Data Analysis and Decision Making - II
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Lecture - 15
DEA

Welcome back my dear friends, a very good morning, good afternoon, good evening to all of you. And as you know this is the DADM - II, which is Data Analysis and Decision Making – II, course on the NPTEL, MOOC series and this total course duration is 30 hours which is for 12 weeks, each week we have 5 lectures, each being for a half an hour and as you know that after each week, we have one assignment. So, we will in totality we will have 12 assignments and the final examination. And as you can see in the slide this is the 15th lecture which is the end of the third week.

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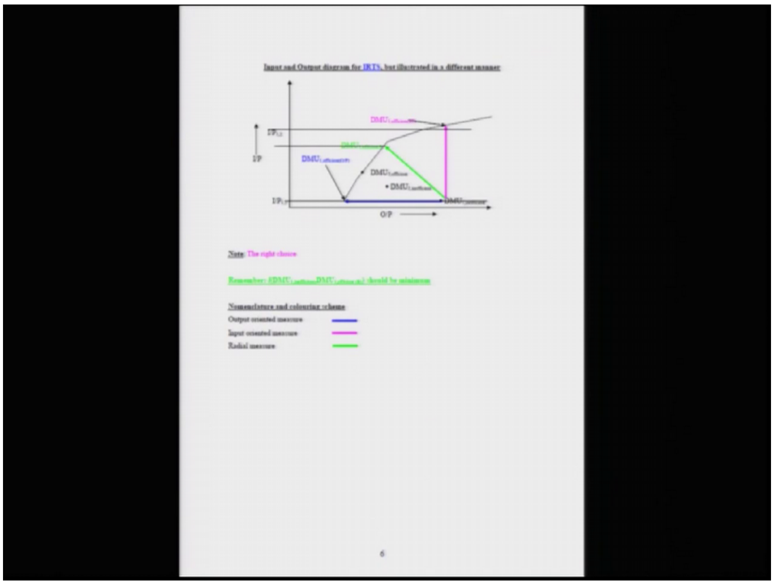

So, if you remember we are considering the different types of return to scale the in decreasing return to scale, increasing return to scale, then constraint return to scale. So, continuing the discussion let us consider the increasing return to scale, but now with the input being measured along the y axis and the output being measured along the x axis.

So, if you have that you are graph would be as shown, where if you consider see for example, this only one issue is that this blue line which you have drawn horizontally on to the left is too close to the y x axis. So, you may not be able to differentiate. But actually do what I have discussed for the decreasing return to scale, and include all the three different graphs one was with input was in the x axis output in the y axis, and other case input was in the y axis and output was in the x axis. Then we had in the combination, then that was basically for the decreasing return to scale; now for the increasing return to scale, again we will have the same combinations. Input once being drawn on the x axis and the next instant being drawn in the y axis and then we will come to the constant return to scale.

So, in this case if you consider any DMU which is inefficient which is where, I am hovering my stylus or this electronic pen. So obviously, they can be three different situations, one for the same level of output what I can do is that because if is it increasing return to scale I would basically technically try to go up which is not right. Because in that case you will be increasing the input and maintaining the same level of output which is not possible; obviously, I will not do that. It is possible, but I will not do that.

In the second case we will consider that we go horizontally on to the left; that means, at the same level of input we decrease the output, and we have the DMU's as shown as the pink and the blue one. The pink one being for the case one is efficient on the efficient front here and then the blue one being again the DMU which is efficient on the efficient frontier.

Another concept can be where you go diagonally; that means, you are trying to increase decrease both the input and then output. So, obviously, in this case we will try to take and the increasing return to scale we will try to basically decrease the output, but maintaining the same level of input such that are efficiency increases.

