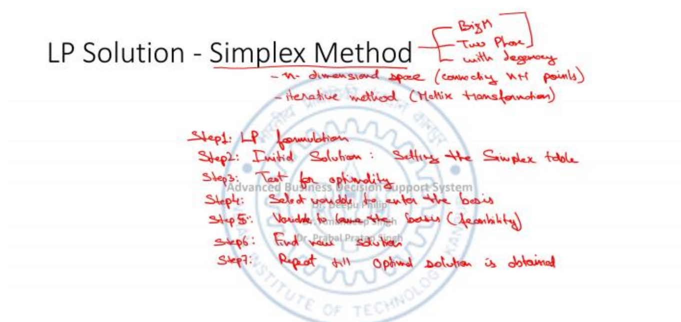
Advanced Business Decision Support Systems
Professor Deepu Philip
Department of Industrial Engineering and Management Engineering
Indian Institute of Technology, Kanpur
Professor Amandeep Singh
Imagineering Laboratory
Dr. Prabal Pratap Singh
Indian Institute of Technology, Kanpur
Lecture 19
Linear Programming Simplex Method

Welcome to the next lecture on Linear Programming. We have discussed the Problem Formulation, the Graphical method in the previous lectures. In this lecture, I will try to talk about Simplex method. As I said in the last lecture, Linear Programming statements do not essentially have only two variables. It can be more than two variables, a number of variables, a number of constraints is also sometimes an issue where dual tree theorem could be used to have transformations.



Simplex methods mathematically means, it is just n dimensional space which is connecting n + 1 points. This is just a mathematical definition of Simplex method. So, Simplex method is very similar to the Graphical method.

Only in the Graphical method, we go for the extreme points of the Feasible solution that we have seen in the last lecture to find the optimal solution. So, in Linear Programming in the LP problems, where several variables are there, it is not possible to graph for more than two variables.

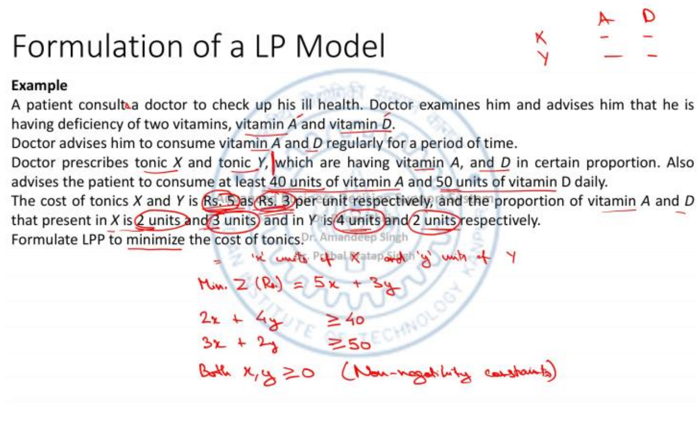
So, this method is used that is Simplex method. It is an Iterative method. Iterative means, we go to the iterations, we do iterations which are generally Matrix transformations. Additional to Matrix transformations, we also try to do small cell calculations as well in the Simplex method. The standard form of the Linear Programming is the most important part.

We need to have a objective function, which is to minimize or maximize then, we have the change coefficient and vary decision variables related to the capacity, that is the b j's, that we have on the right hand side and the constraints, that we have built and non negativity constraints are also part of it.

So, any kind of the problem whether it is less than equal to, more than equal to, exactly equal to or we have also sometimes,   degeneracy in the problems, that could be solved with the Simplex method and the extensive versions of it. Extensive versions could be, I will talk about big M or Two phase or with degeneracy. So, big M and Two phase are the problems when we need to add a sur+ variable. What is Sur+ and Slack variable? This we are just going to talk.

So, having a Linear Programming problem in a standard formulation, we can start this Simplex program by certain steps. Step one is, LP formulation that is, everywhere the step one. Step two is, initial solution, where the points, which I discussed in the last lecture, the Feasible solution, Basic solution, Basic Feasible solution, these are put into a table. When I say Initial solution, Initial solution is just setting the Simplex table. Then, we test for optimality that is, we try to see whether the solution is optimal or not initial solution is generally not the optimal solution.

