Data Analysis and Decision Making - I
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Lecture – 48
Utility Theory Axioms

Welcome back, my dear friends and dear students, very good morning, good afternoon and good evening to all of you. This as you know is the DADM which is Data Analysis and Decision Making number 1 course under the NPTEL MOOC series and this course as you know total number of hours is 30 and which basically gets converted into 60 lectures. Each week, we have 5 lecture, each being for half an hour and my name is Raghu Nandan Sengupta, from IME Department IIT Kanpur.

Now, if you remember that we were discussing about Utility Theory; the difference to just give a small recap. A utility theory the different concepts of utility theory; I know it is a double repetition, but please bear with me. We discussed about marginal utility the U prime, U double prime, the significance of U prime, significance of U double prime considering that you are risk averse person, risk neutral person and a risk lover person. And based on that we further proceeded on to find out the concept of absolute risk aversion, relative risk aversions which is a and r respectively.

Then, we went with the concept of trying to find out a prime and r prime and how a prime and r prime gives us the properties of the utility function. Then, we consider the 4 utility functions the quadratic, exponential, logarithmic and the power. And we saw that using those the functional form, we could form find out U u prime, a a prime, r r prime and depending on the parameter values their range their domains, we could basically see very specifically about risk aversion property using a prime and r prime respectively.

Then and obviously, to make thing life thing simple, I took very hypothetical example for all these 4 different utility functions and drew them in the excel sheet like in the 1st column we have W, then in the 2nd column we have U W 3rd and 4th column which were not shown in the excel sheet on the in the slide; but still I mentioned time and again it was basically U prime and U double prime based on that I found out AA prime RR prime the respective columns.

And it was basically a double check that considering that hypothetical values of W based on which you can find out the other sets of functions, the matched with the corresponding theoretical results which we got for A prime and R prime for all the 4 different utility functions. Then, further on we considered that given the concept of the utility, I want to find out that how I can classify a human being to be the risk lover, risk hater, risk neutral and we considered the concept of certainty value.

Certainty value was innocent and deterministic value where the probability was one such that the expected value based on the functional form of the utility, you can basically balance that means, the exact value of the certainty value. It is corresponding utility would be exactly equal to the expected value of the gamble, the fair gamble because the properties are half and half. And corresponding to W 1 and W 2, the values which you get, your corresponding values of utility would be U W 1 and U W 2. I showed that in the last class in a blank PPT slide. I wrote it down and used different colors to make things much clearer.

Then, I would consider that considering we also discussed the last example which was basically that I want to find out the property of human being whatever (Refer Time: 04:20) function he or she has and we again considered a simple example where along the X axis, you are measuring the values of W. And arbitrarily considering 2 values of a and b, we also assume the utility function was U W is equal to W such that the expected and it was fair gamble. Even though in the problem I did not mentioned it as p n 1 minus p the corresponding probabilities; but we will consider p n 1 minus p as half and half which is the fair gamble tossing a unbiased coin.

So, the corresponding expected value for the values of wealth values of A and B, the utilities are U A which is A; U B which is B and the expected value would be a plus B divided by 2. And then, on the other side of the table we kept a value of certainty C one if C 1 and expected value of C 1 which will be equal to again U of C 1 which is C 1 into 1 big 1 is the probability.

So, that if it is exactly equal to a plus B by 2, we will say that the person will be indifferent between the gamble and fair gamble and the certainty value and before you know the graph paper, you drew a 45 degrees line. So, hence if it is a fair gamble being balanced by a certainty value exactly so, the point will fall on the 45 degree lines.

In case somebody's risk averse, risk lover then obviously, it will mean that the equality sign will be replaced by greater than or less than depending on the way you are trying to analyze, risk averse person and risk lower person. So obviously, from that you will get the utility functions as increasing within increasing trend or increasing with the decreasing trend. The straight line was basically technically increasing at a constant rate. Then, further on the last part of the last lecture which was the 47th one, we considered that few hypothetical values are given and you know the utility functional form of that person, but you do not know the parameters.

So, you basically keep changing the certainty value at the moment the person basically is indecisive or risk neutral between the gamble and the certainty value, will know that the expected value for the gamble fair gamble will exactly match the expected value of certainty value. You get equate them only unknown would be a in a quadratic form considering the example which we have we have considered it in an exponential form. And you will basically find out the parameters of a and proceed accordingly.

So, this was in an essence what we had been covering so, I will consider further on in the 48 lecture and consider things in much details. Now, the axioms of utility functions for the decision makers would be very important for us to understand that means, in case say for example, there are two decisions.

