Quantitative Finance

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Module – 02

Lecture – 07

So, continuing our discussion, we had discussed how to draw the efficient frontier considering four combinations: short selling allowed, not allowed for actual risky assets and short selling which is technically lending/borrowing being allowed for the risk-free interest rates, which is R f. Now, what if we have different another combinations. So, if we consider the slide, which is shown; you have basically either optimizing or trying to basically find out the best return or best risk; risk means you are trying to minimize; return means you are trying to maximize.

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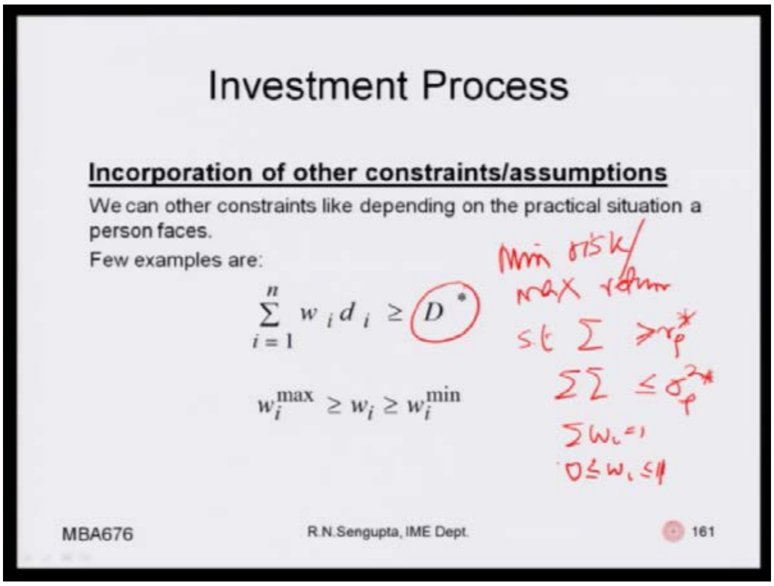

Considering that; if the dividends are greater than equal to some D star value. So, what we actually mean is either you want to basically minimize the risk as the variance or maximize the return, which is expected value. Now, till now, we have considered the… Subject to consideration is that, the summation of the portfolio returns being greater than equal to sum r p star; star means value which changes. Similarly, for the double summation, which is the variance of the portfolio being less than equal to some sigma square star p; p is the suffix; and so on and so forth; that the weights are equal to 1 and weights are either between 0 and 1. If there is no short selling, if there is short selling; obviously, that changes. So, another case can be where it is greater than some fixed D star value of dividend, which we want for our case. Or, it can be say for example, there are cost involved with respect to buying and selling stocks. So, obviously, those things can be considered. And, rather than solving the nitty-gritties of that, we will… Once the overall course content for the options are over, we will include few of the interesting problems with their actual formulation and also with their solution techniques for some of the datasets considering from the stock markets throughout the world. Like we can consider NSE, which is the national stock exchange; we can considered the BSE; we can considered the DAX; we can consider the CAC; we can consider the Dow Jones, so on and so forth.

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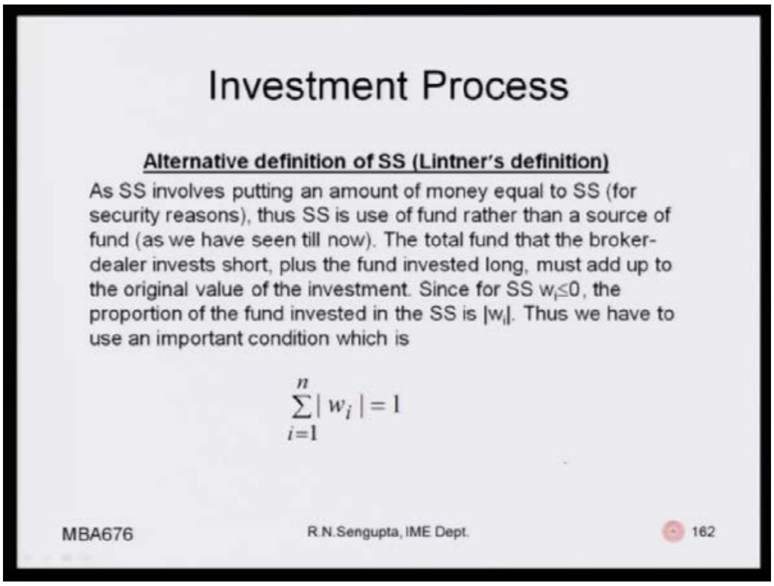

Now, another alternative solution method has been proposed by Lintner. Now, if you remember; why? Why we are going to consider that? Let me step back few lectures or one-on-one – one and a half lecture before. We considered that, the concept of short selling was there. And, we also considered that, in order to find out the actual weights, we normalize them and find out the normalized weights and the sum is 1. Now, as SS, which is short selling involves putting an amount of money on equal proportions for the short selling, Lintner's method proposes that, if you borrow an asset and a financial asset from a person; obviously, there has to be some risk for the other person. So, in order to compensate that, there should be some collateral. So, considering that collateral concept is there and if you want to incorporate that as a short selling concept as well as an optimization concept, Lintner has proposed a model, where the sum of the mod of the weight should be equal to 1. So, rather than only the weights, we should basically consider the mod of the weights is equal to 1. And, obviously, there are solution techniques for that; but, we will consider a very simple solution technique based on which you can solve that problem.

