$ is closed in K, for every compact subset K of X.

(3) The third condition is in terms of the continuous functions. A function f from $k(X)$ to Y is continuous, where Y is any topological space, if and only if for every compact subset K of X, the restricted function, (which is the same thing as taking the inclusion map η_K of K into X and then follow it by the given function f, that map) from K to Y is continuous for all compact subsets K of X.

So, these are very easy to verify. If you do not know what is the co-induced topology, you can take any one of them as the definition. Often, we will use (1) or (2) and sometimes (3) also.

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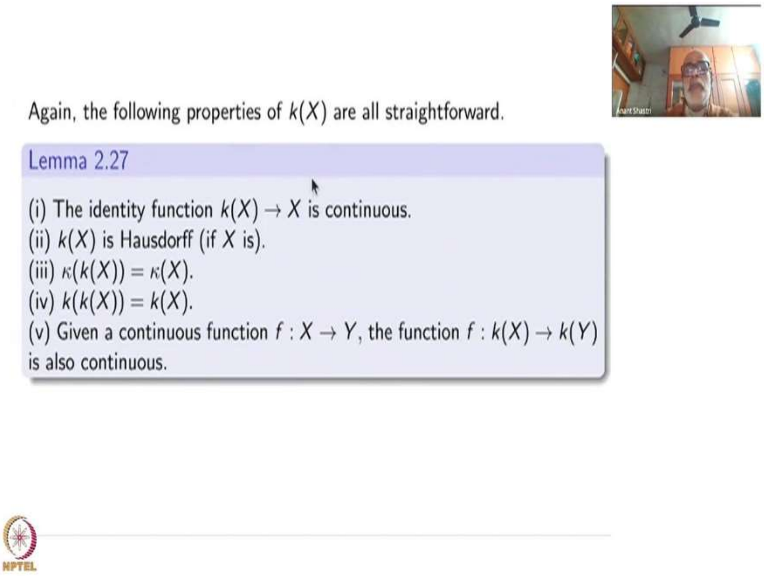

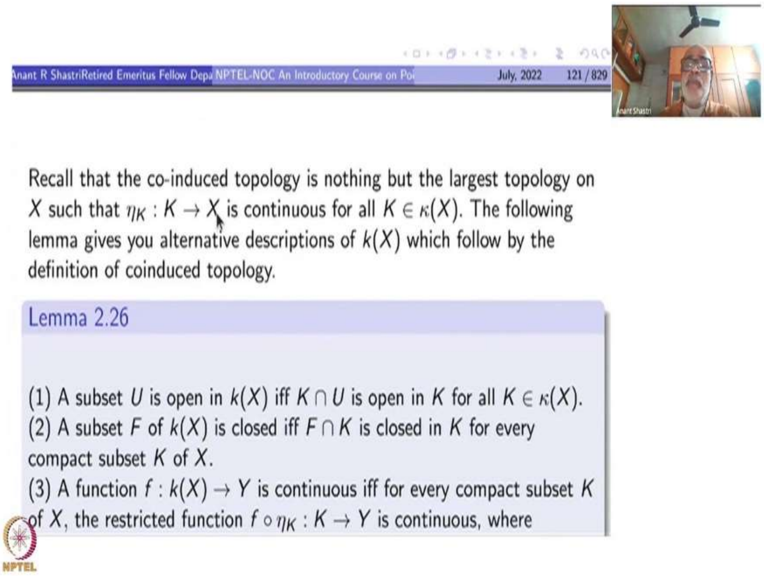

Now, the following properties of $k(X)$ are all straight forward.

(i) The co-induced topology is always finer than the topology with which you are started. In this particular case, see X is already a topological space and η_K is the inclusion map from a compact subset K of X. So, before talking about compact subsets of X, X must have a topology already. η_K are continuous if you take the topology on X to be the original topology. But $k(X)$ is the one with the maximum number of open sets with this property. It is the largest topology with this property. $k(X)$ may have more open subsets than X. Therefore, what happens is that the identity function from $k(X)$ to X is continuous. This means for instance,

(ii) $k(X)$ will be Hausdorff if X is Hausdorff, why? All open sets in X are open in $k(X)$ as well.

(iii) If you take the family of all compact subsets of this new topological space $k(X)$, that is not changed, it is same thing as the family $\kappa(X)$. That is $\kappa(k(X)) = \kappa(X)$. So, this comes as a surprise, a pleasant surprise, because if there are more open sets than the original topology, then there is a danger that some compact subset in the orignial topology may be compact in the new topology. But here it is ok. That is the good thing about this particular topology $k(X)$.

(iv) The fourth property is an easy consequence of (3) viz., take some space X, you can take k of k of that space. You do not get anything new than $k(X)$ itself.

(v) The fifth properties is: given a continuous function from X to Y, then underlying spaces of $k(X)$ and $k(Y)$ are the same as X and Y. Therefore, you can talk about the same function f from $k(X)$ to $k(Y)$ is also continuous wit the new topologies on both sides.

So, finally, you may ask why do you need it at all. So, that will be explained now.

(ii) So, the identity function from $k(X)$ to X is continuous I have explained that already. Then (ii) follows one because $k(X)$ has more open sets than X, every subset if it is open here, then it is open here as well. So, you can use the same open sets to serve in $k(X)$ also for separating given distinct points.

(iii) The third one needs a little more explanation. Take a compact set here in $k(X)$. Under the inclusion map, which is continuous its image begin itself, will be compact in X also. also, so that is the beauty. So, all compact of subsets here are compact here, fine. But suppose something is compact here in X. Why it is compact here in $k(X)$? That is what you need to understand.