So, the decision maker would be able to rank them. In the sense the utility coming out from A and the utility coming out from B can be (Refer Time: 07:25) equated in the sense that I am indifferent between the utilities for both the decisions. The second option can be I prefer A with respect to B; hence, the utility of A would be better or greater than B and if I prefer B with respect to A, then I will say that I prefer decision B because the utility of B is better than utility of A.

So, in this case if you have so obviously, ranking would be very logical and the person would basically be able to analyzing A and B 2 decisions. Now, consider I bring another third decision A B C. So, in case say for example, I am saying that the overall utility of A; A is better than B that is U A is greater than expected value of U A is greater than expected value of U B.

And if the expected value of U B is greater than expected value of U C, then we will can always say that the ranking nomenclature would be followed where A would be greater than B would be greater than C. As it is written in the second bullet point which means the utility of A is better with respect to utility of B which is further on better than utility of C so, that the ranking can be done.

Now, remember one thing when I am mentioning the utility, it basically means utility specific; it can change the ranking can change depending on their utility. But, once you had decide on the utility, this ranking concept of A being greater than B, A being better than B; B being that better than C would also always prevail.

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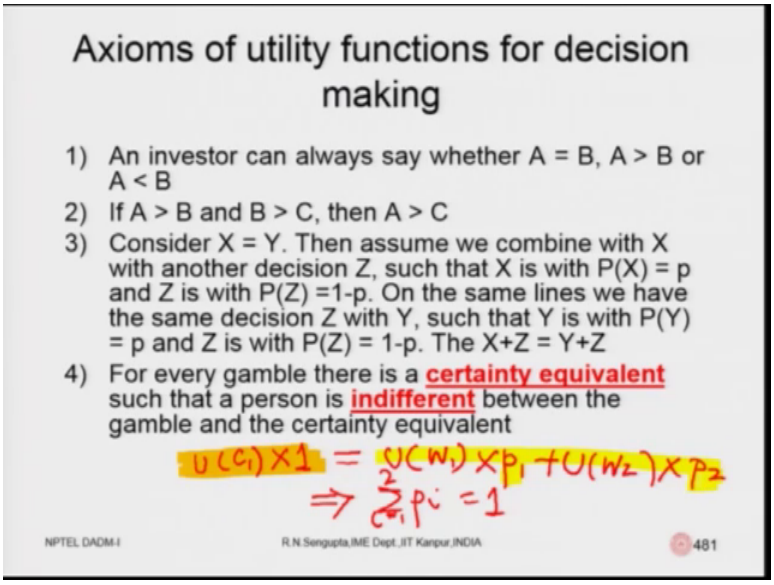

Now, consider there are two decisions, this would not make much sense in the practical aspect, but when we solve the problems very small problems which we have solved or the assignments which are given they be would be almost on the same line. You will be able to appreciate the essence of utility, expected utility fair gamble the concept of certainty value and all these things.

So, consider X and Y decisions are such that X is equal to Y, then we will assume that means, we can combine X with another decision Z such that the X will have basically a probability of P. So, that means, we are combining X with an another decision Z and we basically assign some probabilities p and 1 minus p, it could have been basically half and half also. It could have been say for example, 0.8 and at 0.2 also. So, that does not matter. So, I am taking probabilities of p and 1 minus p.

So, on the same lines, we have another decision we take the same decision Z with Y. So, we are combining X with Z and X with Y and such that the probability of Y is also equal to p. So, in that case we can say that the probability is an obviously, Z is equal to 1 minus p.

So, we are combining X and Y in some proportions with Z another decisions and the probabilities remain the same for both the cases. If it is the same mean means X and Z are being combined corresponding to the probabilities of p and 1 minus p; similarly Y and Z would be combined corresponding to the probabilities of p and 1 minus p.

So, in that case we can say the ranking or the decisions between choosing X plus Y and Y plus Z; X plus Z sorry X plus Z and Y plus Z would basically be of equivalent value or equal value because initially we have considered the decisions X and Y are same. So, you can consider there are 3 decisions, 4 decisions, 5 decisions so, in whichever you combine them, if the corresponding probability is the same considering the initial decisions give us the same utility the ranking would be maintained.

You maintain the ranking, I would not use the word ranking; I would say the equivalence between the decisions would continue to be remain the same provided the following assumptions which I had just mentioned; would be true which means one is X is equal to Y initially and the corresponding probabilities of combining Z it happens in the same proportions.