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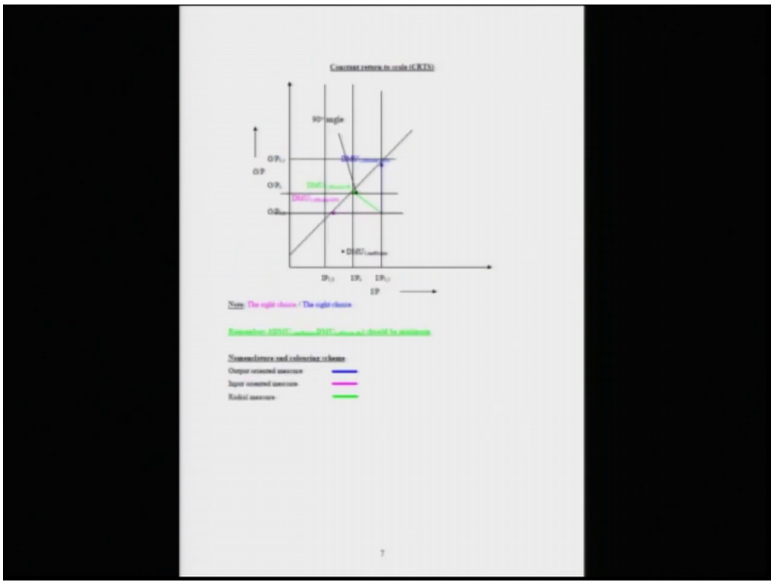

Now, consider the constant return to scale. So, in the constant return to scale obviously, it is a straight line with respect to the input or output being drawn on the x axis or y axis whichever it is. So, in this case we have output along the y axis input along the x axis, if you reverse diagram in the sense that you have input along the y axis and output along the x axis the graph would or we straight, in means, in the sense it means that for one unity increase in the input the output also increases in the same way, such that the constant return to scale is maintained.

So, in this case if you have a DMU is inefficient. So, in this case the DMU can go either vertically up horizontally on to the left or right depending on which type of diagram you are looking at; that means, output on the y axis or input on the y axis. You can go horizontally on to the left or horizontal onto the right or in the case you can either go vertically up or vertically down depending on how you want to reach the efficient frontier. So obviously, in this case you will the concentrate on either the input and the output to maintain your same efficiency.

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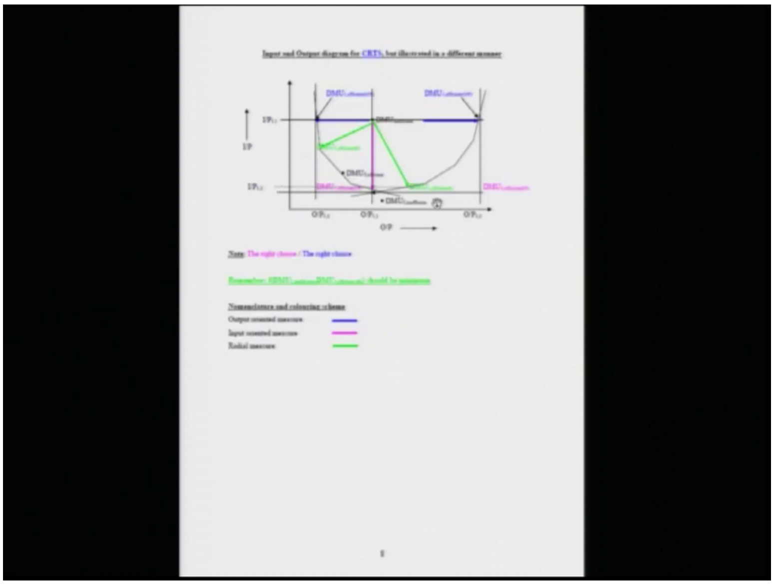

Now, consider the case for changing return to scale where the return to scale varies. So, in this case you have a the diagram is like this where you have the graph. And, here remember I am measuring the input along the y axis, output along the x axis and my actual graph is like this. So, one set is decreasing, another set is increasing. So, if you see the DMU which is inefficient where I am hovering my stylus. So, it can either go horizontal on to the right, horizontal on to the left which are the blue lines or you can come vertically down.