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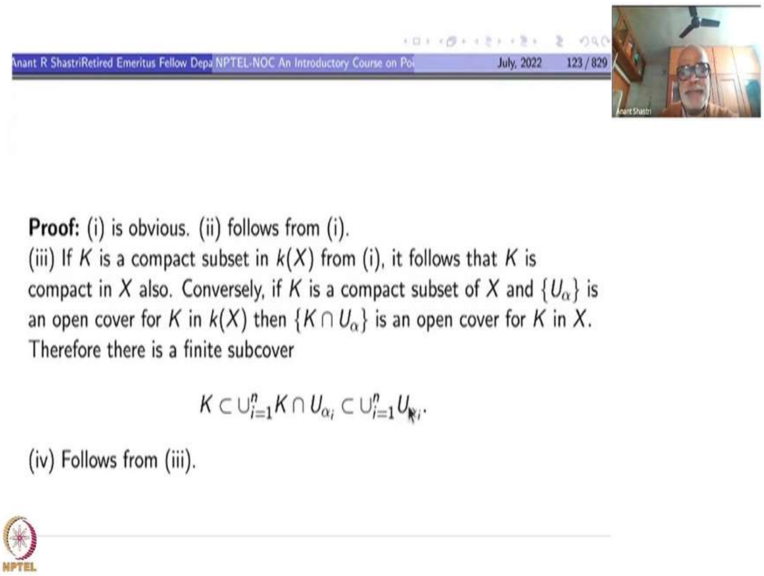

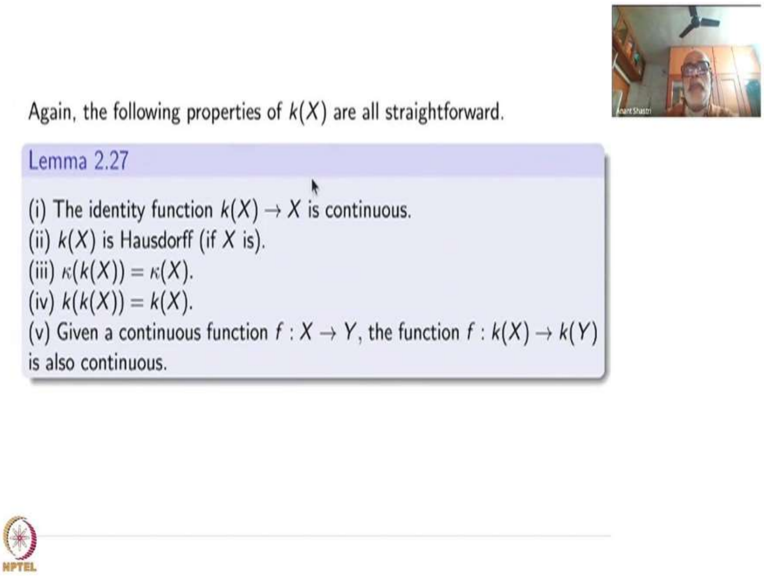

So, let K be compact subset of X and $\{U_\alpha\}$ be an open cover for K in $k(X)$. What are open subsets of $k(X)$? They have the property that when you intersect with any compact set in X, you get an open set in that compact set.

So, each $K \cap U_\alpha$ will be open in K. Obviously they will cover the entire K. K itself is a compact subset of X. I started with U_α which are open subsets of $k(X)$, but now $K \cap U_\alpha$ are open subsets of K, where K has the subspace topology from the original topology on X. Therefore, they will admit a finite subcover for K. But now just the corresponding finitely many members U_α's. That will give you a finite cover for K. So, K is contained in the finite union of i ranging from 1 to n of $K \cap U_{\alpha_i}$, which is contained in the union of i ranging from 1 to n of U_{α_i}.

So, once the compact subsets of $k(X)$ are the same as that of X, the two families of functions the same. Therefore, $k(k(X)) = k(X)$. So, (iv) follows from (iii), since the family $\{\eta_K : K \to k(X)\}$ is the same as the family $\{\eta_K : K \to k(X)\}$.

So, finally, I will have to verify (v) what is (v)? Start with a continuous function f from X to Y. Now, pass on to $k(X)$ and $k(Y)$, these are new topologies, what is the relation? If you take $k(X)$ to X, the inclusion map that is continuous, X to Y is continuous. So, after composing, I get $k(X)$ to Y, the same function f, thought of as function from $k(X)$ to Y, that is continuous. But I have to show that $k(X)$ to $k(Y)$ is continuous.

Now, $k(Y)$ obviously may have more open sets than Y. For those open sets, extra open sets, why their inverse images under f are open subsets inside $k(X)$. That is what we have to check.

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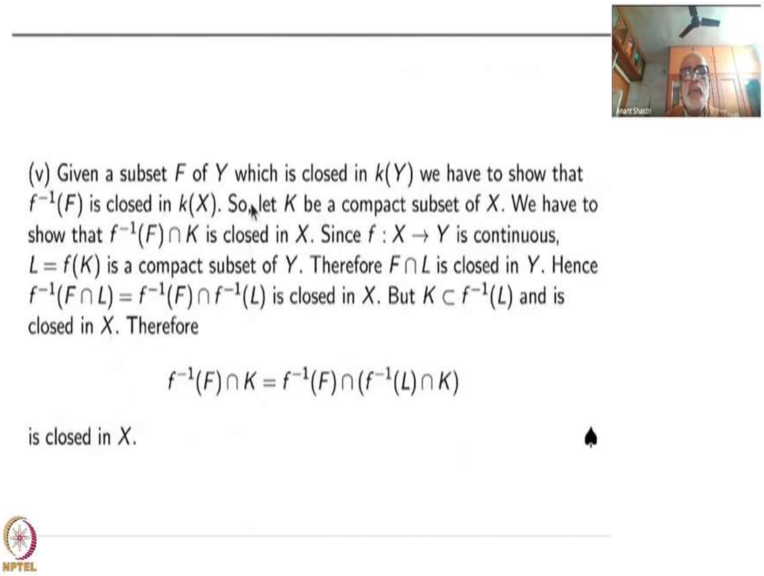

So, instead of open sets, you can do it with closed sets. So let me do it with closed sets here. Given a subset F of Y which is closed in $k(Y)$, we have to show that $f^{-1}(F)$ is closed in $k(X)$, what is the criterion? Take a compact subset K of X, intersect it with $f^{-1}(F)$, show that that is close in K, that will do the job. So, we show that $K \cap f^{-1}(F)$ is closed in K.

