Data Analysis and Decision Making - II
Prof. Raghu Nandan Sengupta
Department of Industrial & Management Engineering
Indian Institute of Technology, Kanpur

Lecture - 23
AHP

Welcome back my dear friends and students, a very good morning, good afternoon, good evening to all of you. And this is the DADM, which is Data Analysis and Decision Making-2 course, under NPTEL MOOC series. And as you know this course is for total spread over 12 weeks, which is total number of lectures would be 16 total. And each week we have 5 lectures, and each lecture being for half an hour. So, after each week, we have assignments. So, now we are in the 5th week.

As you can see this is the 23rd lecture which is the third one in the 5th week. And my name is Raghu Nandan Sengupta from the IME department, IIT Kanpur in India. So, if you remember we were discussing about the way how you formulate or understand the AHP, which is Analytical Hierarchy Process. And initially, I had given an example how Ram and Sham can take independent decisions, whether to join A or B or C which is IIMA, IIMB or IIMC, depending on placement potential and academic reputation.

And then we need when you combine the overall decision making criteria's for the parents, obviously the whole situation may change where we would like to basically have a combined best decision with respect to all the decision makers, where local optima local based solutions may not be allowed. Considering that Ram wants to maximize his placement potential idea as well as academic rigor idea also, similarly for Sham.

Then I said that another example can be very good example can be for to buy a house, I give examples, how you can consider all those criteria's both subjective and objective. Then I came to the example of buying a car and for buying a car, you had the criteria's as fuel efficiency, then safety and all these things were there, and we had few cars for that.

Now, each time I have been discussing that you will basically compare all the criteria's amongst themselves and rank them and give scores. And this if you remember in the last lecture weeks consider the scores being one, where you give a score of one is to one if

both the criteria's are the same intensity level or the difference between them is also on minimal, and you rank them at the same level.

Then the next level of score was 3 is to 1 is to third one-third, then next was 5 is to one-fifth, 7 is to one-seventh, 9 is to one-ninth. Now, these values of 9 and its reciprocal which is 1 by 9 or any value x and is reciprocal 1 by x would denote that the value of x you gave to that alternative or that is decision provided you are comparing the alternatives or comparing the decisions rest whatever the case may be. And you will give that high score to that decision alternative, where you want to put your decision, want to put your bet or you like that decision or like that alternative.

So, it is basically like this consider. If I give a score of 9 for A, and one-ninth for B for B, it will mean that I want to take A with a level of propensity of positive propensity of 9. And if I am forced to take the decision B, then my level of propensity or my liking of taking the decision is one-ninth. I have to take the decision somehow for some reason, then my overall so called benefit would be one-ninth that means, the level of benefit would be very minimal.

Now, you may be thinking that why does one have to take a decision, which may not give the best benefit. Say for example, you are a government bank or you are a government working in a government company, and that company or a bank has to open its branch in some very remote rural area in India. So, obviously the bank's main motivation is profit, if you consider it in a in a very realistic sense.

But, in the case for public undertaking bank, profit motivation may not be the main factor, because it may be that you have to serve the people in those rural districts people are in a position to get a loan, education loan, car loan, housing loan or you have you are able to give them lockers or they are able to open their fixed deposits or recurring deposits, whatever you are giving some services. So, when it comes to the service for the nation, obviously then profit motive may not be important. So, you have to basically analyze that problem accordingly.

So, the values of x and 1 by x would give you the propensity of liking, and not that liking for the decision of the alternate if you are taking. Now, for the case which were discussing, if you remember there were three alternatives, and similarly four alternatives and similarly, there would be three criteria's. And you would rank the combination of the

alternatives, and the combination of the criteria's in order to basically make a ranking, and then combine them. So, this type of ranking would be done for person 1, person 2, person 3 that means, person 1, person 2, person 3, and the decision makers for the buy buying that car, and you will combine them later on.

So, first I will only go through one person decision, how it is made. Now, the process of solving it is that you will consider the equation of if you remember when we solve mathematical problems, the concept of eigenvalues and eigenvectors come. So, Eigen values, Eigen vectors would basically give you the level of importance for each and every criteria, you will try to assign such that some sort of independence concept can be build up.