So, there are two graphs, but there almost similar they overlap each other, so you are seeing only one pink line. And another can be you can go radially such that you go take the least distance and reach the curve which is basically 90 degrees of the point which is the tangent. So, it can either go on to the left bottom corner another can be on the right bottom corner, so obviously, when you take the green line then you are trying to basically maintain a proportion of the input output in order to basically increase the efficiency.

While if you go horizontally on to the right or the left you are basically trying to concentrate on maintaining a same level of input, but trying to basically increase the or decreasing the output depending on how the things are. So, if it is a decreasing return to scale so obviously, maintaining the same input you will definite try to decrease the output such that is your efficiency increase. In the other case when it is increasing return to scale you will try to just reverse the situation.

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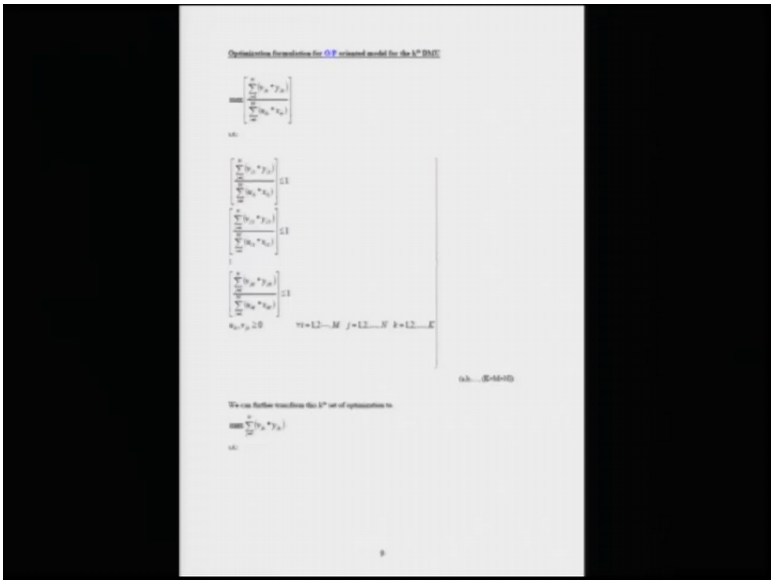

Now, we will consider the actual optimization problem and come back to the solution of that later on how you solve it. So, considered that what we mean by efficiency when we are talking about efficiency, we always talk about the ratio of the output of the input. And when we are talking of the efficiency, actually we will try to maximize the efficiency, so if there are different type of DMUs, we will try to basically maximize them or put from the highest to the lowest such that the ranking would be easy so we can compare them.

Now, another picture can be which I will come later on to the solution or to the formulation, let me use the word formulation another can be that why not basically consider the in ratio of the efficiency. That means, the ratio of the input to the output and then basically try to rank them from the lowest to the highest such that if I rank them from the highest to the lowest considering the output to the input I should technically get the same type of result. So, these are the two approaches which I am going to follow or which you will follow.

Now, consider that is an output oriented model considering the k-th DMU k is that small k-th so obviously, small k ranges from 1, 2, 3, 4, till capital K, so obviously, where I am taking in arbitrary any DMU for our consideration. Now, instead the output oriented model so obviously, the emphasis would be on the output as such and I will discuss that how it can be done. So, consider the efficiency of that k-th DMU. So, if I consider the efficiency of the k-th DMU the ratio would be what is the ratio and what I want to do I will discuss.

So, the ratio is basically the multiplication of the wage for the output multiplied by the way the output divided by the weights of the input multiplied by the input. That means, I am taking the summation in the in the denominator you will have the summation of all the vs into ys for all the combinations of the output which you have for that k-th DMU only, only k-th because we are not going to consider any anything else. So, if there are 10 DMUs I am consider go into consider arbitrary say for example, the fifth one and once you understand that then I will explain them how it can be extended for the first, second, third, fourth, sixth, seventh, eight, ninth and tenth. It can be, fifth is just an arbitrary example it could have been anything else also.

So, we will take the ratio of the sum in the numerator you have the sum of the multiplication terms of which is basically the output and the output widths which is v j k multiplied by y j k. Now, here k is the DMU number and j is basically the subscript pertaining to the output only. Now, in the denominator what we have? We have the summation of u i k into x i k, where again k is the DMU number and i is the input based on which we are trying to consider.