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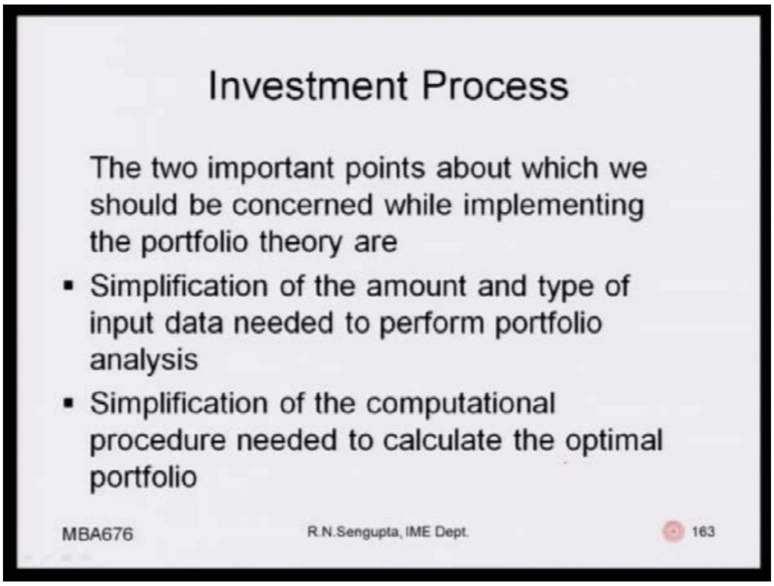

The two important points about which we should be concerned while implementing the portfolio theory or trying to basically analyze the best optimum portfolio or trying to find out the efficient frontier; either or trying to find out the best set of solutions; or, trying to find out the minimum variance point or the maximum variance – the maximum risk return concept, which is Q; we should basically consider the simplification of the amount and the type of input data needed to perform portfolio analysis. And, simplification of the computational procedure is needed to calculate that.

Now, remember till now, even though stock market prices are non-deterministic; that means, they are changing; we have considered very simply that a priori the values are given, and based on that we are doing our calculations. Later on we will be – just relax this consideration; like I said, we will consider dividend policies to be true; we will consider non-deterministic concepts or stock prices; hence, the returns. We will consider some distribution of the returns as well as the parameters of the distributions of returns are changing. We will consider buying and selling of stock are there as parts and parcel of our investment policies. So, once we understand the general formulation, we will see that, they can be done for variety of problems.

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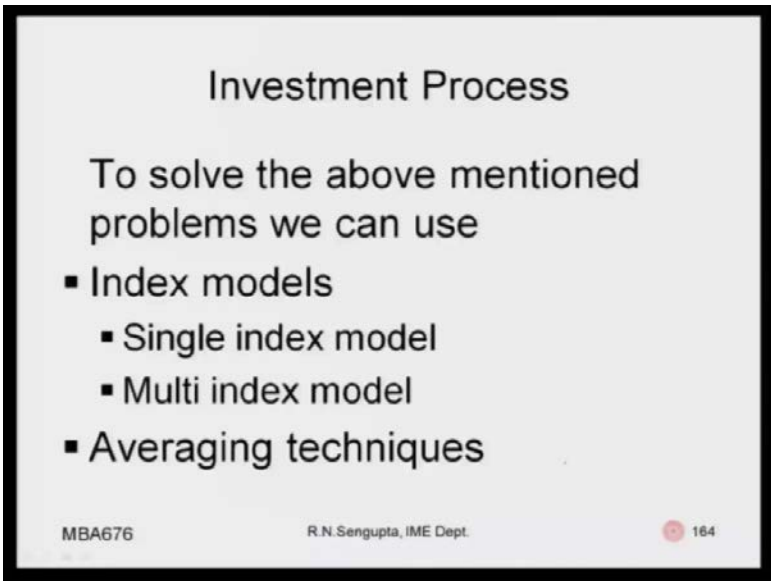

Now, to solve the above the mentioned problem sets or all what we discussed; initially, we will consider the single index model and the multi index model and few of the averaging techniques.