That is, why we try to see the optimality whether the maximization or minimization is done or not.   Then, we try to have next step that is, we try to select variable to enter the basis. Now, this basis, I will show you in a table, then we try to see the feasibility or the variable to leave the basis that is feasibility. Then, we try to find new solution and we try to repeat the procedure, repeat till optimal solution is obtained.



To demonstrate the Simplex method, let me take an example directly. So, there is a problem statement given here in which, it is given a factory manufactures, two products, these products are A and B on three machines X, Y and Z.

Product A requires 10 hours machine X, 5 hours of machine Y and 1 hour of machine Z. It is directly given and requirement for product B is 6 hours, 10 hours and 2 hours of machine X, Y and Z. Contribution for A and B is given definitely. Now, we have two variables again here.

So, I will denote the small letters as my variables. So, A represents product A and B represents product B. We can see the contribution is rupees 23 and rupees 32 here. So, that means, we have to maximize the profit. So, done the objective functions becomes maximize Z = 23 times product A + 32 times product B.

This is subjected to the constraints. So, let we have three constraints which are corresponding to the machine X, Y and Z. So, for machine X, it is 10 hours of A + 6 hours, it is given here 6 hours of B should be less than equal to the given values. Maximum machine hours variable are 2500 for X, 2000 for Y and 500 for Z. X, Y and Z, so this is 2500.

Similarly, for the machine Y, we have 5 times of A from here + + 10 times of B is less than equal to 2000 and 1 A + 1 hour, 1 A + 2 B is less than equal to 500. So, both of them are A, B are greater than equal to 0. So, this is a problem formulation for the given problem statement, that is here. So, we need to maximize the profit.

LP Solution - Simplex Method

$$
\begin{aligned}
10a + 6b + 1S_1 + 0S_2 + 0S_3 &= 2500 \\
5a + 6b + 0S_1 + 1S_2 + 0S_3 &= 2000 \\
1a + 2b + 0S_1 + 0S_2 + 1S_3 &= 500
\end{aligned}
$$

LP Solution - Simplex Method

Objective function co-efficients

Basic variables values

Programme (Basic variables)	Profit per unit In Rs. C_b	Quantity in Units.	C_j 23 a	32 b	0 S_1	0 S_2	0 S_3
S_1	0	2500	10	6	1	0	0
S_2	0	2000	5	10	0	1	0
S_3	0	500	1	2	0	0	1
Z_j	Zero = C_b · quantity		0	0	0	0	0
$C_j - Z_j$ = Opportunity cost in Rs. Net evaluation row.			23	32	0	0	0

Outgoing Profit →

Opportunity cost →

When you need to maximize the profit, we try to use a term known as Slack variable. So, what is Slack variable? We will try to see it is an imaginary product, which requires specific hours of machine X, Y or Z alone. So, come to this point, let me try to see this table here. So, to start my table, the table is given here, the Simplex table we need to understand, what are the Slack variables S_1, S_2 and S_3. Let us try to see this. Now, here you can see the inequality expression that is, there is less than equal to that means, 10 times of unit A and 6 times of unit B that means, 10 hours for product A and 6 hours for product B are there for machine X, which has a capacity of 2500, this is less than equal to.

So, to solve this, we need to take off the inequality sign, we will put it as 10 A + 6 B + S_1 = 2500 that means, S_1 is a Slack variable, which when added, takes off this inequality sign. So, whatever the value is lesser than 2500 that is covered by S_1. Similarly, 5A + 20B + S_2 = 2000. Similarly, 1A + 2B + S_3 = 500. So, if I put this in this way, I can even say, the equations are in such a way, that we have 10A + 6B + $1S_1$ + $0S_2$ + $0S_3$ = 2500.