So, if you will not to maximize the ratio of the output to the input what are the constraints? Now, the constraints are very simple. So, what we are trying to do? We are trying to maximize the efficiency of the kth DMU corresponding to the fact that the ratios of all the DMUs, ratios means the output to the input ratios of all the DMUs should always be bound less than equal to 1. So, what we I am trying to do is that I am trying to increase the efficiency of the k-th one, but at the same time trying to basically ensure that in the constraints the efficiency of all the other DMUs including the k-th one are all less than equal to 1.

So, in the objective function we have the maximization should one and in the constraints we have the ratios of the output to the input which is the efficiency and all being less than equal to 1. Now, let us look at this problem. If you look at this problem, this is a non-linear problem because this is a ratio. So, I have to basically converted in to a simple linear optimization problem such that we are able to solve. So, what they do?

So, before I go let me again repeat it, you are taking the maximization of the ratio of the output to the input for the k-th DMU, k-th that is the arbitrary and the such this constraints are all the efficiencies for all the capital K number of DMUs where you are taking the ratio of the output to the input and each ratio is basically less than equal to 1. Now, this I said is an non-linear formulation, so I need to basically converted into a linear formulation so this is what I do.

I take out the denominator in the objective function forcefully put it as 1. So that means, I am pegging the in input to a ratio 1 and in that respect I am trying to increase the output. So, once I see the objective function it now becomes maximization of the summation of v j k into y j k, where again j is basically the output number, k is the input number and what happens to the constraints is what I go into again discuss.

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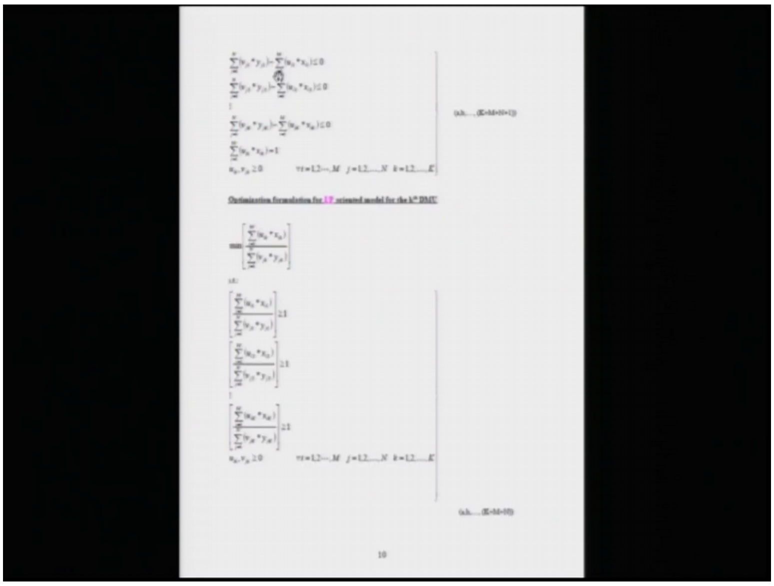

So, the constraints are like this. The first one if you remember the first constraint was basically summation which was less than equal to 1 was the summation of v j 1 because the as the first DMU multiplied by y j 1. This is the first DMU and j is as you know I am repeating it is the output number divided by summation of u i 1 into x i 1, which was less than equal to 1.

So, what I do is that, I take the denominator on to the right-hand side and again bring it; so now, it becomes simple less than equal to I have constraints with known a denominator. Then I bring the right-hand side on the less than equal less than equal to sin

I bring it on the left-hand side and I do it repeatedly for all of them. Apart from the last one where last one means before the k-th one, where I am basically trying to put that input bundle as equal to 1 because I am trying to maximize the output such that that I am putting a restriction on the input you should be 1. This is basically I am trying to rationalize it or normalize it.

