Advanced Business Decision Support Systems
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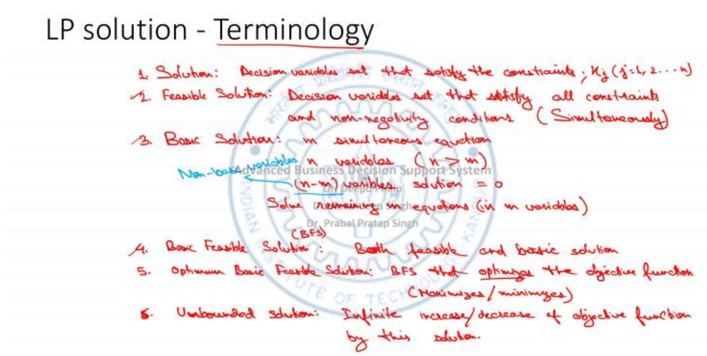
Lecture 17
Linear Programming Graphical Method

Welcome back to the course on Advanced Business Decision Support Systems. We are discussing Deterministic Modeling in this week. I have discussed about the Linear Programming, I have discussed about general classification of the modeling and Linear Programming falls into the Deterministic Modeling, where data is certain, where linearity is there, where we try to see that the contributions of each or the coefficients, of each of the variables, that we are taken should be additive.

And, contribution per unit for each of the individual variables that we are taken should be equal to the sum of the overall contribution that it is taken, divisibility is also one of the assumptions. Let us try to see the solution to the Linear Programming problem statements. A few statements we have discussed in the last lecture. I will try to now talk about the Graphical Solution to the Linear Programming.



Now, certain terminologies to be given before we talk about the solutions. Terminology when I say, we know what are decision variables, we know what are the objective functions, we have donated them with the specific letters xij, aij, cj, bi, all these limitations were given. Now, when I talk about solution, there are certain terms I will keep on using here.

I will talk about solution to the problem, I will talk about basic solution, I will talk about feasible solution, the solution that is feasible, then basic feasible solution. What are these terms? Let us try to see those. Number one is Solution. Simply it looks like the problem is completely solved. So, the solution is set of the decision variables that satisfy the constraints.

That is when we are trying to have a solution to the objective functions subjected to the constraints that we have mentioned. Any set, that set can assume value xj where, j varies from 1 up to n. So, whenever they try to satisfy the constraints, this is just known as Solution. Now let us be more specific. What is Feasible solution? Feasible solution means, any set of value of decision variables, that satisfy the constraints and the non-negativity conditions as well.

The non-negative conditions were given in the linear programming formulation that I mentioned. So, that means, the decision variables set that satisfy all constraints and non-negativity conditions and these two are done simultaneously. This is known as Feasible solution. So, in the whole range of Feasible solutions there could be some solution that could be I will come to the word optimal. Optimal solution will come later.

Let us now try to first see feasible solution is there. What is Basic solution? Basic solution, from where the things start. Now, Basic solution. Basic solution means, if there are m simultaneous equations and we have n variables. Here, number of variables is greater than the number of equations.

So, for this kind of the set, the n - m variables equal to 0, the solution obtained by n - m variables, which is equal to 0 and we solve remaining m equations, which is m equations in m variables. Now, this is called the Basic solution. Now, n - m variables whose value was not taken into account in the basic solution are known as Non-Basic variables. So this n - m becomes non-basic variables. Now, this is called the Basic solution.

Now, the basic solution is known as Non-Basic variables. For instance, if we have 6 variables and we are having only 4 constraints. 4 constraints cannot help us to solve the overall problems scenario where 6 variables are there. At most, 4 variables could be solved. That means, n - m variables, the 2 variables which were not solved are known as Non-Basic variables and the Basic solution is that is taken from solving the 4 variables only.