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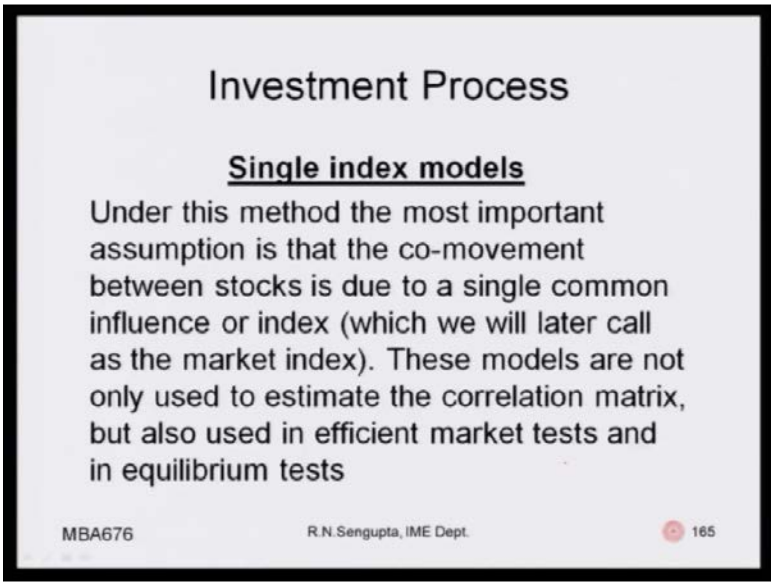

Now, in the single index model, which as the word says, there are some single index or some single variable, based on which you can find the prices of the stocks, terms and based on that you can proceed. Assumption – the most important assumption is that, the co-movement between the stocks is due to on a single common stock, which is the index we will consider. These models are not only used to estimate the correlation matrix, but also used to find out the efficient market test and try to find out the best optimum solutions of the portfolios such that we are almost equal to the market as we basically formulate our portfolio considering different types of stocks are there.

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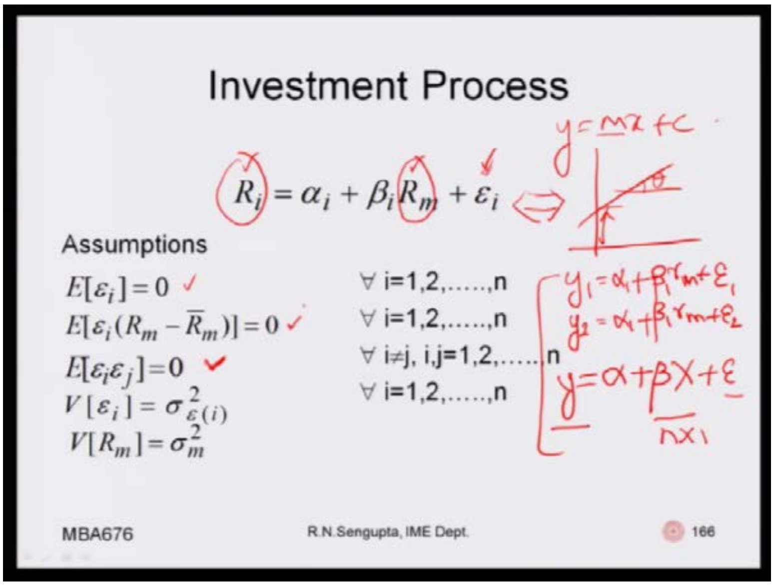

So, if we remember, in class 10 and 12, we have considered the equation of a straight line as equal to y is equal to mx plus c. Now, if we had drawn that curve, it was like this; where, this value – the height where the line cuts, the y-axis is known as c. And, if we consider the tan of this angle, which is the slope, this value is given by m. So, what we are trying to find out is that, given different values of x, which are along the straight line, we want to find out what are the values of y. Now, in general, we would consider that, over and above the movement of the stock prices or over and above the movement of some random variables, there is some white noise also. So, in the concept of simple linear regression, multiple linear regression or simple linear regression, we consider there is a x which is a random variable, which is independent and known to us. We have the datasets. And, there is another random variable y, which is a dependant variable such that there is some relationship between x and y's such that given x, we are able to find y.

Now, our main task is basically to predict or estimate or forecast the value of y to the best possible extent; which means that, if we have set of x's for different time periods, we also have the set of y's for those same time period. Now, consider – given that information, we want to find out some relationship functions between x and y such that in the future, if we have x; that means, given n number of x, we have already n number of y's. Now, in the second stage, use the information of those n number of x; try to find out the information for relationship between x and y. And then, when we get the n-th plus 1, n-th plus 2, n-th plus 3 x, we are able to predict forecast or find out the predicted value or the estimated value of y to such an extent that, some rules are met.