So, now actually it when we started we had basically k number of constraints and the k number of constraints continuous to remains the same, but in this case the constraint have taken a change. They are now placed in a different format where all the k minus 1 constraints which are there with does not consider the k-th DMU are of this form. What is this form? It is summation of v j k, v j whatever the k DMU is multiplied by y j small k minus summation of u i k into x i k there should be less than equal to 0. And the last k-th k-th constraint would be such that the ratio for the case when you are taking the input as 1 will be ensured.

Now, what is the change we are having? The change we are having is that we are slowly trying to convert the non-linear optimization problem into a linear optimization problem with the same number of constraints and same number of objective function. So, objective function gets converted for a non-linear ratio to linear ratio point 1, and the constraints which you are also non-linear or a slowly converted into linear constraint for k minus 1 everything is less than equal to 0. And for the case when we put the bring the k-th one we are forcefully putting the input as 1. So obviously, now we have one linear constraints.

Now, let us consider the input-oriented model. So, it in the input-oriented model what we do is that we try to maximize, in the in the in the output-oriented model we are try to minimize. Now, in the input-oriented model we will try to minimize, so the minimize in the sense that we will try to minimize the ratio of the input to the output and the constraints would be such that rather than trying to put them as less than equal to 0 we will try to basically put them as greater than equals 0 and do the same conceptually fundamental change in the problem such that it gets converted from a non-linear problem to a linear problem. So, this is a how I do.

Objective function is minimization of the ratio of the input to the output so obviously, in the input which is the numerator we have summation of u i k in to x i k and in the denominator which would basically be for the output we have summation of v j k into i j k that is point 1.

Point number 2, for the such that constraints will have all them of them is greater than 1. So, what we have is basically again the ratios, but the ratios as they being greater than 1 what we have what we are doing is that trying to take the ratio of the input to the output such that the input to output is definitely greater than 1 and we put it like this. The first constraint is summation of u i 1 into x i 1 divided by v j 1 into y j 1 that is greater than 1. Similarly, for the second one if the summation in the numerator is summation u i 2 in to x i 2 divided by v j 2 into y j 2. And continue it for the one number of k number of DMUs.

Now, again the question happens is they are all, all the non-linear of objective function and non-linear constraint, so you need to basically convert them into a linear constraint and linear objective function we do the same thing. Here now we want to minimize the input the because the input is an input oriented model, but keep the output value as 1; that means, in the first case when you took the output oriented model we fix the input at level 1 and try to basically push the objective function which is the output as far as possible as highest possible, that was case 1.

In the case 2, when we take the ratio of the input to the output we basically put the output a level as 1, normalize as 1, and try to basically push down input as well as possible such that we get the data sufficiency in both the cases. So, now, we basically have a minimization of summation of u i k into x i k which is for the k-th DMU and the constraints are like this. The first constraint, so what you do is that you take the denominator for all the constraint to the right-hand side, then you bring that the right-hand value to the left-hand side and hence it becomes greater than 0. So, the equations are as follows, apart from the last one I will come to the last one again, in the similar way as we did for the output-oriented model.

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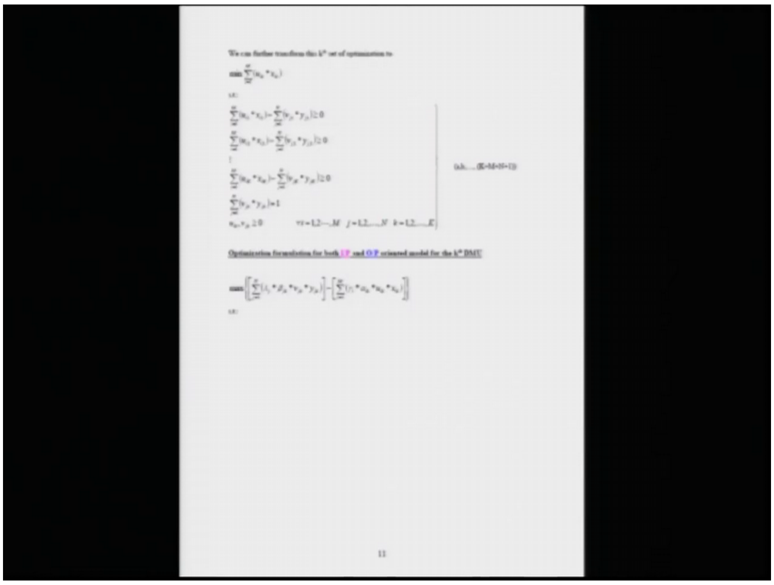