Now, f is continuous from X to Y. Therefore, if you take $L = f(K)$, that will be a compact subset of Y. Therefore, by the criteria for f to be closed in $k(Y)$, $F \cap L$ is closed in L. Hence,

$f^{-1}(F \cap L)$, which is $f^{-1}(F) \cap f^{-1}(L)$, that is closed in X, because f is continuous from X to Y. But now, K is already contained in $f^{-1}(L)$, because L is nothing but $f(K)$.

Therefore, $f^{-1}(F) \cap K$ is same thing as $f^{-1}(F) \cap f^{-1}(L) \cap K$. Taking intersection further with K does not change the set. Therefore, this intersection with K is closed in K. Since this true for all K, $f^{-1}(F)$ is a close subset of $k(X)$. That is what we wanted to show. So, that completes the proof of this lemma.

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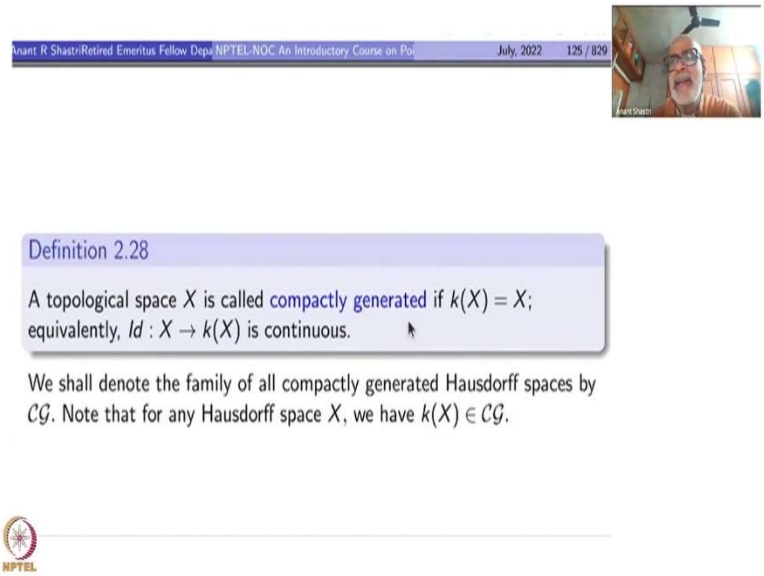

So, we know something about this $k(X)$. Out of a given space X, we are getting some other space, what is this space, how to identify it, what are its properties, quite a few things. Many properties are already listed. Now, we will name this $k(X)$ and then start studying it further.

The topological space X is called compactly generated if $k(X)$ is equal to X. So, this is a general definition I am making. Remember that this is the same thing as saying that identity map from $k(X)$ to X which is already continuous, is also continuous in the other way round. That is its inverse is also continuous. That means it is a homeomorphism. That is just the same thing as saying $k(X)$ is equal to X. In other words, the two topologies are the same. So, such spaces are called compactly generated.

Remember that in this section, we are all the time working inside Hausdorff spaces, though, so far, we have not paid much attention to that, but we will keep insisting that we are using Hausdorff spaces. Only then we will have a notation $\mathcal{CG}$ for the family of compactly generated Hausdorff spaces.

So, this is another subcategory of $\mathcal{H}$. In particular, starting with any X in $\mathcal{H}$, $k(X)$ is always compactly generated and is in $\mathcal{H}$. So, you see that from a general space you are getting some special space. Obviously $\mathcal{CG}$ is a smaller family than $\mathcal{H}$.

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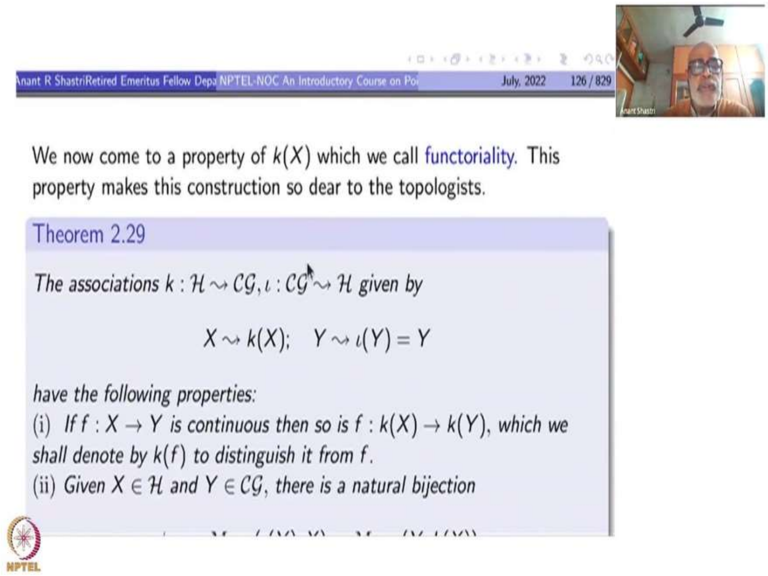

We now come to the property of $k(X)$ which we call functoriality or people also keep referring to this as canonical property. So, this property makes this construction quite dear to topologists. The association k from $\mathcal{H}$ to $\mathcal{CG}$, and ι from $\mathcal{CG}$ to $\mathcal{H}$, I do not call it a function or a map, because its domain and codomain are not necessarily sets. But that is why I have used this twiddling arrow here instead of straight arrow, straight arrow to we are using straight arrow for indicating functions, so here, I am only calling them association.

For all matters of our importance they do behave like functions. (Logical problems will arise if you insist that, $\mathcal{H}$ is a set. That is not a set. The collection of `all' objects, as soon as the word `all' is there we have to be careful.) So, consider these two associations given by X going to $k(X)$, and Y going to $\iota(Y) = Y$; ι is just the inclusion of the smaller family $\mathcal{CG}$ into the larger family $\mathcal{H}$. They have the following very close relations, nice properties.