So, say for example you have if you have three vectors, and if they are not orthogonal to each other, you will try to basically take them their component like f cos theta, and sin theta in the directions of that particular plane, where you will break them into orthogonal such that the if you add up the orthogonals, then the effect of to the cumulative force in one direction would not have any effect on the on the orthogonal direction. So, what you are trying to do is that trying to break up this a level of importance of these criteria's or alternatives in an ascending orders or a descending order in whichever you look such that you are able to find assigned scores on them, when you normalize them that normalization factor will I will come to that later, if you remember I had mentioned that a few times in the last lecture.

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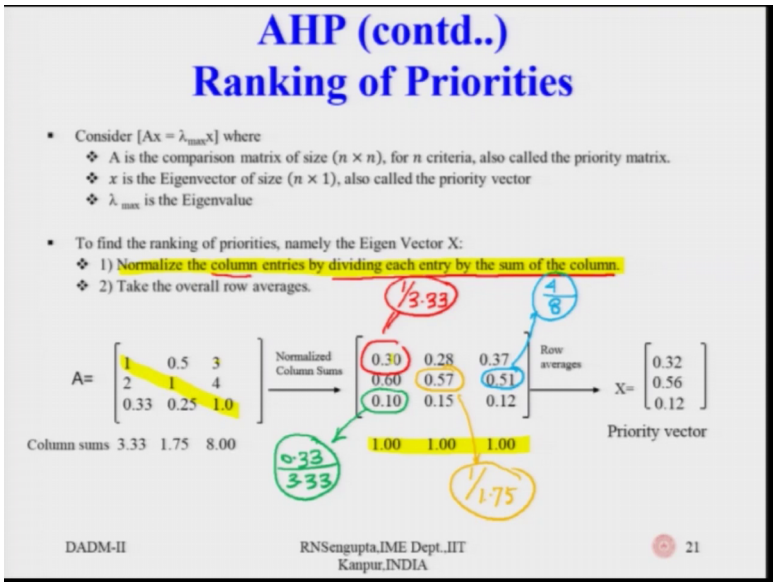

So, you will consider Ax, what is A, and what is x, I am going to come to that with this problem. So, I will just discuss the algorithm, and then come back to the problem. So, we will consider Ax is equal to lambda max into x, where lambda is basically the Eigen values and the vectors which you have, where A is the comparison matrix of size n cross n.

So, that if you remember when we are compared the alternatives or the decisions or the criteria's, you had basically n cross n corresponding to the fact that the principal diagonal is one, and the of the diagonal element, obviously it would not be a mirror image of each other. But, if you give a score say for example, at one for A when compared to B, if you give a score of say for example 3, you will give a score of one-third, when B is compared to A. So, as I had just discussed in the last class.

So, A is the comparison matrix of size n cross n, for n criteria, n decision, whatever it is and it is called basically call the priority matrix that means, what is the priority values, you are giving for each and every criteria or each and every decisions. x would basically be the Eigen vectors of size n cross 1, corresponding to the matrix A which you have, and it is basically called the priority vector based on which you are able to prioritize the values of each and every criteria in the orthogonal direction. And lambda max would be the eigenvalues, which will have based on which you will try to find out the scores for each and every decisions on the criteria's.

So, in order to normalize again I am saying, I am just repeating the steps of the algorithm so for solving how you solve it, I will basically consider that to a simple problem. To find the ranking of the priorities, namely the eigenvectors X which will be of size n cross 1, because n being the number of decisions. You normalize the column entries by dividing each entry by the sum of the column. So, this is very important, I will try to basically highlight it. And discuss few things verbally if required, I will come back to this problem solution in the next class, which would be in the 24th lecture.

So, you normalize the column entries by dividing each entry by the sum of the column. So, I will basically highlight this, and also highlight the word column, and dividing each entry with the sum of the column I will highlight that. Now, when you are trying to normalize, what you are trying to do is that you are trying to basically put a relative weight such that for a score of say for example 0.34 or a score of 0.43. We can easily say that which criteria which decision would be higher rank with respect to the other criteria, so that normalization score will give us.

Now, the point is mentioned that you normalize the column entries, it need not be true, you can also normalize the row entries also because, if you remember it is basically n cross n matrix, which you have which is the square matrix 0.1. Point number 2, you are you it says that you are dividing each entry by the sum of the columns.