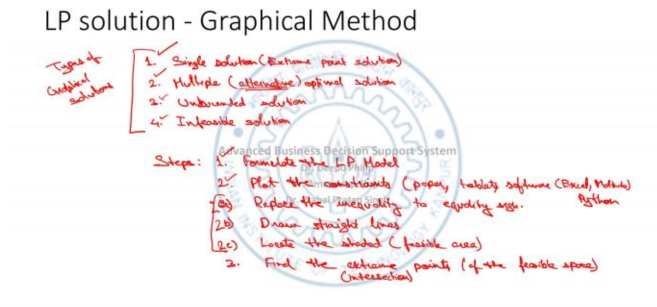
The 2 variables are not taken into account. So, this we will be more clear with when we come up with the solution of the longer problem statement using the simplex method. So, when we are clear with Basic solution, let us try to combine Basic and Feasible solution which is known as Basic Feasible solution. Very common term, when we are going to solve using the simplex method. Basic Feasible solution means, the solution which is Basic solution and also a Feasible solution.

I would say, both Feasible and Basic solution. That means, it satisfies the constraints as well and it also have the solution based upon the m variables which are taken into account for solving the problem. So there are certain types. It could be degenerate, it could be non-degenerate. So, generally we hope to have the Basic Feasible solution, which is non-degenerate, that is m Basic variables have non-zero or positive values that is, we target to get in the solution when we try to get through the simplex method.

Now, finally what we are targeting to is Optimum Basic Feasible solution. Which means, just Basic Feasible solution that optimizes. So, this is, I would say, for Basic Feasible solution I will put it as 'BFS'. So I will put BFS that optimizes, optimizes what? The objective function that is known as Optimum Feasible solution. This optimization could be again it maximizes or minimizes depending upon the condition that is given.

Then, there could be a condition when the solution is unbounded that we will just see when we try to have a solution using the Graphical method as well Unbounded solution. Which means, the solution can increase or decrease infinitely the value objective function and there is no bound over it. For instance, if in the graph, if the area is given above or away from 0, it can go to infinite values. So, it is I would say, infinite increase or decrease of objective function by this solution. So, these are few terms, I will keep on using the term BFS that is Basic Feasible solution.

Solution in General, Feasible solution, Basic solution, Unbounded solution would just maybe come one or more times, when we talk about or discuss about the unbounded solution only.



Let us now try to see what is graphical method? Graphical method could be used when we are trying to solve for two variables either two basic variables at least. Because graph is a two dimensional figure, where two variables could be plotted. So, when we try to solve using Graphical methods, there are certain steps associated with it and there are kind of the conditions or the solutions that we are trying to get through it as well. Number one is the kind of the solutions that we get is single solution.

Single solution or we might also call it when we try to plot the graph, we call to say the name for it as Extreme Point solution. Then, we will say, that the solution is not other than the single solution. The objective function whether it is maximized or minimized the same value that is the optimal value that we have taken from the solution, could be obtained for more than one points. In this case, it is known as the Multiple or Alternative optimal solution. We will talk about the unbounded solution definitely in the graphical method in which, the maximization or the minimization is unbounded that is, it can take infinitely high or infinitely low values with no limits.

And, also, Infeasible solutions could also be seen on the graph. Types of graphical solutions. So, when So, when we try to plot a graph, there are certain steps like, the for the Problem Formulation, we first develop or formulate the LP model. And, try to put it in a legitimate manner, so that the equations are very clear and the objective function is very clear and two variables are there. Now, we try to formulate the LP model, then we plot the constraints.

So, we will try to formulate the Linear Programming that means, we try to plot the constraint on a paper or may be using any of the softwares, excel could also be used. I would say paper, then you use the tablet or may be the softwares, it could be Excel, Matlab or any that you would like to pick Python, that we will discuss in this course itself as a coding language. Now, plotting the constraint itself has certain steps that is, we first replace the inequality to equality sign and then we will try to write the equation.