Start with any continuous function f from X to Y, inside this larger family $\mathcal{H}$, larger category. So, take a continuous function like this, then with the same underlying sets, but topologies is different, f is continuous. So, to distinguish this `new' avatar of f from f, we will just denote it by $k(f)$, because now we are thinking of f having both its domain and codomain having the compactly generated topologies, i.e., the topology co-induced by the collection of compact sets, so that is why we will denote it by $k(f)$. As a function it is f itself.

Given X belonging to $\mathcal{H}$, and Y inside $\mathcal{CG}$, there is a natural bijection,--- So again, I have used the word natural, I will explain it to you later, but not in the statement of the theorem. So, what is the natural bijection? It depends upon Y and X of course. It is between the sets of all continuous functions from theses spaces, $\iota(Y)$ to X, on one side and Y to $k(X)$ on the other side. When I write $\iota(Y)$, I have to think of this space Y as an ordinary topological space, though it is actually a compactly generated one.

When I come here on the other side, Y is actually compactly generated space, and the ordinary topology on X is replaced by the compactly generated topology. Maps here and maps here are related by this $\psi_{Y,X}$, given by taking any h from Y to X into $k(h)$. However, then $k(h)$ is from $k(Y)$ to $k(X)$, but we already know that $k(Y) = Y$. So, this $\psi_{Y,X}$, new symbol here, it is just the map h going to $k(h)$. So, we assert that this is a bijection, so, let us understand that. The first part, we already seen. It is a restatement of the fifth statement of the previous lemma.

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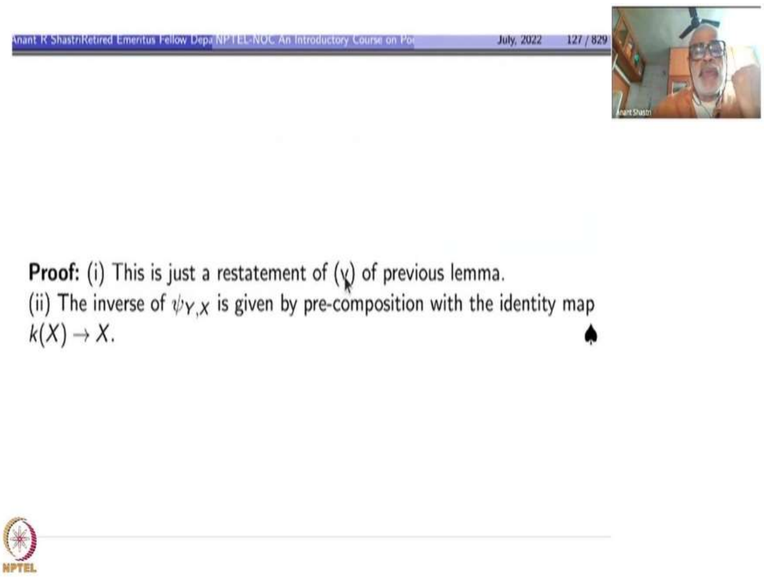

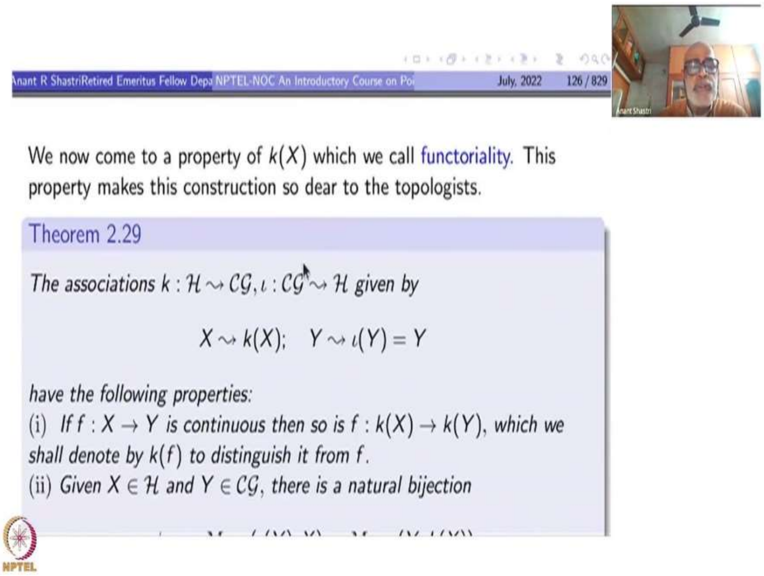

For the second part, all that I do is to tell you what is the inverse of $\psi_{Y,X}$. Given any continuous function g from Y to $k(X)$, pre-compose it with the identity map $k(X)$ to X, which we know is continuous and which we denote by i_X, temporarily. So $\lambda(g) = i_X \circ g$.

So, how do you see that, λ is the inverse of $\psi_{Y,X}$? If $g = k(h)$ for some continuous function h from Y to X, $i_X \circ g$ is clearly h, since as set functions, g and h are the same.

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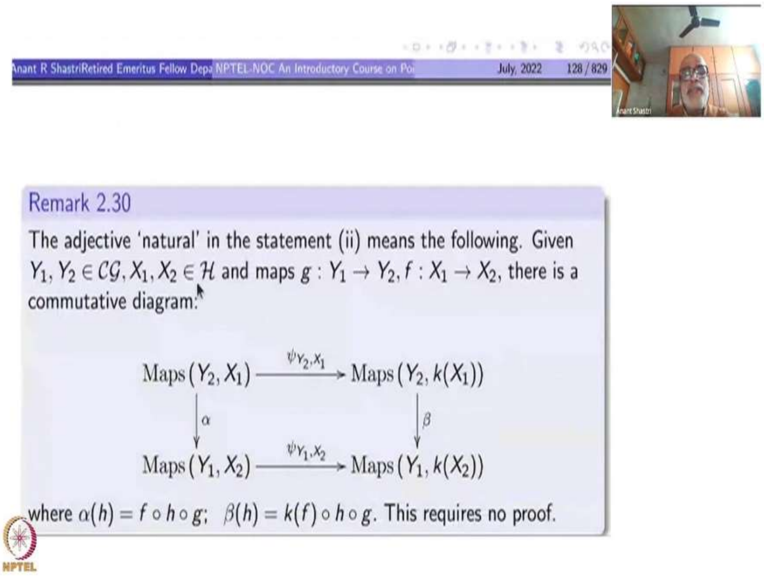