Now, the fact is that you can also use different type of normalizations scheme such that the normalization scheme will give you whatever utility function, you are trying to utilize. So, say for example if you are using the quadratic utility function which we have discussed in the initial and few lectures, then you will remember that I did mention that the quadratic utility function, and the normal normality of returns have a one to one correspondence.

So, if I think that normality of returns is true, then obviously we will consider the quadratic utility function to be true. And try to utilize the normalization based on the fact that you are using the quadratic utility function. Quadratic utility function is basically of that form a x squared plus b x plus c, where x is basically the wealth. In case it is a logarithmic utility function or an exponential utility function or an power utility function, you will use that utility function to normalize the rows or the columns accordingly.

So, the fact which you are going to use now, the normalization of the columns column entries by dividing each element by the sum of the columns, we will follow that for this problem. But, the problem overall flavor can be changed, if you can basically use a different normalization technique correspondingly point 1.

Point number 2, obviously you may be quite inquisitive too and may be tempted to ask that what happens if there are three different persons, who have three different types of utility function. So, if that is the case, so in case for this case when you are buying a car, the person 1, person 2, person 3, which I mentioned would have basically different type of priority matrix. And when they use the priority matrix for all the criterias or all the decisions, they would basically try to rank them accordingly different giving different scores point 1.

And point number 2, when you normalize whether you do it through the column or whether you do it through the rows, the normalization would be done according to the utility function. And when you combine them, obviously the answer which will get will depend on the combined utility function, which you have for all these 3 persons. But, I will only stick to the case, when the normalization is done based on a utility function which is same for three different persons, whether you do it in the row or the column does not matter. But, we will try to stick to the simplistic assumption that the utility function is same for all the decision makers, who are taking the decision. So, this is a simplistic assumption, I know that what we will follow that.

You will take the overall row averages in case if it is a column or normalization which you are doing that means, normalize along the rows, obviously then you will take the average along the columns. So, what you do is that initially, so the priority matrix was this 1 1 1 is the principal diagonal, when you comparing; so, criteria 1 with criteria 1, criteria 2 2 with criteria 2, criteria 3 with criteria 3.

And when you basically compare the values of criteria 1 to criteria 2, you will either have a value of two or in other case I am taking this even numbers also in this case, you would have a value of 2, and other side it would be 1 by 2 which is 0.5. In one case, when you compare criteria 1 with criteria 3, you give a score of 3. And when you go to the reverse direction, it will be basically a score of one-third. And when you give a score, when comparing c 2 to c 3, you go give a score of 4.

So, hence when you reverse the decision of c 3 to c 2, the value would be one-fourth which is 0.25. When you normalize the sums along this column, so if we add up this is 8, 1.75, 3.3, so when you divide, it will be 1 by 3.3. 2 by 3.3, you are doing an and point 3 by 3.3, so the values which you which comes out, when you normalize them is basically I am only reading the values. So, let me make it clear.

So, this value would be equal to if I consider this value 3.33. If I consider this value, this should be half, so 0.51 if one should not be there. If I consider this value, so once you basically have the normalized values, check what is important is check the sum is 1, so 0.37, 0.51, 0.12, so obviously it would be 1. Then here 0.28, 0.57, 0.15 the value is 1, 0.36, 0.30, 0.6, 0.1 is 1.

In case if you have normalized along the rows, then obviously the column averages would have been formed. So, again if you find out the averages row averages, it will be the row first element 0.3 plus 0.28 plus 0.37 divided by 3. Next will be 0.6, 0.57, 0.51 divided by 3. Next value is 0.1 plus 0.15 plus 1.12 divided by 3. So, the values which comes out are basically this is very easy to find out, because the difference between 0.12 and 0.15 is plus 3. And difference between 0.12, and 0.10 minus 2, so obviously the value would be almost equal to 0.12.

So, if you find out the priority vectors, the scores are corresponding to the rows are I am only read the first two decimals 0.32, 0.56, and 0.12. So, you can at least understand what is the relative overly importance, you are slowly trying to assign, when you are taking these criteria's into consideration. Another point which I am repeating, so please bear with me. This course which we had given which is the priority matrix A, obviously will be different for person 2. So, once you have the priority vectors, the values would definitely be different. So, keep that in mind.