The constraints are summation of u i 1 into x i 1, that is basically the bundle of in inputs minus summation of v j i into y j i which is basically the bundle of outputs. So, difference between them should be greater than 0, and we do it for all the DMUs apart from the k-th one because of the k-th one we have ensured that the summation of v j i, v j k into y j k is equal to 1 because we are trying to minimize the bundle of inputs. That means, as I repeated I will repeat again. In the first case we basically fixed the input and try to push output as far as possible or as a high as possible, in the second case we fix the output and try to push down input as low as possible in order to in both the cases try to maximize the efficiency.

In the optimization formulation for where both input and output are utilize will basically try to do a very simple this is not the actual way how the DA is solved, but we will try to basically take a very in a different approach and tray try to take a multi objective problem where we try to maximize the bundle of say for example, outputs and minimize the bundle of inputs which we take. So, it can be maximization of one function and minimization on the other function or it can be maximization of the output minus the in the input. This is because the minus sign with the maximization sign will basically make it minimum one.

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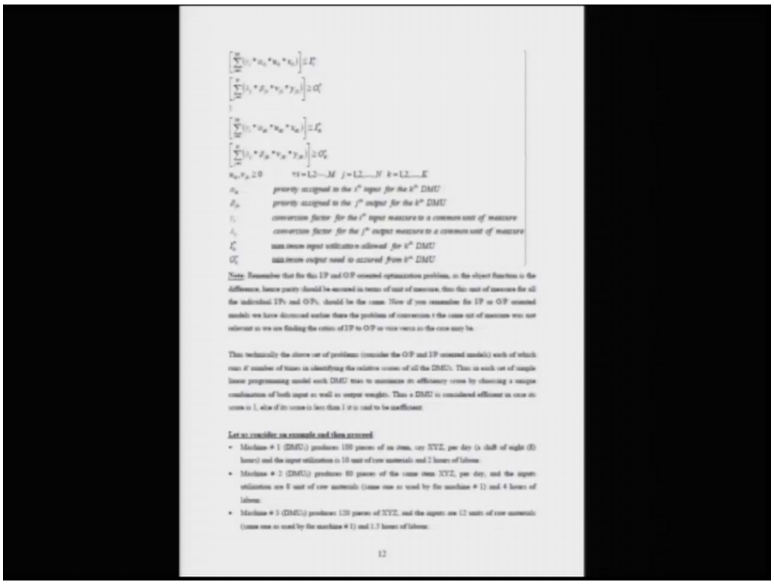

So, what would be the constraint these constraints would be basically fix both to the input on the output. So, if I consider the optimum ratios of the maximum values of the inputs for each and every DMUs as i 1 star, i 2 star, i 3 star till i k star, similarly for the outputs I have o o 1 star, o 2 star, o 3 star, till o k star. So, obviously, this k values are the suffix, sorry. So, I when I mean i 1, it basically i suffix 1 star is the maximum input I can utilize. Similarly, when I am o suffix 1 star is the maximum output which I can get from DMU 1. Similarly, the suffix changes depending on which DMU I am going to consider.

So, if you consider the input on the output. So, it will be; I am not going to go with the details because the problems will be very similar when we solvent and will be kept simple. So, what we ensure is that the bundle of the inputs for the ith one in my DMU should be always less than equal to i 1 star and the bundle of outputs for the first DMU should always be a greater than equal to v 1. This is the objective functions we want to have it will be greater than equal to o star, where i i star or i values are the maximum input utilization and allowed for k, k th DMU and the o star is the minimum output needed to a or assured from the k-th DMU.