Similarly, 5A + 6B + $0S_1$ + $1S_2$ + $0S_3$ = 2000 and 1A + 2B + $0S_1$ + $0S_2$ + $1S_3$ = 500. I have put it in this way because I needed to get to this table in which, you can say, we have 1, 1, 1, this is a kind of an identity matrix, that I have here. So, these were my variables which are given decision variables. So, these are put here, the decision variables 10, 6 and we have an identity matrix here. So, each simplex table or the initial simplex table will always have the identity matrix.

So, this first column here is known as the Basic variables columns and this is known as Basis. This is known as basis, where the variables would keep on entering from the given table. Right now, the table is put in such a way, that Slack variables are completely contributing to the objective function and the objective function value here, ZJ value is 0 which is the product of the profit into quantity. So, this is equal to column CB into quantity. Second column that you could see this is, the Objective Function Coefficients.

The third column represent the Basic variables values. Which means, in this case, S_1 will attain the value 2500 and the column fourth and fifth are made decision variables that we can see directly. These columns 6th, 7th and 8th, 0 should also be here. These are my Slack variables. Regarding rows, these rows are my outgoing profit.

Because I am maximizing the profit, the outgoing profit here is 0, which is the product of the profit per unit into number of units directly. Then, the net evaluation row is also there, which represents the net evaluation under each column, this is also known as the Opportunity cost. So, the first column is Program column once again. In this, there are 3 basic variables S_1, S_2 and S_3. The second column is a Profit column.

The profit column represents the profit coefficient for basic variable which is, CB which is corresponding to S_1, S_2 and S_3. The profit corresponding to S_1, S_2 and S_3 in the problem statement that we can put here is 0, 0, 0. The third column is the values of basic variables in the program solution that is, these are the quantities or units currently being produced which are entered here corresponding to S_1, S_2 and S_3. So, in any program, this total profit is the product of the profit and quantity in any of the simplest Now, these are the variables S_1 and S_2 and S_3 which will leave the basis and the decision variables needs to enter it. So, there is a procedure to do that we will try to see.

So, we need to maximize the profit. So, how do we maximize the profit? Let us try to see. Now, outgoing profit is there, the important rho here is the net evaluation rho which is also known as the Opportunity cost. Opportunity cost means, what is the opportunity to improve our Simplex program. So, in this case, let me come to the table once again here.

LP Solution - Simplex Method

Max. $Z = 23a + 32b + 0S_1 + 0S_2 + 0S_3$

Key Element

Programme (Basic variables)	Profit per unit In Rs. C_b	Quantity in Units.	C_j 23 a	32 b	0 S_1	0 S_2	S_3	Replacement ratios
S_1 ✓	0	2500	10	6	1	0	0	2500/6 = 416.7
→ S_2 → b	0	2000	5	10	0	1	0	2000/10 = 200 ← Outgoing value (Limiting ratio)
S_3	0	500	1	2	0	0	1	500/2 = 250
Z_j	0×2500 + 0×2000 + 0×500 = 0		0	0	0	0	0	
$C_j - Z_j$ = Opportunity cost in Rs. Net evaluation row.			23	32	0	0	0	

2000/10 5/10 10/10 0/10 1/10 0/10

Row3 → 200 0.5 1 0 0.1 0] ×6

Row2 → 2500 − 6(200) = 1300; 10 − 6(0.5) = 7; 6 − 6(1) = 0; 1 − 6(0) = 1; 0 − 6(.1) = −.6; 0 − 6(0) = 0

LP Solution - Simplex Method

	Problem variable.	Profit in Rs.	Capacity	C_j 23 a	32 b	0 S_1	0 S_2	0 S_3	Replacement ratio
Row2 →	S_1	0	1,300	7	0	1	-0.6	0	1300/7 = 185.7 ←
Row3 →	b	32	200	0.5	1	0	0.10	0	200/0.5 = 400
Row4 →	S_3	0	100	0	0	0	-0.5	1	
	Z_j	32×200 = 6400		16	32	0	3.2	0	
→	$C_j - Z_j$	= net evaluation		7	0	0	- 3.2	0	

So, here this product that is 0 into 2500 + 0 into 2000 + 0 * 500 = 0. So, as of now, the objective function that is there, that has a value 0, which means the objective function, that could be written once again that is, we try to maximize G equal to $23A + 32B + 0S_1 + 0S_2 + 0S_3$ in which because A and B are not in the basis, the value is 0. So, we have the Opportunity cost here. It is the cost, what is the difference between the contribution here and the value of Z that we have obtained for the specific variable which means, there is an opportunity to change or improve our profit maximum.