Now, for every gamble there is a certainty equivalent; obviously, that would be true because if the it is if it is not a fair gamble, if the wealth's are the. So, called investments are W 1 and W 2 whatever the units are they are numbers. So, the corresponding utilities would be U W 1 and U W 2 whatever the utility; quadratic, logarithmic, exponential, power; it does not matter. So, when we combine and try to find out the expected value; what will we have? We will basically have probability considering for W 1 as P 1.

So, P 1 into U of W 1 plus p 2 into U of W 2, where P 2 is the probability corresponding to W 2 such that the sum of probability of P 1 and P 2 is 1. So, when we find out the expected value, again I will repeat it will be P 1 into U W 1 plus P 2 into U W 2 that would be equated to the deterministic events.

So, consider the fair gamble has a value of C 1. So, U of C 1 into 1 because the probability is 1, it will be exactly equal to this values which I have taken. So, if I consider U of C 1 multiplied by 1 which is the corresponding probability of U of W 1 multiplied by P 1 plus U of W 2 multiplied by 2 p 2 such that summation of p i; i is equal to 1 to 2 is equal to 1.

So obviously, in this case this concept of utility would be would be coming out. So, this is with the certainty value. So, so for every gamble, there is a certainty equivalent such that the person is indifferent between the gamble which is on the right hand side. This is should not be, I should not highlight this; oh my mistake.

So, this should be equivalent to the expected on the left hand side which is the certainty value. So, C 1 would be found out corresponding to balancing the or equating the utility coming out from the certainty value and the equivalent and the utility coming out from the fair gamble. So, fair gamble would be the case when in case this P 1 value is half and P 2 value is half.

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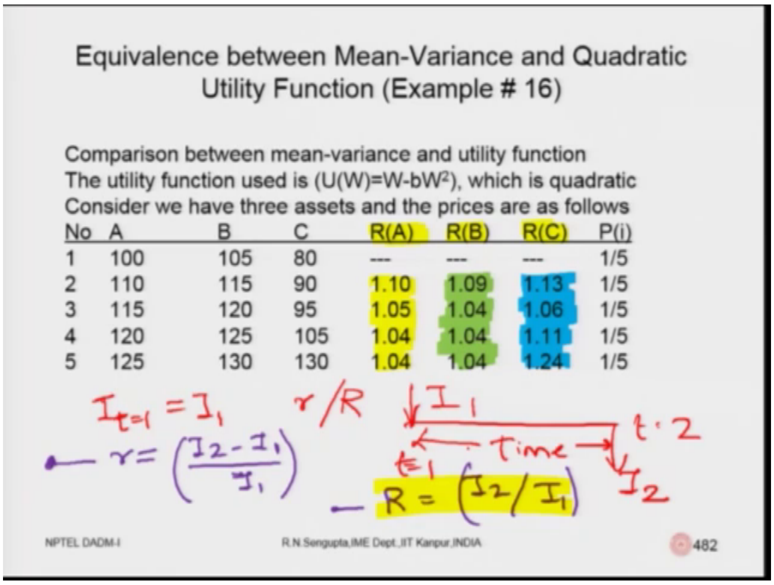

So, let us consider on the concept of utility, but now it is a quadratic one and the here is the important concept mainly from the investment point of view or decision sciences. So, comparison between the mean variance of the utility theory; the utility function is a quadratic which is W minus b W square so obviously, b can be any value.

Now, consider on the other hand, I am trying to basically find out the returns based on the utilities or return based on the investment which I am making or returns based on the decisions I am making. So, in one hand, I am trying to basically find out the returns in other hand, I am trying to find out the so called expected value based on the fact the utility is quadratic.

Now, in that case it can be proved in and it can be found in good economics and good game theoretic book or utilitarianisms book that for the case when the utility function is quadratic, it is equivalent to the case when we consider the return to be returns to be normally distributed or vice versa.

So, this relationship is if and only if so if it returns again I will repeat, if the returns are quadratic the sorry my mistake is the utility functions are quadratic, then we rest assured the returns based on which we are trying to do the calculation would be normal or else if the returns are normal. Then be rest assured the utility function based on which we are doing in the calculations would be quadratic. This would not be true for other type of distributions.

So, based on that we are basically trying to show that without going to the a theoretical proofs with a simple example so, this is the example number 16. So, in the 1st column we consider the numbers of the investment and the prices and in the 2nd 3rd and 4th column, we consider the prices of three assets which is A B C.