So, coming to the adjective `natural'. What is the meaning of natural isomorphism? It is the following thing. Though the symbol $\psi_{Y,X}$, indicates that it depends on the domain and codomain, the construction of these maps that does not depend upon what domain/codmain you have. All the time, start with a map and apply k functor. That is why this is natural, that is the meaning of this one. But technically we have to express this, as below, namely, given Y_1 and

Y_2 inside $\mathcal{CG}$, X_1 and X_2 inside $\mathcal{H}$, and maps g from Y_1 to Y_2, f from X_1 to X_2, these are totally arbitrary, and yet, we have the relations given by the commutative diagram here. ψ_{Y,X_1} and ψ_{Y_1,X_2} these are defined already, what is the relation between them? Wherever you have maps f and g, start with a map h from Y_2 to X_1, here, you pre-compose and post-compose, look at $f \circ h \circ g$, that will be from Y_1 to X_2. Starting with a map from Y_2 to X_1, (I do not know whether I have written compositions correctly, ok that is correct), take $\alpha(h) = f \circ h \circ g$.

Similarly, here see, it is $\beta(h)$ is $k(f) \circ h \circ g$. The assertion is that $k(\alpha(h)) = \beta(k(h))$. This is completely obvious, since both are equal to $k(f) \circ k(h) \circ k(g)$.

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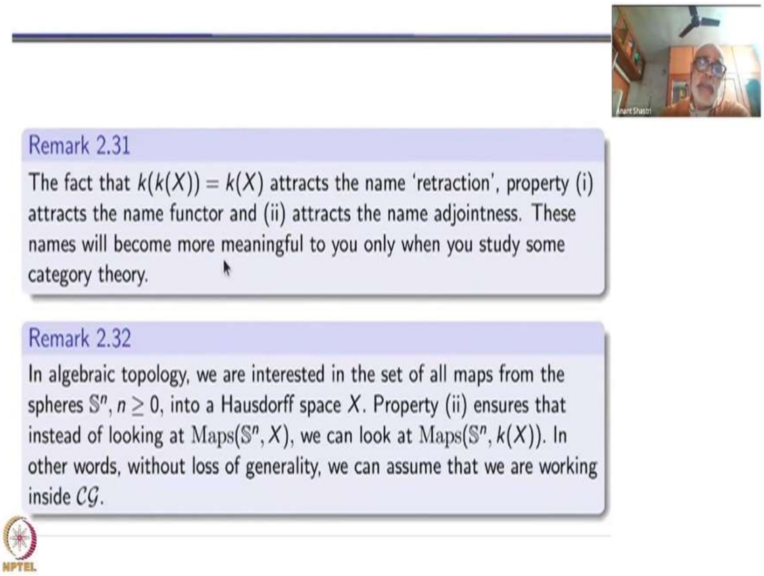

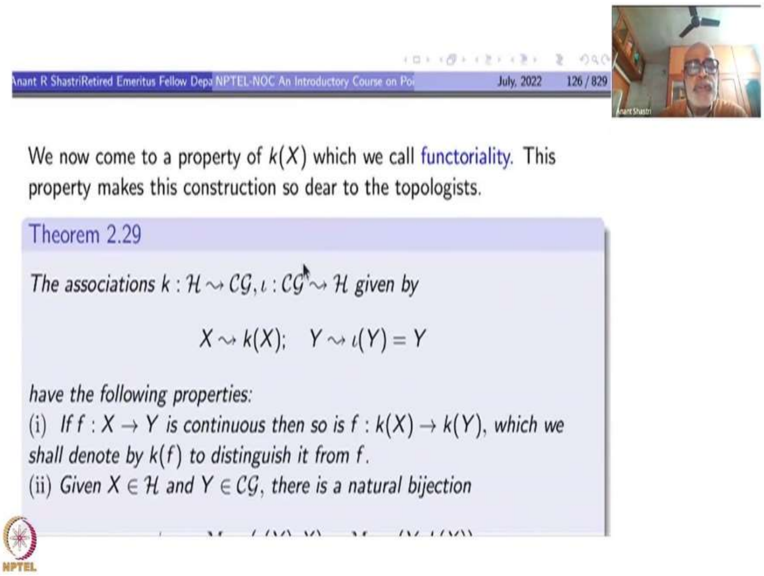

The fact that k of $k(X)$ is $k(X)$ that allows us to call k as a retraction. It is as if $\mathcal{CG}$ is sitting inside $\mathcal{H}$ and we are coming back to $\mathcal{CG}$, and the association is identity on $\mathcal{CG}$, so that is the meaning of `retration'. If it is X is already $\mathcal{CG}$, then $k(X)$ will be X itself. So, in any case $k(k(X))$ is $k(X)$. So, that is $\mathbb{R}^2$ is $\mathbb{R}$, that the property of a retraction as a map.

Property (i) above attracts the name functor, namely not only the objects have been associated, the functions are also associated correspondingly. So, with certain properties which we do not want to go into detail, so that is the name `functor' given in general. Together I have called it a retraction functor. Property (ii) describes something called adjointness. The two associations k and the inclusion, they are adjoint of each other, this is a wonderful property.

Now, finally, I want to come to one important, very-very important use of this idea, namely this functor k, namely compactly generated. In the study of algebraic topology one of the central things that we do is to study maps from compact spaces into some space, especially from the spheres. You know all the spheres are compact. Maps from a sphere into X, or maps from sphere cross closed interval into X, these are of prime importance in algebraic topology.

Now, you start with any space X, then if you look at the set of all maps from $\mathbb{S}^n$ to X, it is the same thing as the set of all maps from $\mathbb{S}^n$ to $k(X)$, why? Because $\mathbb{S}^n$ is already compact, therefore it is compactly generated, therefore, $k(\mathbb{S}^n) = \mathbb{S}^n$ itself. So, studying maps from $\mathbb{S}^n$ to any topological space X is the same thing as studying maps from $\mathbb{S}^n$ to $k(X)$. That means what? Without loss of generality, we can assume that $k(X)$ itself is compactly generated. So, that will help in lots of problems, namely, constructing maps etc, becomes easier, since you have to do things only on compact subsets. This is not all. I have just introduced this one, it will help you in the study of homology functors also.