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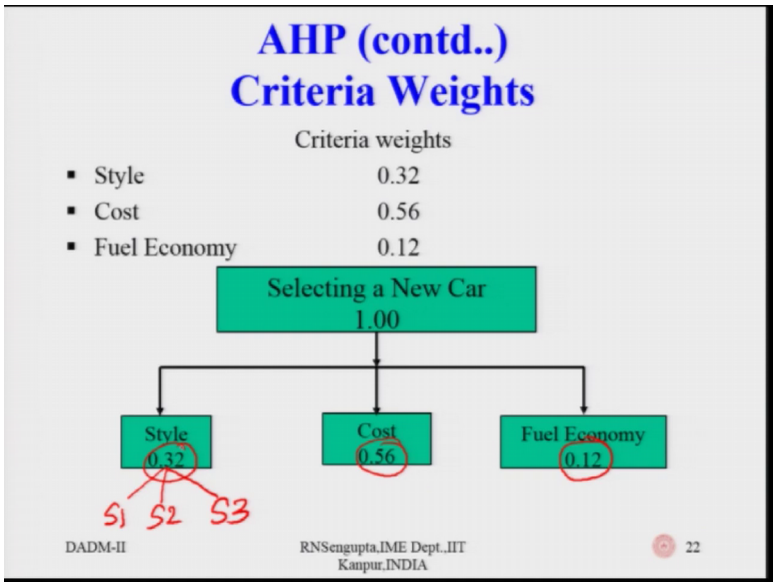

So, once you have the criteria weights, I am only doing the criteria weights, I have not brought the alternatives into consideration So, style gets if the if is in the first row style, second row was cost, and fuel economy was the third. So, the weights are the criteria weights are coming out to be 0.3 to 0.56, 0.12, which are the values which we have here 0.32, 0.56, 0.12.

Another point if you remember, I did mention in the last class which was in the 22nd lecture that the style, cost, fuel economy are the primary levels, there are no tertiary levels. In the sense style does not have different criteria's, which can be broken down into the next hierarchy. Similarly we do not have any cost being broken up into next category or fuel economy.

In that if it was there, say for example style had three consider it is as S 1, S 2, S 3, then you will basically compare S 1 with respect to S 2 with respect to S 3, and you will also compare S 2 with respect to S 3 such that you will have a 3 by 3 matrix. And then the principal (Refer Time: 19:17) again for the matrix which you are drawing will be one. And the of the diagonal elements would be basically one value would be x, other value would be 1 by x depending on the importance which you want to place, when you are comparing S 1 with respect to S 2 or S 1 with respect to S 3 or when you are trying to compare S 2 with respect to x S 3.

Now, you will again have that priority matrix, have that priority vectors whether you normalize according to the row or column that will depend on whichever policy your falling important thing is that. When you are trying to basically consider such different priorities for the same person, you are not going to change the utility function, and the normalization concept.

So, if you are considering the sum of the rows or the sum of the columns, and then try to basically find out the relative ratios, ratio's means that element value divided by the sum of the rows, which you have all the sum of the columns which you have, then you will try to basically use the same concept of normalization for the bad person for different type of criteria's of the decisions you are going to consider, because the persons utility function would not change in between, so that is very logical.

So, the values of 0.32, 0.56, 0.12, and these weights if style was there, obviously they would be another level with S 1, S 2, and S 3 each having weight such that the cumulative weights would give us some information of what 0.32 which is already written.