The price of A changes from 100, 110, 115, 120, 125 which is in the 2nd column; similarly corresponding to 3rd and 4th column, you have the changes for B and C assets for which the values are 105, 115, 120, 125, 130 respectively for B and the corresponding values for C would be 80, 90, 95, 105 and 130.

Now, based on these values of the prices of the assets A B C, we need to find out what is the returns? Now the returns can be found out in 2 ways; one case can be I am not going to consider this value, I will just give a simple concept. So, consider your investment in time t 1 is equal to at 1 is equal to 1 consider this as 1.

So, the suffix would denote the timeframe and the corresponding to that considering the interest rate is r or capital R. I will come to what is r or capital R within few seconds and consider the timeframe is such that you invest I 1 and in the timeframe of after 1 time period.

So, this is equal to 1, this is equal to 2 so, this is the time, this is equal to I 2. So, in this case if you want to find out r so, it uses separate color. So, you want to find out r will be equal to I 2 minus I 1 divided by I 1, then you can find out the percentage of this and in case if you want to find out R so, capital R it will be equal to I 2 by I 1. So, you can find it out accordingly so, here the values which are given R for R A R B and R C which has basically the value which is given here. It could have been done for small r, also no problem you would basically get the same result, but I am just trying to make it very specific.

Now, based on that when we want to find out the values of for returns for A B C for a case it will be first value 110 would be 110 1.1 would be 110 divided by 100; second value would be 115 by 110; third value would be 120 by 115; fourth value would be 125 by 120. So, the corresponding values are obtained for R A. Similarly, when I find out the returns of 115 divided by 105; 120 divided 115; 125 divided by 120 and 130 divided by 125, you have the returns for B which is I am just highlighting with the green color.

And when I consider the returns for C investment on asset prices, the returns would be capital R would be 90 by 80, 95 by 90, 105 by 95 and 130 by 105, the values are which I highlight by the blue color. And also considered hypothetically the corresponding probability is based on which the prices occur which is the 2nd 3rd and 4th column the probabilities are one-fifth. So, which means for each A B C, the probabilities are equally dispersed which is uniform discrete distribution.

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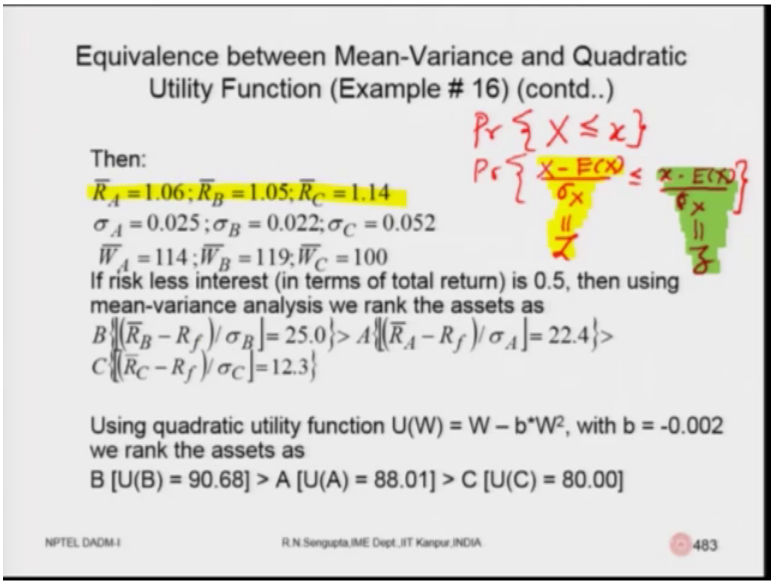

Now, based on the returns R A R B R C, I find out the expected values of returns for A B C which is R bar suffix A R bar at suffix B R bar suffix C. So, R bar basically means the expected value. So, you find out the expected value for the returns for A, returns for B and returns for C, they come out to be 106, 105, 114.

So, these are the returns capital R so, you could have found out the small returns r also. In that case the formula as per the formula U 2 is basically I 2 minus I 1 divided by I 1 so, you basically plug in the values and basically solve it. The corresponding value of sigma are given by C 0.025, 0.022 and 0.052 and consider that the weights we are considering are as given by and the weights average are which you want to find out consider them 114, 115 and 100.

Now, if you want to find out the risk or the interest rate in terms of the total returns and it has basically less than equal to it has to be either greater than or less than 0.5 something so that is the threshold value. So, you will try to utilize the mean-variance theorem. So, mean-variance theorem or the concept will be used will be probability of X being less than equal to x or will be right will be required to find out product of capital X is greater than equal to small x.