So, opportunity cost means for not including a particular profitable variable in the program, the manufacturer has to lose the amount equivalent to the profit contribution variable that means, if manufacturer does not include this variable, it will lose 23 rupees per unit and if it does not enter this value, the manufacturer will lose 32 rupees per unit that means, the maximum opportunity available is 32.

So, the maximum positive value from the Opportunity cost rho, that is the net evaluation rho is taken as our key. Now, out of this key rho, we need to select the key element that is out of these three elements with values 6, 10 and 2, we need to select one, how do we select them? We try to calculate the replacement ratios. Replacement ratio means, we try to divide each of the quantities which are given by the corresponding element in the selected key rho with maximum opportunity cost that means, here 2500 by 6 which =

416.7 and this is 2000 by 10 = 200 and this is 500 by 2 = 250. Now, we have already selected the key column.

We need to see which of the elements would go away with or with enter the basic variables column. So, this gives me a ratio that, how much a basic variable could be manufactured while using or for having the B variable entered in the basis. You can see 416.7, 200 and 250 are the values which are corresponding to S_1, S_2 and S_3. So, which values to be selected here? We need to select a key value here.

The key value has to be the minimum value which is known as the Outgoing value. Outgoing value, which is a minimum value. Why this value is minimum? Because this number is limiting the ratio if we try to manufacture even 201 units, my this machine y would not allow. So, 200 is a minimum value, which is limiting my ratio. So, this is the limiting ratio outgoing value or I will also call it as Limiting ratio.

So, to again recall Opportunity cost, is there in the bottom row which tells the opportunity to increase the profit that is the maximum opportunity is to be taken into account. So, opportunity is 32 rupees per unit that is taken because if we do not take this opportunity, we lose rupees 32 per unit.

So, maximum positive value is taken from the row that is the net valuation row here. And, now we try to see the capacity using this row, what capacity could be manufactured, that capacity is taken by total capacity available that is 2500 hours are there, which is divided by the change coefficient of this specific decision variable, which turns out to 416.7, 200 and 250.

So, the minimum value here is 200. That is, this is the limited value maximum that, we can manufacture is 200 here. Other more than that cannot be manufactured because machine y limits that number. So, this is the outgoing value. So, we try to have an intersection of this row and this column, that is this number 10.

Now, this 10 turns to 1, so you can see wherever the Slack variable is there. The corresponding coefficient here is 1 and there are 2 other coefficients with value 0 and 0. That is why, it makes this as Identity matrix. So, similar analogy is to be applied to this. So, we need to divide this whole column by this value that is, the value of the key element.

I will write it as a key element that means, we try to divide 2000 by 10 which = 200, we try to divide 5 by 10 = 0.5. We try to divide 10/10 = 1, 0 by 10 is 0, 1 by 10 is 0.1 and 0 by 10 is 0.

This becomes my new row 3 here. That means, the row with the Slack variable S2, which was there. Now, this will become my decision variable B, which enters here. Now, corresponding elements in the column should be 0, 0 here, this 6 should turn to 0 which means, your row 2, let me come to this table. Now, you can see B has entered here and the corresponding profit 32, which was there. This has come here profit per unit and these are the values, which are taken by dividing the corresponding elements by the key element.

This was the key element 10 here. So, each of the values from the previous table are divided by 10. Now, from where does this value 1300 comes to 1300? Because this value stuff was to be turned to 0 in row 2 and in row 4. Also, the value this was to be turned to 0. These 2 values to get this value 0. How is it obtained? This 200 was there and we had here previously 2500, 10, 6, 1, 0, 0, 0.