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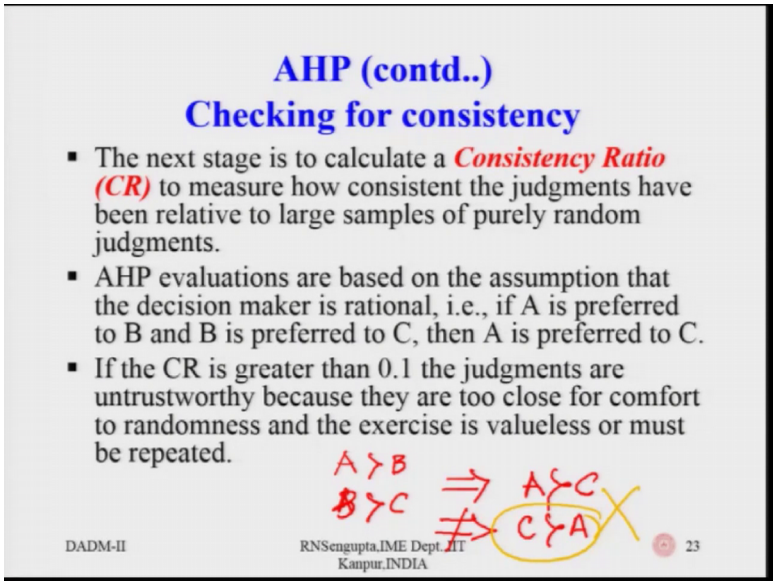

The next stage of the work of the algorithm, which will use for all the decisions all the criteria's compared is basically to calculate the consistency ratio or CR. Now, consistency basically ratio means, the judgment which you are going to make as an individual, how consistent is it or whether there is in a discrepancy or logical flaw in the

decision making process. If you remember lastly, I did mention, if I am saying that A is better than B, and B is better than C, then obviously the later decisions would always lead to the fact that the A is better than C.

So, if there are such inconsistency in the sense that in the later stage, I have marked that C is better than A, those inconsistencies in a very simplistic sense the inconsistency which I said would come out such that we will basically eliminate the decisions for which the overall answer may be biased. So, let me consider continue reading. The next stage is to calculate a consistency ratio to measure how consistent the judgments are made by the decision maker. And how the judgments have been relative to a large sample of purely random judgments with the person is making. So, obviously the person will be making many many decisions, so with respect to that how unbiased or how consistence the decision is.

AHP or analytical hierarchy process evaluations are based on the assumption that at the decision maker is rational. So, if A is preferred which I just said if A is preferred to B and B is preferred to C, then A would be preferred to C. So, in case if A is B, and B you see, so obviously we should have this. So, it cannot be it would not lead to the fact that this is true this would never be allowed, this is not allowed, I am just cancelling it.

If CR is greater so obviously the consistency ratio ranking is already there from a table. So, it says is that if CR is greater than 0.1, the judgments are untrustworthy that there is some discrepancy in the logic, because there are too close for comfort to randomness about the exercise, which is will be valueless or must be repeated as required.

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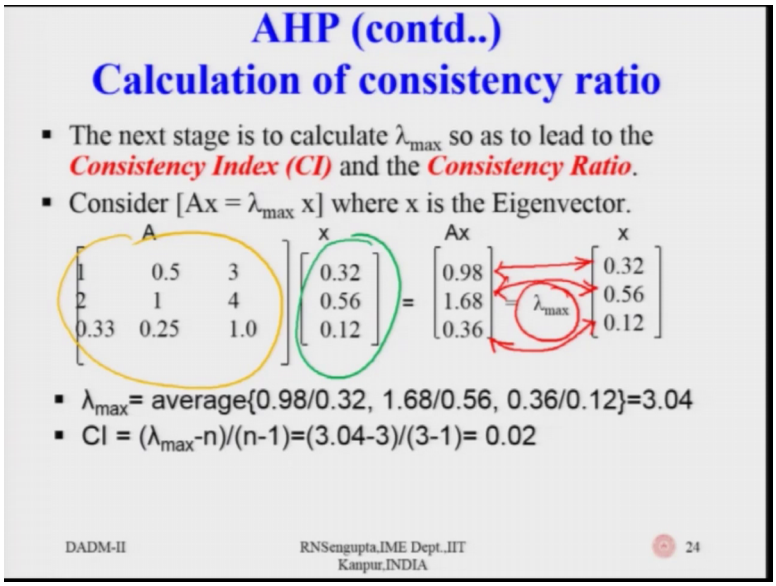

The next stage, once we do this ranking of the priority is fine on the priority vectors is to calculate the lambda max who has to lead to the consistency index, and compare that with the consistency ratio, and make a decisions accordingly. So, consider we already have that Ax A is the priority, x is the eigenvalues lambda is eigenvectors, so which Ax equal to lambda max x, where x is the eigenvectors.