So, in both the cases you will use simple concept of standard normal distribution. So, how do you reuse? I should basically sorry I should it is this so, this space. So, sorry let me continue probability of X minus this E value is basically the expected value X divided by sigma of X less than equal to x minus expected value of capital X divided by sigma X. This value closes that bracket closes and the value basically should be coming out to the probability.

So, this technically I will just highlight it this becomes capital Z, this becomes small z. So, these signals can be proven and this becomes capital Z and this become small z. So, given these values of alpha, given this values of small z, you can find the other values and do your calculations accordingly.

So, with the return of 0.5, I basically put in the mean-variance theorem, I want to find out it to be greater than. So, what I do is that probability R B minus that. So, this 0.05 that is free interested which is there in the banks such that if we go to the bank invest, you will basically get return which is minimum of this. So, you want to basically analyze your investment with base based to the risk free interest rate.

So, corresponding to B A and C, the values come out to be about 25, 22.4 and 12.3. So, it means that when you rank them, corresponding to that B would be better than A would be better than C corresponding to the probabilities which you found out. Now, let us stop here and go back to the utilizing the quadratic utility function because, I mentioned that quadratic utility function and normal distributions based on the returns are equivalent.

So, let us take the quadratic utility function W minus b W square with B as 0 minus 0.002 plug them in the values solve them. So, it would change, but the values were almost the relationship between A B, A B C would come out to be the same. So, once you find out the utility of B comes out to be 90.68 which is better than utility of A which is 88.01 which is still better than utility of C which is 80.

So, if you compare B is better than A is better than C considering the case of quadratic utility functions you are using. Similarly, when you do the calculation using the normality returns, the values also come out to be the same which is B is greater than better than A is better than C. So, you both wise you basically try to verify the fact that the concept of a quadratic utility function and normality distributions are interlinked in the general sense.

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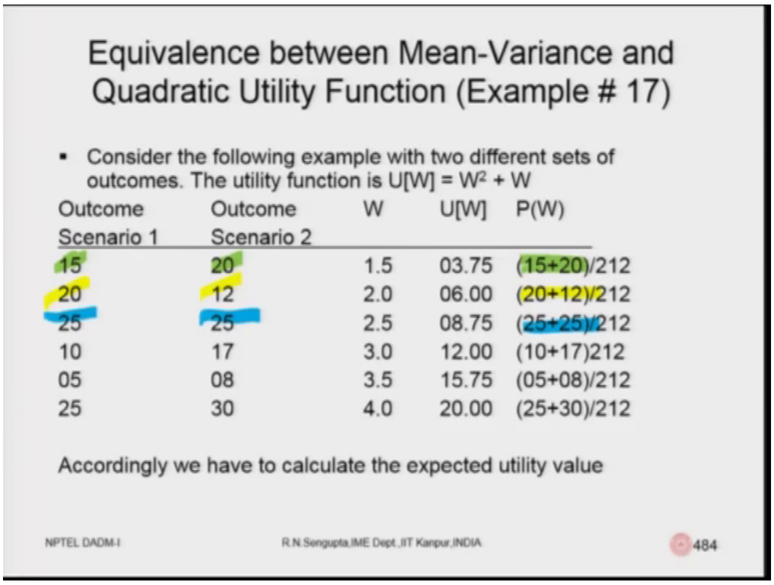

So, consider the following example with 2 different sets of outcomes so, they are as follows. The utility function is given by a quadratic utility function W square plus W; I am just taking an arbitrary value. And considering there are 2 scenarios; scenario 1 and scenario 2 or 2 different tournaments or 2 different ranking system whatever it is. The outcomes of the scenarios for the first one are given in the first column starting on 15, 20, 25, 10, 5 and 25 and outcomes for scenario 2 are given by 20, 12, 25, 17, 8 and 30.

Now, in the 3rd column I basically write the values of W arbitrary values based on this I can find out U W which is quadratic one with the certain value of B and then, we can find out the probabilities corresponding to this utility. So, if I consider say for example, scenario 1 and scenario 2.

So, what I would basically try to do is that try to basically find out the relative frequency of with the number of events which basically affects scenario 1 and on scenario corresponding to sorry which basically is affected from the situations of scenario 1 and scenario 2 and you want to find out that what is the relative frequency.

So, if you find out so, for combined effect being there combined effect in the sense, you I want to find out the relative frequency. So, the relative frequency for utility 1.5 with a the wealth over 1.5 with the utility of 3.75, the corresponding probability is basically 15 plus 20 divided by 212.