So, row 2 how is obtained? We try to multiply this by 6 because this value is 6 here and subtract it from the existing row 2 that means, we get 2500 minus 6 times of 200, which is 1300 and here, we have 10 minus 6 times of 0.5 that means, 10 minus 3 = 7. The third one, we have 6 minus 6 times of 1 = 0 then, we have 1 minus 6 times 0 which = 1 only.

Then, we have here 0 minus 6 times into 0.1 = -0.6 and for the last value, we have 0 minus 6 times of 0, which = 0. So, this value 1300, 7, 0, 1 minus 0.6 and 0 is entered 1300, 7, 0, 1 minus 0.6 and 0 similar iteration is taken for the row 4, this is my row 2, this is row 3 and this is row 4 and here my maximization problem that is, the total value of Z turns to 0 into 1300 + 3200 * 200 + 0 * 100.

This is 32 into 200 = 6400. So, 6 rupees 6400 is the profit, that is given for this problem statement. In the present scenario, that is when we have the tabular or the first iteration is completed. Now, we need to further see whether we do have further opportunity or not. So, in the next Opportunity cost row, we can see, still we have the number 7 here, which is a positive value. That is still we have an opportunity, that is rupees 7 per unit of A could be increased in the profit.

So, this becomes my key column now. Now, to find the replacement ratio, these are all similar iteration that I did in the previous tabular. So, this becomes 1300 by the corresponding change coefficient of the decision variable, 1300 by 7 is equal to 185.7 and 200 by 0.5 = 400.

So, here the limiting case is 185.7, that is maximum capacity that I have is 185.7, which could be 185 or so is the maximum that I could use the number of hours from the machine X. That means, column with decision variable A and row 2 is selected with Slack variable S_1. So, 7 becomes my key element or the next variable to enter the basis is A. So, we do the similar iteration and we get to this table here.

LP Solution - Simplex Method

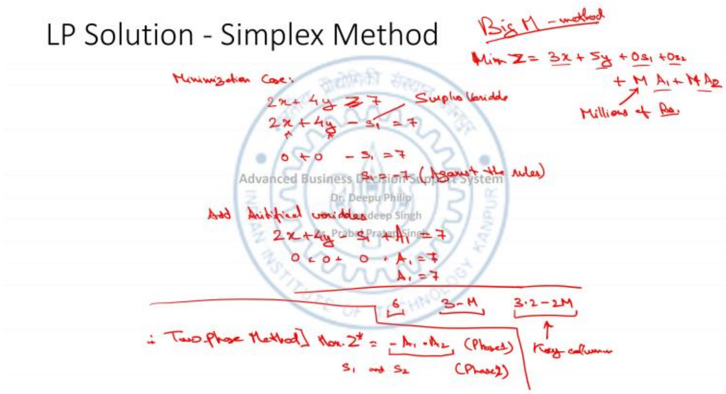
Problem variable	Profit in Rs.	Capacity	C_j 23 a	32 b	0 S_1	0 S_2	0 S_3	Replacement ratio
a	23	185.7	1	0	0.143	−0.086	0	
b	32	107.14	0	1	−0.07	0.143	0	
S_3	0	100	0	0	0	−0.02	1	
Z_j	23 x 185.7 + 32 x 107.14 = Rs.7,700		23	32	1	2.6	0	
$C_j - Z_j$	Net evaluation.		0	0	−1.0	−2.6	0	

a = 185.7, b = 107.14, Max. Z = Rs 7,700

Now, in this, table A is entered in the basis, B is entered in the basis, corresponding profit per unit is given and corresponding calculation of the value of Z could be taken that is, 23 into 185.7 + 32 * 107.14. This value is equal to rupees 7700 which is larger than the previous value 6400. Now, do we still have further chance of improvement? That could be again taken from the Opportunity cost row in which Cj minus Zj is given, you can see all the values are 0 or negative.