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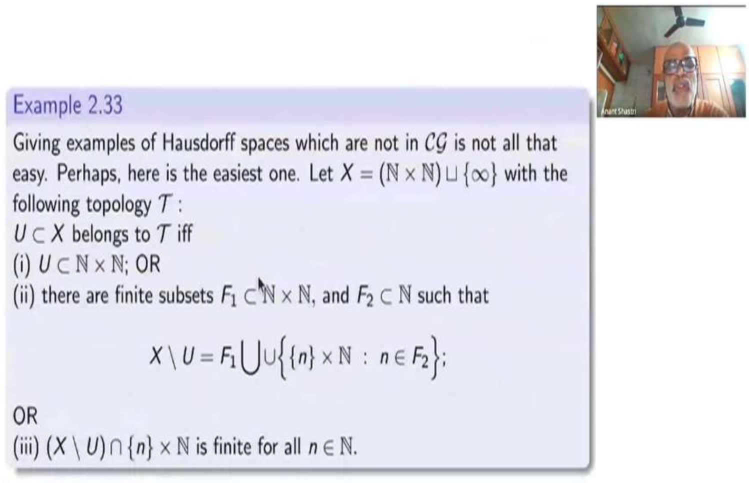

So, here is an example in which you must notice that giving an example of a Hausdorff space which is not compactly generated is not all that easy. That is good, because that just means that a lot of Hausdorff spaces are compactly generated already, so that is a happy situation. However, we will give you an example which is not all that difficult, but it is not all that easy also. So pay attention.

So, here is an example of a Hausdorff space which is not $\mathcal{CG}$. To start with $N \times N$, where N is these set of natural numbers, with the discrete topology. So, $N \times N$ is also discrete space. Take one extra point, disjoint union with extra point, I denote it by infinity, with the following topology. Now, I have defined topology on the whole space X. So, a subset U of this X, X is what? $N \times (N \cup \{\infty\})$, a subset U of X is open if and only if, U is a subset of $N \times N$, (that is one condition according to which every subset of $N \times N$, is inside $\mathcal{T}$).

Or, second condition is that there are finite subsets $F_1(N \times N)$ and $F_2(N)$, such that $X \setminus U$ is equal to the finite subset $F_1(N \times N)$ union of all singleton $\{n\} \times N$, where the singleton n ranges over the finite set F_2. (In other words, take a finite subset of $N \times N$ and finitely many vertical lines, take their union, if $X \setminus U$ looks like this, then the U is open).

Or third condition is that $X \setminus U$ intersects $\{n\} \times N$ in finitely many points. I am again looking at vertical lines, a vertical line intersection with $X \setminus U$ is a finite set for all $n \in \mathbb{N}$.

So, there are three different types of open sets. I will repeat, the first one gives all subsets of N cross N, because they are all open subsets because this is a discrete space. The second and third give conditions for the complement of U. The third one is easy to remember, viz., intersection with every vertical line must be finite, it may be empty that certain lines may not be intersecting, but whatever intersects, that intersection must be finite, such a set will be taken and its complement in X, will b etc as an open set.

The second condition you have to understand carefully. Here, in the complement I am allowing full vertical lines, but only finitely many vertical lines. Along with them some more finite subsets of $N \times N$. So type (ii) and (iii) are especially designed for neighbourhoods of infinity. If a subset U does not contain infinity, then condition (i) tells you what you have to do, what you have to do? Nothing. Any set which does not contain infinities is automatically open. (ii) and (iii) will give you neighbourhoods of infinity. So, this is somewhat unusual, so, I have I am going through this carefully here.

So, do you understand now that the description of this topology? It is very easy to check that this is a topology, all that you have to do is neighbourhoods are correct, neighbourhoods of infinity are correct, because this part is already a topology, this is a discrete space. So, why? Suppose you take two of them here, U_1 and U_2, what will be their intersection? The complements will be unions of such two things by De Morgan law. Union of finite subsets F_1 and F_1', F_2 and F_2', will come, so such unions will be alright.

Or it may be like this, again intersection with each N is finite, if union of two such finite-finite-finite. Suppose you have one here, one here, then what happens? You are allowed from finite subset, some finitely many things are there fine. So, rest of them are all finite what? Rest of them are all finite here. So, this condition will take care of this one, $\{n\} \times N$ for these things are N, it is fine, did not intersect, its complement did not intersect, but if it contains some of them it is filthy as this one.

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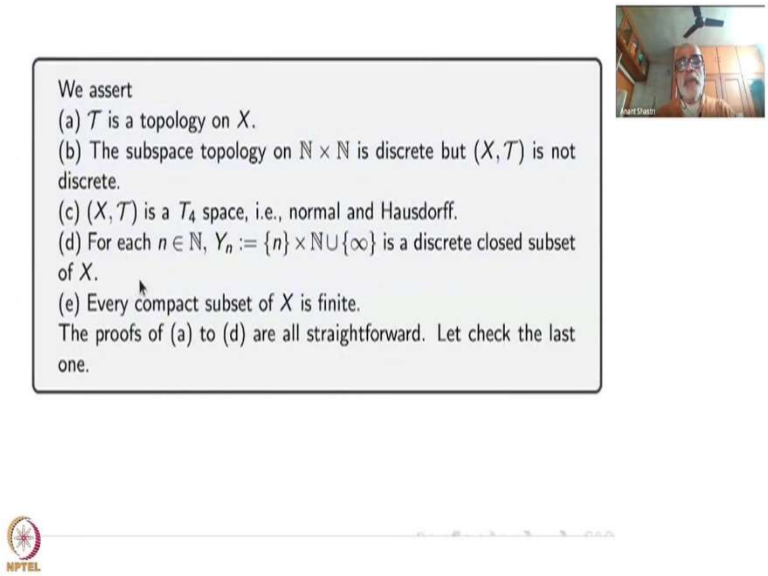