So, obviously you had this was already known to you priority values, these values of x you have already found out. So, multiplying them would basically lead to you the fact that what is the value of lambda max, so lambda max you want to find that we change the color it will be better. So, the value of lambda max.

So, we will try to basically find out the average. So, the average would come out to be the values, so because the lambda max values which you are going to find out would be a vector. So, the vector values comes out to be 0.98 by point 0.32, 1.68. So, these are the division of these terms. So, this is 0.98 by 0.32, 1.69 by 0.56, and 0.36 divided by 0.12. The average of that comes out to be 3.04 or consider 3.3 value.

Now, as per the concept the consistency ratio CI, which you have to calculate would be the difference by lambda max minus n and divided by n minus 1. So, this is the formula which is given by (Refer Time: 25:35). Now, this n minus 1 which we are going to take even though that would not mix in the immediate sense. If you remember in DADM-1 in statistics, we had consist considered that the value of sigma hat which is the estimate for

the variance provided mean was not known we found out the formula like this I would not write it, but I will slowly repeat it.

It will be 1 divided by n minus 1, and in the bracket you have summation of x i minus x bar whole square that squared. And you will sum them up find out the sum, and divide by n minus 1. This n minus 1 is basically the degrees of freedom, because you are losing one degrees of freedom based on the fact that you are trying to utilize the overall sample whole set of sample x 1 to x n. In order to basically calculate, what is the best estimate for the population parameter, which is the mean value which you are trying to replace and you utilize the sample mean.

So, is this is in a similar way, you are trying to basically use the sample estimate to find out the value of CI. And compare that value of CI, which is consistency index and take a decision accordingly.

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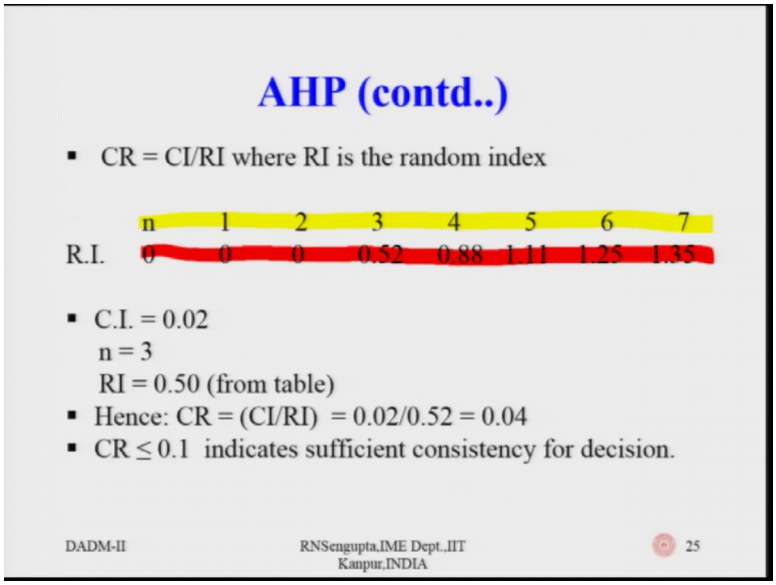

So, the consistency index is 0.02. And consistency ratios so would be values would be given. So, consistency ratios is given by CI divided by RI, so RI values are a random values index which is given based on the size of the number of decisions or number of alternatives which are there. So, if the values of n are given in the first column. So, we obviously, n can be 0, 1, 2, 3, 4, 5, 6, 7, it can be more also. The corresponding values of RI's are given random index.

They are as you can see from the value starting 0, 0, 0.52, 0.88, 1.11, 1.25, 1.35 and once you find out the values of CI comes out to be because it is already calculated as 0.03. n was 3, because there were three criteria's. So, for from the table, you can find it is 0.52 consider is 0.5. And once you have the CR value, it comes out to be 0.04.

So, as the consistency ratio is less than 0.1, so indicates sufficient consistency of the decisions such that you are certain the decision with the decision maker is has some same influence of logic. With this I will end by this 23rd lecture, and continue more and discussions about AHP in the 24th class. And try to wrap a page in a 24th, and start a new topic t in the 25th which will be the last lecture for the 5th week.

So, have a nice day, and thank you very much for your attention.