So, this becomes our key criteria. For a maximization problem, these values should be 0 or negative and that means, there is no opportunity left to increase the profit further. So, my solution for this problem statement becomes A = 185.7 and B = 107.14 maximizing my profit Z, which is equal to rupees 7700. So, this becomes my Simplex method given the number of variables, the number of variables could be more as the number of variables increase the slack values would also be added accordingly.

As I said, the number of constraints has to be larger than what number of decision variables, that we have, here we had two decision variables and we had three constraints. So, in this case, my basis is still having S_3, which is having no value here. Now, A and B are my basis which are also included. The number of variables should be lesser than the number of constraints or equal to the number of constraints to solve any multi-equation problems. There could be multiple cases here in the Simplex method for example, the case is minimization.



Now, in the case of minimization, what happens because it is minimizing Z and the constraints if are given in greater than equal to form. For example, let me consider a constraint which is given as $2x + 4y \geq 7$. So, we cannot add a slack variable, we need to subtract something from it. That is subtraction, that is taken from it is known as Sur+ variable. This becomes $2x + 4y - s_1$. S

o, this becomes my Sur+ variable here. So, we have to subtract a Sur+ variable. Whenever we subtract a sur+ variable and suppose, if I allocate value 0, 0 to x and y x_0, y_0 this becomes 0 + 0 minus s_1 = 7, which turns out that s_1 value becomes -7. This is against the rules of the Simplex model.

So, to cover up this, what we need to do, we add Artificial variables. So, this s_1 is my Sur+ variable, Artificial variable means, we try to make the equations in such a way, that it becomes 2x + 4y minus $s_1 + A_1 = 7$ and this A1 can have value maybe, if everything else is 0 here, this is 0. So, A_1 can attain value 7. That means, it can have a positive value which is okay right.

Now, this A1 has otherwise no physical significance, so to include this A_1 in the minimization case, in the Objective function, Now, my objective function here becomes minimize Z = 3x + 5y + 0 sur+ variable 1 + 0 sur+ variable 2 + A1 and A2 * M.

So, this method is known as Big M method. I am not going to resolve this method completely here, going to the time limitations, this M means millions of rupees or dollars is a penalty. If M is there in the Objective function, penalty if A_1 or A_2, any of these are there in the Objective function that means, A1 and A2 cannot exist in the Objective function, so the penalty would be given. So, using these 1, 2, 3, 4, 5, 6 variables, an initial simplex matrix is generated. So, here because it is a minimization problem, the opportunity has to be a negative value. So, maximum negative value in the net evaluation row is taken and maximum negative value means, wherever M has a negative sign corresponding to it, that will be taken.

For example, in the Net Evaluation matrix, something like this is there- 3 - M is 1 second, here it could be something like 6 third value is 3.2 - 2M. So, what is the most negative value here? Most negative value is, where minus 2M is there. So, this becomes my key column. So, I put all the Arbitrary values here just to make you understand that, how these matrix were, you can work on this, you can refer to the books given in the reference section of this.

The books that I have referred majorly for this lecture are, Operation Research by Wayne Winston, Operation Research by JK. Sharma, Operation research by Ramamurthy. The maximum of these problems are taken from those only. Other than big M method, there is another method known as Two phase method.

When I say Two phase, there are two phases to solve the minimization problem. The phases are solved in such a way that initially, we try to only solve the maximization problem for A1 and A2. So, here what we try to do? We only add the Artificial variables and first try to solve a maximization problem, we call it maximize Z = minus A_1 and minus A_2.

We try to solve this matrix first and try to get a solution where, A_1 and A_2 both have attained negative values in the decision matrix and in end of phase one, A_1 and A_2 are eliminated. Eliminated means, these are not included in the solution and without A_1 and A_2, only with S_1 and S_2 in phase two it is taken. So, this is phase one, in phase two, this is a very crude introduction to the two phase method and the Big M method, we will try to provide notes in the description section of the videos that you try to watch here.

I will also like to solve the Simplex method using the MS Excel program as I said, let us move to the MS Excel program. We will try to see, how to solve the Simplex program or Simplex model using MS Excel solver. Thank you.