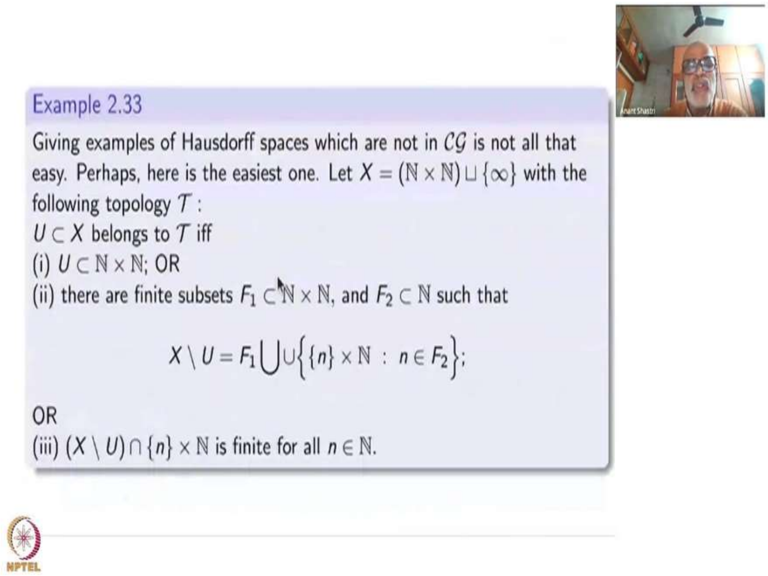

So, take a topology like this, that is going to give you something, what is that? So, $\mathcal{T}$ is a topology on X, that is fine. The subspace topology on $N \times N$ is discrete, that is also fine, because we started with $N \times N$ as the discrete space, but $(X, \mathcal{T})$ itself is not discrete, not all subsets are open. A neighbourhood of infinity cannot be arbitrary set, that satisfy this condition or this condition, so it is not discrete, that is obvious.

Third thing is here, $(X, \mathcal{T})$ is actually a T_4 space. It is Hausdorff and it is normal. Once again, because of $N \times N$ is already discrete space, we have to verify Hausdorffness for points when one of them is infinity. For example, any point here can be separated from any point here, because you can take this point itself as an open subset. Also since every point of $N \times N$ is also a closed set, it follows that its complement is neighbourhood of infinity. So, Hausdorff space is over. I will leave it to you to verify normality. The hint is that you need to consider the case when one of the closed set contains infinity.

The fourth thing is for each n, look at Y_n equal to the the vertical line $\{n\} \times (N \cup \{\infty\})$. This itself is a discrete closed subset of X. All points of $\{n\} \times N$ they are open, that is fine. So, if I show that $\{\infty\}$ is open here then this will be discrete. Why is $\{\infty\}$ open inside this subspace Y_n? Because you take singleton infinity union all other vertical lines, $\{m\} \times N$, $m \neq n$.

By property (iii) that will be an open subset in X. Its intersection with Y_n will be just singleton infinity. So, therefore, this singleton infinity is open here. So, that proves that Y is a discrete space. Why this is a closed subset? Its complement is some subset of $N \times N$. Every subset of $N \times N$ is open. So, Y_n a closed subset, with the subspace topology being discrete. Final thing is very important one here. Every compact subset of X is finite. So, let me I have already explained (a), (b), (c), (d), let me prove (e) completely here.

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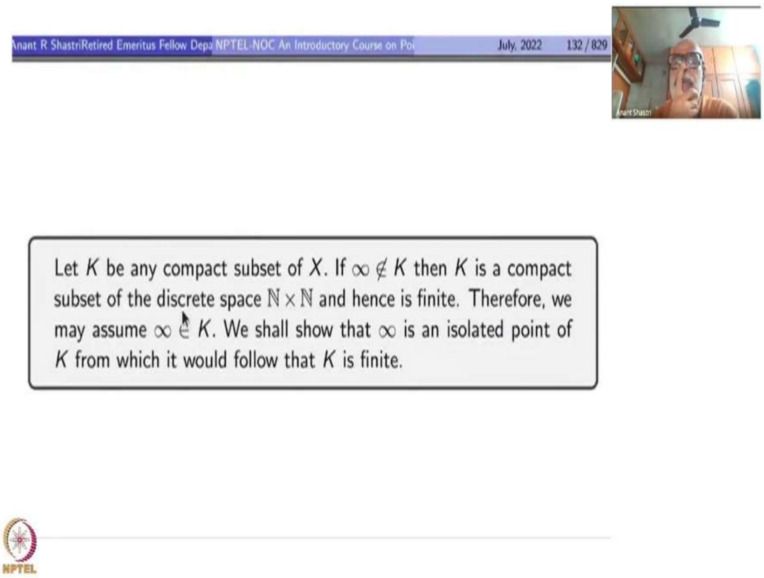

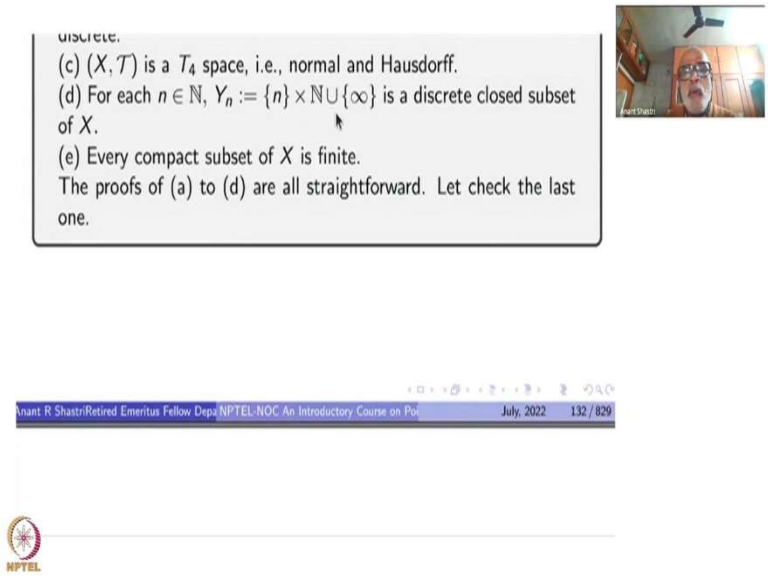

Let K me any compact subset of X. If infinity is not in K, then K is a compact subset of the discrete space $N \times N$. In a discrete space we know that every compact subset is finite, so we are done. Now let infinity be in K. Then we claim that infinity itself is an isolated point of K. Isolated singleton points are open subsets of the spaces. If you remove an isolated point, the remaining set will be a closed subset and hence compact because closed subsets of compact subsets are compact. But then that will be inside $N \times N$, so that is finite. Therefore, K will be finite.