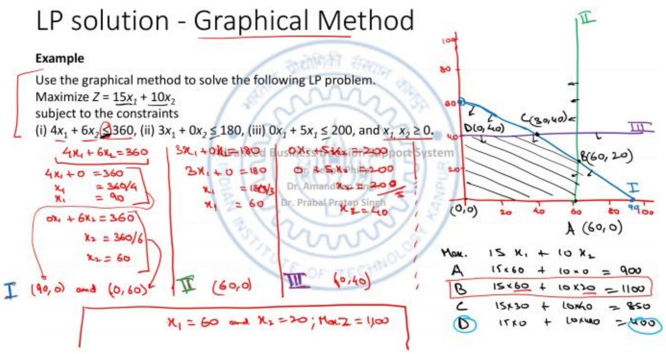
Then, we try to draw straight lines, choosing this equality sign how to draw straight lines in the constraint itself x_1, x_2 value would be given we try to put 0 value for x_1 and try to calculate value of x_2 that gives me the y coordinate. Then, we try to put 0 value for x_2 and try to calculate value for x_1 that give me the x coordinate for this straight line would be given.

So, we draw straight lines, this is we are just trying to plot the constraints and finally, we try to find the shaded area, we try to locate the shaded or I would say Feasible area. Now, here what happens is this inequality sign, which was given, based upon the inequality sign, we try to say, whether it reaches towards 0 or goes away from 0. If it is less than equal to 0 that means, it is towards 0. If it is greater than equal to 0, it goes away from 0. So, then we locate the shaded area, that is we get the Feasible area.

Now, third point, because we need to identify the single solution at least, multiple solutions could come as an Alternative, Unbounded and Infeasible solutions are not even helpful if those come. So, we need to find the Extreme points or I will also call it as Intersection points. These are taken of the Feasible area or Feasible space. We determine the coordinates of each Extreme point of the Feasible solution. Compute and compare the value of each objective function, whichever extreme points give me the maximum value or the minimum value is my final optimal solution.

Your word Solution sometimes, when I say, word Solution it is Optimal Solution I am trying to mean, Optimal solution is gotten more than one Optimal solutions could also be gotten that comes the case too, when we have Multiple or Alternative Optimal solutions. Now, let us try to see an example to draw the graph for that and try to see how the Graphical method is used. Now, this Linear Programming solution using Graphical method is to be taken for this problem statement, that is given, is not even the problem statement. We have completely formulated the problem. So, the first step is to Formulate the Linear Programming model that is already discussed in the last lecture that is taken care of.

Next point is, plot the constraints in which, first step is replace the inequality to equality sign, draw straight lines, locate the shaded area. Let us try to work on point 2a, 2b and 2c are given here.



These inequalities are given, where it is given $4x + 6x_2 \leq 660$. Less than equal to is very important to identify which side of the line we need to mark as a shaded area or the feasible area. Now, let me try to convert this into the equality sign.

Number 1 is $3x_1 + 0x_2 = 180$, number 3 equation is $0x_1 + 5x2$ is equal to 200 and these are the non negativity constraints. Now, to solve the first constraint ,second constraint and third constraint, I will try to put value of x_2 as 0. $4x_1 + 0 = 360$ which means, x_1 = 360 by 4 and $x_1 = 90$. Similarly $0x_1 + 6x_2 = 360$ which means, x_2 = 360 by 4 and $x_1 = 90$. Similarly $0x_1 + 6x_2 = 360$ which means, x_2 = 360 by 6, which means, $x_2 = 60$.

So, my equation 1 gives me 2 points to draw the line which are x_1 90 and x_2 0 and x_1 0 and x_2 0 and x_2 60, this is taken from here and this is taken from here. In these two already, $0x_2$ is given. So, that means we have $3x_1 + 0 = 180$ which means, x_1 = 180 by 3 which means, $x_1 = 60$. So, if I put this as 0, it does not mean anything. So, only x_1 60 is another coordinate that we have got.