Similarly, when I do it for the 2nd 3rd, 4ths and so on and so 4th cases of W and U W; U W basically remains continuously remain the same as the quadratic utility function. The probabilities for decision 2, decision 3, decision 4 so on and so forth basically combines in the numerator the number of such cases which have been favored for the wealth W which is 2. And similarly, I want to find out what is the number of cases which has been favored under scenario to 1 for wealth of one point a wealth for 2 and once you want to compare doing that.

So, what you have is basically 15 plus 20 which basically is sum of this. So, this 20 plus 12 is basically sum of this, 25 plus 25 is basically sum of this and you continue doing it. So, the corresponding probabilities are found out, corresponding utilities are found out according to that find out the expected value.

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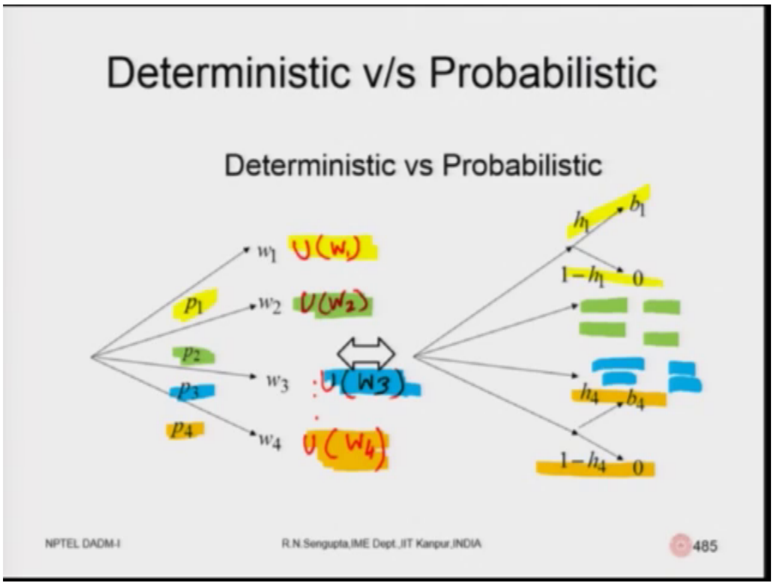

Now, this diagram basically shows the deterministic and the probabilistic nature of utility. So, in the left hand side, you have a not a gamble and a decision tree which is certainty certain probabilities and certain so called outputs with respect to the W's. So, obviously, you will convert them into U W so, and considering there is only one utility functional only.

So, you will find out the utility here, utility here, it comes corresponding to utility here. So, based on that once you find out the utilities you multiply the utilities, probability here then, this value multiplied by p 2, I missed one I will right it again. This will be U W 3 then, based on that I want to basically. So, this would green would has come now it will blue P 3 U 3 this is wait ha U 3, then and you run the color P 4 U 3 basically U 4.

So, you want to find out the expected value, what you do is that multiply the utility and the probability. Now what I am further doing is that in order to find out the equivalence is that I break up that utility or W 1 wealth into a fair gamble. So, the fair gamble values I will use the yellow color in for the right hand side. So, if it is yellow so, it basically h 1 into b 1 plus 1 minus h 1 into 0 would basically give me the so called value which is a U W 1 because that is equivalent.

Next, when I go to the second one which is not shown; second one will basically have a fair gamble with the probabilities say for example, I am considering h 2 and 1 minus h 2 and the values are on the arms are b 2 into 0. So, obviously, in that case the expected value for the gamble would be exactly equal to U W 2.

Similarly, for the third case, we have which you need to find out it will be U 3 into p 3 which will be equal to the equivalent probability. And finally, if we find out the expected value of h 4 into b 4 plus 1 minus h 4 into 0, the value comes out to be equivalent terms with respect to the utility U W 4.

So, you have to basically understand how you basically place it. So, what we have seen in this diagram using even though I use different colors. So, basically we are trying to be analyze the expected values of the utilities multiple of the utility functions which is utility function multiplied by its probability is being broken down into a case, where you want to basically find out the corresponding gamble; even though I being using this word fair gamble, but it is the gamble.

So, technically U W 1 value would be equal to the expected value for the gamble which is h 1 into b 1 plus 1 minus h 1 into 0 and we will continue considering this for each and every arm. So, with this I will end on the this lecture and continue more discussions among utilities and what are the different type concepts of other ideas how you can basically analyze your decision making we will be considered in the further lectures.

Thank you very much and have a nice day.