So, let us see why infinity is an isolated subset inside a compact subset K. The proof is similar to what we did here, but a little more thing we have to do. Just examine the definition carefully.

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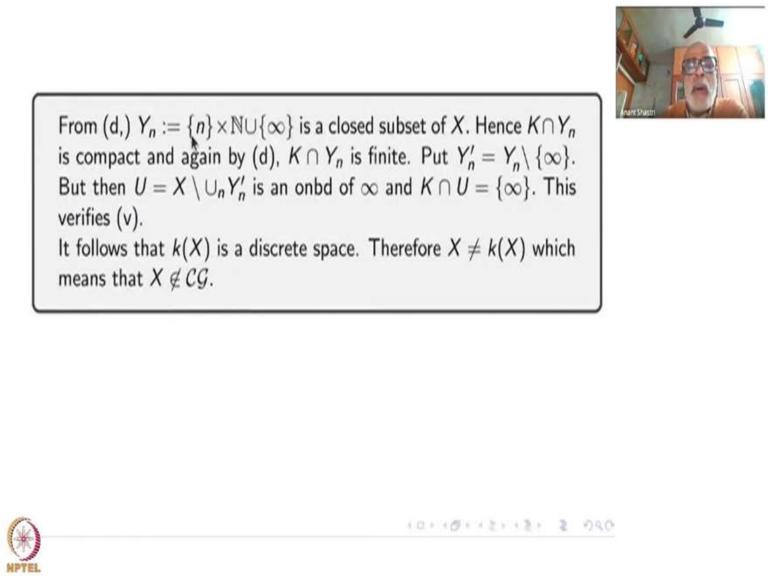

From (d), $Y_n = \{n\} \times (N \cup \{\infty\})$, is a closed subset of X, that is what we have seen. Hence $K \cap Y_n$ is compact because K is compact, intersection with Y_n will be a closed subset of K. Again, by (d), $K \cap Y_n$ is finite, why? Because we have just now seen that this is a discrete space.

Now, take $Y_n' = Y_n \setminus \{\infty\}$, throw away infinity. So, now we are inside $N \times N$. Now put $U = X \setminus \cup Y_n'$. By (iii), this U is an open neighbourhood of infinity. Moreover $K \cap U$ is just $\{\infty\}$. See, what I have did, I took K to be a compact set, intersected it with each Y_n, that is a finite set, take Y_n' as $Y_n \setminus \{\infty\}$, so threw away infinity. Then these Y_n' are closed subsets of K, because, infinity was an isolated point of Y_n, and hence Y_n' are compact.

So, $U = X \setminus \cup Y_n'$ is an open subset. It is a neighbourhood of infinity, infinity is still there, because I have thrown away infinity from Y_n, so infinity is not in any of them. So, when take the complement again, infinity will be there. So, U is an open subset because I have thrown away close subset. So, $K \cap U$ is $\{\infty\}$, so that is open subset of K. So, that proves $\{\infty\}$ is an isolated point of K. This verifies (e)

So, it follows that $k(X)$ is a discrete space, why? Because we have proved that every compact subset of X is finite and hence discrete. Now take any singleton set. Its intersection with any compact subset K will be open in K. That means all singletons are open in $k(X)$. That means every subset of X is open in $k(X)$.

So, $k(X)$ will be a discrete space. X is not discrete, therefore, X is not equal to $k(X)$. Just by adding one single point, the compactly generated-ness has gone away. You may wonder, $N \times N$ is a discrete space, is it compactly generated? Yes of course. Any discrete space is compactly generated, singleton sets are compact there. All finite subsets are compact there, that is fine. But X is not compactly generated. That X is not a member of $\mathcal{CG}$.

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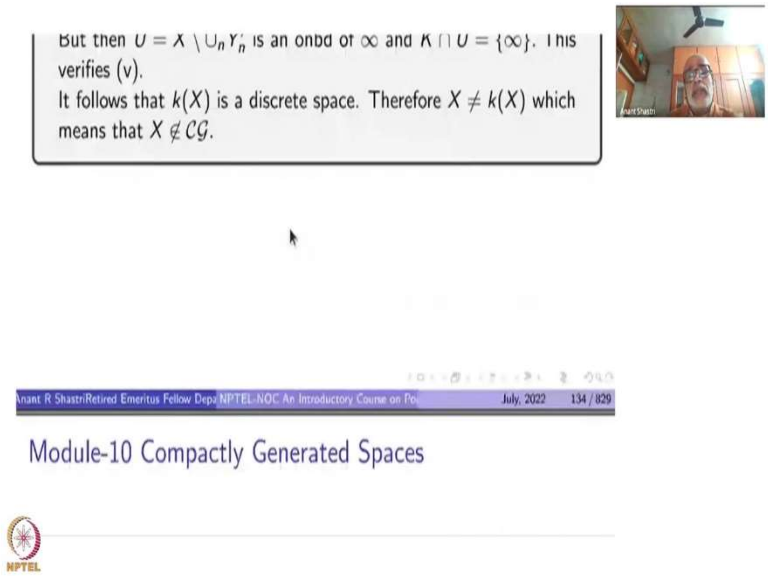

So, next time we will study compactly generated spaces a little deeper. I have already told you why they are important and so on. Thank you.