So, the thus technically the above set of problems considering both output, input and combination of input output models each of which runs k number of such times identifying the relative scores of the DMUs. And thus, in each set of simple linear optimization problem because we have been able to convert this non-linear optimization problem using this transformation into simple linear optimization problem.

Thus, in each of the simple linear programming model each give you tries to maximum the efficiency score by choosing an unique combination, using a unique combination of both input as well as the output which, thus the DMU is considered the efficient in case it is its course is basically less than 1 if it is one it is efficient, but I will come to the efficiency cause of later on. And else if this code is less than 1; obviously, it is said to be in efficient if it is equal to 1 we will call it in efficient one. But obviously, efficient one would also have a different connotation which I will come to that later.

So, let us consider few simple formulation problem. Let us consider an example and then proceed machine one, produces a 100 pieces of an item say xyz per hour, per day a shift of 8 and the input you utilization is 10 units of raw materials and 2 units of labor. Machine two, which is DMU 2 produces 80 pieces of the same item xyz per day and the input utilizations are 8 units of raw materials and 4 units of labor. Machine 3 which is DMU 3 produces 120 pieces of xyz and the input and the labor utilizations are two 12 units of raw materials and 1.5 labor hours. So, this is how we will formulate.

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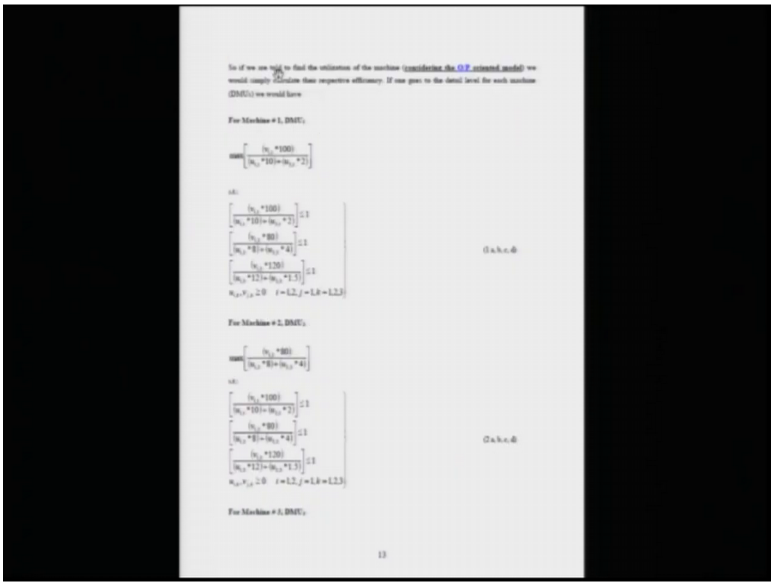

So, if you are told to find out the utilization on the machine considering the output-oriented model, you simply calculate the respective efficiency considering the output-oriented model when you put the output in the numerator the input in the denominator and try to basically maximize that. While in the input-oriented model we will try to put the ratio of the input to the output, fix the output of the level of 1; 1 is basically normalization one and try to minimize the input. So, for machine one; obviously, it will be maximization of v 11 into 100 which is for the output. And the and the input you have because there are two things which is raw materials are labor, so they will work they were basically a 10 and 2.

So, it will be u 11 into 10 plus u 21 into 2 which will come into the in the denominator. Such that is constraints obviously, they would basically be only similar line. So, the such that constraints I would be now three because there are 3 DMUs and they would be as follows. And the only first read the numerator and then read the denominator and each of these three questions would be less than equal to 1 because you are now trying to do the output oriented model where output will be increased subject to the constraint that subjected to the idea that input would basically be fixed at the level of 1 unit and you will try to increase the output as for as possible.

So, the constraints are in the numerator is v 11 into 100 while the corresponding denominator of the first equations would be u 11 into 10 plus u 21 into 2. So, if the second constraint numerator is v 12 into; so this 1 into I am repeating are in the suffix v 12 into l t and the corresponding denominator is u 12 into 8 plus u 22 into 4. And the third constraint the numerator is v 13 into 120 and in the denominator you have u 13 into 12 plus u 23 into 1.5 and each of this constraint would basically be less than equal to 1. So, and the similarly when I go the formulation for the second, and the third I will only concentrate initially on the objective function and then basically give a very general feedback about the constraints.