For equation 2, we have a line that says $x_1 = 60$. 60 and 0 is the point that we have got. For the third one, we have $0x_1 + 5x_2 = 200$ which means, $0 + 5x_2 = 200$, x_2 = 200 by 5 where, $x_2 = 40$ and for the third constraint, we get this value as 0, 40. So, we have got the coordinates to plot the lines.

Now, the values are varying from 0 to 90. So, I can maybe have a scale up to 100 in x axis and maybe up to again 100 or maybe 60 or more than 60 up to 80 in y axis. So, let me try to plot it here. Let me say, x_1 = 20, 40, 60, 80, 100 with almost similar sizes 20, 40, 60, 80 and 100. So, this is the first line 90, 0 and 0, 60 when y is 0, x is 90. Here, this is the point 90 and when x is 0, y is 60, this is the second point, this is my constraint 1.

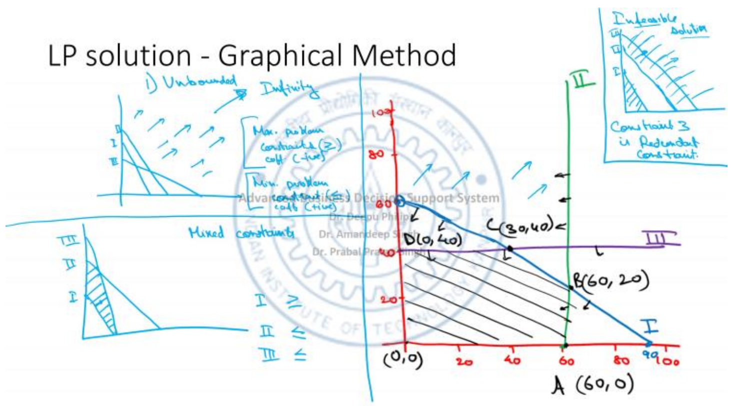
I will color the constraint 1 blue. If I join this line, this is a straight line which is representing my constraint 1. For the constraint 2, it is giving y_0 and x_{60}, x is 60 here and y_0. So, this is just a straight line, where y is 0. That means, I go straight high and this is my second constraint.

For the third constraint, we have x_0 and y_{40}. So, this is the second constraint. This is my constraint number 3, I have constraint number 2 here and 1 touching both the axis is my constraint number 1. Now, let me try to shade the Feasible area now. Now,. the feasible area, you can see this sign for constraint 1 is less than equal to 360. Second sign is also less than equal to, third sign is also less than equal to. As I mentioned in the beginning, this is my origin, when the sign is less than equal to the final b_i values, which are given. So, we go towards 0 that is, towards origin, which is greater than equal to, we go away from origin. So, we go towards this direction from here and also from towards the region from here and towards the region from here as well. So, this is my shaded area which means, I have got solution points which are the extreme point of this area.

So, 0, 0 is the origin value, as one of the extreme points which I put it in the objective function, maximise z which turns to 0, which is also one of the values, that we need to put because we need to pick all the extreme points. This is one point. Second point is this point, where value of x is 60 and y is 0. Let me call this as point A. Now, there is one point here where, the value of x is 60 and value of y is 20.

I will call it as point B. There is another extreme point to the Feasible solution, that I have got. Now, this is my Feasible solution here the shaded area. Now, here this point is almost, if I calculate it numerically because I am drawing through pen, small misalignment could be there, but this point is 30 and 40 and the point D is x_0 and x_0 is x_0. So, this is the point and y_{40}. Now, these points have been obtained and we can now try to put all these values in my objective function which is, maximize $15x_1 + 10x_2$.

So, let me try it for the value of A, B, C and D. For A, it is 15 times 60 + 10 times 0 = 15* 6 are 900. The extreme point B which is 15 times 60 +, 10 times 20 which is equal to 900 + 200, which is 1100. For the point C, 15 times 30 + 10 times 40. So, 15 * 30 = 450 and 10 into 40 = 400, 450 + 400 = 850. The point number D is 15 times 0 + 10 times 40 = 400.Now, out of these values, we have seen that, the maximum value is taken from the point B, which is 60 and 20 for the x1 and x2 values which means, the maximum value of Z is, we get value of x1. This is the solution as 60 and x2 as 20 to get maximum Z is equal to 1100. There are no units given here, it could be maximizing the profit, it could be maximizing the production or anything. So, this is a general single solution using the graphical method Now, let us only try to discuss the other methods or other solutions that have given which are multiple solutions, unbounded solutions, and infeasible solutions.



Now, this graph is reproduce here just to understand that different kind of solutions that have given in the one of the previous slides.

Suppose if this is a minimization problem. In the case of the minimization problem, if suppose graph is something like this, I have a line like this as one of the equations, then there is a second line, then there might be a third line, something like this. Depending upon the constraint that is given, the solution that is away from the origin is marked. So, for this line, let me call it line 1, line 2 and line 3, for constraints 1, 2, 3, the solution is marked away. So, it is greater than or equal to something, so this area is marked. So, in this case what happens, since the solution is marked away, the solution becomes unbounded.

It is going towards infinity, so unbounded solution. Second point is suppose at certain points, point B and C gives the same value, something like that, maybe B and C, could have given values 1100 both, the equation could have been formed in such a way, that would be called as the Multiple or Alternative solutions. So, constraints could also be mixed, like in this case, as I have drawn, a few constraints could be greater than equal to, a few could be less than or equal to.

So, then what we get, a solution something like this, if this constraint 1, 2 and 3 are given, constraint 1 is greater than equal to, constraint 2 is less than equal to, constraint 3 is less than equal to, then constraint 1 will give me the area towards this direction. That means. away from the origin, and constraint 2 or 3 are giving me the area towards the origin, then this area would be shaded.

So, these are known as Mixed constraints. Now, comes the Unbounded solution as it is mentioned here, generally Unbounded solution, a very general tips for this to identify is broadly. For the maximization problem, the constraints would have greater than equal to sign in general, or for the minimization problem, the constraints would have less than equal to sign. Or maybe negative coefficient in the maximization problem, the coefficients are negative, or in the minimization problem, the coefficients are positive. Even given all of these, still we have to find the solution to confirm whether the solution is unbounded or not. These are general tips which generally tell us okay, this could be the Unbounded solution, it is not for sure. These are the few conditions.

Next comes the Infeasible solution. Infeasible solution means, when the solution is not even there, we have plotted everything, the constraints are not even trying to solve anything. In those case, what happen, it is something like this. I will try to draw Infeasible solution here. We have our axes here, and there is one constraint here, and there is another constraint here, 1 and 2.

The constraint 1 is less than equal to, that means, this become the Feasible area. Constraint 2 has greater than equal to, that means this become the Feasible area. So in this case, the solution becomes Infeasible. Even, this is one more term, that I could put here, Redundant constraints. First is Infeasible solution.

Now, suppose, there is another constraint 3 which is here, which is also greater than equal to. This constraint 3 is not even playing any role because constraints 1 and 2 are covering these things. So, that means, constraint 2 is also telling this direction or so, constraint 3 here becomes redundant. These are the few examples that to make you understand what is Graphical method, and we have tried to solve a very simple Graphical method problem statement. I have another graphical method statement here, which also have 3 constraints.

It is $5x1 + x2 \geq 10$, $x1 + x2 \geq 6$, $x1 + 4x2 \geq 12$. This is a minimization Graphical method constraint. Try to solve this by yourself. I will try to solve this problem statement in the excel demonstration. So, please before coming to the excel demonstration, try to plot this on your papers and try to get the solution, so that, you understand the demonstration given in Microsoft Excel better. Thank you.