So, a machine to the DMU 2, the maximization would be this is the maximization problem sorry I should I have repeated it, but I am sure you understand that because this being an output-oriented model. Maximization would be v 12 into 80 and in the numerator you have u 12 into 8 plus u 22 into 4. And the constraints remains exactly as they are only remembering then when you convert this non-linear optimization problem to the linear optimization problem be careful that for which of the constraints are you trying to basically put it into 1. Because if it is a first optimization problem then it would be corresponding to the first of constraint, if it is a second optimization problem it will be corresponding the second constraints and so on and so forth.

So, as there are 3 DMUs you will have basically three optimization problem and I will and you will basically do the changing of the constraint with respect to the first, second and the third. So, we are going to just read the initial non-linear formulation simple case for the third DMU. We have already completed the first and the second.

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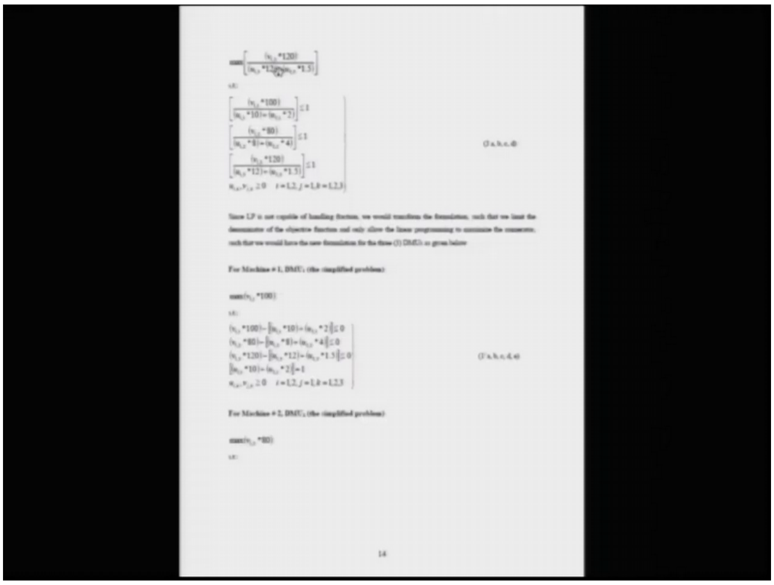

The third DMU will be maximization of v 13 into 2, 120 which is the numerator and in the denominator you have u 13 into 12 plus u 23 into 1.5 and the constraints exactly remains the same. Now, the question obviously, as we have already discussed is the in the conceptual frame work that we need to basically convert this objective functions for 1 2 3 or considering these three optimization problems 1 2 3 into the linear case. So, this is what we do. In the first case we will remove the denominator and put the denominator as 1, so obviously, it will be as this. So, what is the objective function? Objective function was initially was the ratio of output to input. So, input is removed put into the constraint, so hence it will be maximization of v 11 into 100.

What happens to the input the constraint? So, you have basically you are taking the denominator on to the right-hand side because it will less then type and then again bring it to the left-hand side. So, now, you have a linear problem which is v 11 in to 100 and minus the time which is basically the case for the objective function is because for the constraints you are taking the denominator onto the right-hand side which I am just repeating bring it to the right-hand side. So, it becomes v 11 into 100 minus u 11 into 10 plus u 21 into 2 less than 0.

Similarly, you do for the second one. For the third one, and then extra constraint which comes which I thought I would basically mention it when you are doing the formulation is that the denominator which you had for the objective function for the first DMU, now, basically comes as a added constraint in the first optimization problem and we will basically repeated doing for the second DMU and the third DMU.

So, with this I will end this 15th lecture which is the third week of classes over and continue discussing more of the DA part here and then later on go into solving the DA problem.

Have a nice day and thank you very much.