Now, if I ask you very simply that, what rules would you like to basically meet for your prediction; obviously, very intuitively, everybody will say that, our predicted value or the estimated value should be as close as possible to the future actual value of y; which means the difference between the predicted value and the future value, actual value is y – should be as low as possible; which means that we want to basically minimize the error. Now, the word error is being used here which is the proxy of the overall variance we want to basically minimize. So, our main task is basically to find a model such that we are able to find out the set of relationship function between x and y such that the overall error is minimized to the maximum possible extent.

So, let us consider a very simple equation between – relationship between x, which is the return for any particular stock and which is y, which is basically the market. If you remember in the last slide, we discuss that, there is some variable, which is independent, which is the market, which gives us the best amount of prediction, which is happening between the return of a particular stock and the market. Now, here R i's or small r i‘s are the return of particular stock with a suffix i meaning the stock number. And, R m is basically the conglomeration of all the market, which you have, which is known as index. As we said that, it can be the bum index; the national stock exchange can be say for example, Dow Jone industrial average; can be NIKKEI from Japan, can be FTSE from UK, so on and so forth.

Now, there are few assumptions. And, if you go to the fundamental principle of linear regression; we consider that, intrinsically, x has a distribution, which is normal; x means the R m, which we have. So, R m has a normal distribution; which if you remember, we mentioned that, if the utility functions are quadratic; then only, you can say the returns are normal. So, we will consider R m as normal distributed with some mean and some standard deviation. We will have the returns of the stock R i's as also normal distributed. So, if you can recollect from very basic concepts we have done in engineering or in MCom, BCom, we will consider the normality or normal distribution as properties such that any convex combination or normal distribution will always result in a normal distribution. So, what we are trying to do is that, we are trying to find out some relationship between R I, which is normally distribution with certain mean and certain standard deviation. Along with that, you want to find out what is the independent variable, which is R m, which is the market, which is also normal distributed.

Now, if you find out this – the equation along with the graph which you had drawn for y is equal to mx plus c; then, alpha is to c; beta is to the slope. And, this epsilon, which you see here in this single index model, which is not there appearing in the case for the simple equation – y is equal to mx plus c; it basically means white noise; that means, there are some fluctuation and this white noise over which we have no control; and this white noise basically gives you the overall randomness in the overall process. Over and above that randomness in the values of x, which is R i or values of y, which is R m – value of x – value of x would be basically R m and value of y would basically be R i.

Now, the assumptions are very simple. You consider in the long run the expected value of the white noise is 0; that means, epsilon, which is the white noise; if you add up the values, they add up in the long run as 0. We will also consider that, there is no covariances existing between the market and the white noise, because that is the second equation. We will also consider that, there is no correlation coefficient existing between the i-th white noise and the j-th white noise. And, finally, we will consider another two, which we will consider is that, the variance of the white noise, which is given is time independent, which may not be true in actual relationship. And, we will consider the market as a variance, which is known as the market variance or market risks as such for the market only, which is given by the variance sigma square suffix m.

Now, if you look at the third one and if you have basically n sets of equations; in the sense, i have y 1 is equal to alpha 1 plus beta 1 into R m plus epsilon 1. Then, next case will basically of y 2 is equal to alpha 1 plus beta 1 R m plus epsilon 2. Now, remember this 1, 2, 3, 4 for y is basically the data point, which is changing from 1, 2, 3, 4, 5, 6, 7, 8, till 9. While epsilon is also in the same case; but, the differentiation, which we have we want to highlight is basically alpha 1 – would basically be the so-called c value for y is equal to mx plus c for the first stock. While beta 1 would be basically the market risk or the beta value for the first stock. So, if we have 10 different stocks, the equations would be – they would be n such equations for the first stock; n such equations for the second stock and so on and so forth. So, we will consider the first stock only and that should basically give us all the information and all the concepts how we are able to solve the problem.

Now, given this, our main task – if we basically denote it in a matrix notation and format; what we have is basically y, which now is a matrix is equal to alpha. Again it is a constant value remember that. It is again a vector; y as such is a vector. And, you have basically a vector of beta into x plus some vector, which is basically epsilon. So, what we have? y is a vector of cross – n cross 1 depending on the number of ratings which you have; and, beta is also – when beta into x also multiplied is basically n cross 1 matrix depending on the data size. So, obviously, you can find out what are these values, which you have for beta; what is that rank of that; what is the size of that, so on and so forth. Similarly, you will have epsilon.

Now, if you write in the concept of vector notation and you want to solve them; then, generally we know there is a method of Gauss-Jordan multiplication, where the concept of determinant or the value of the determinant is very important. Now, in the value of – finding out the value of the determinant, we want to basically find out the inverse and so on and so forth. Now, finding out the inverse, important concept comes is that, rank of a matrix. So, rank of a matrix is a very simple concept, where the number of rows and number of columns are such that all of them are independent. So, if they are not independent; as finally, when you are trying to solve the equation in a simple linear equation format, technically you should have basically n number of variables. This n and what n – what we have written are different. So, this is the first n, which we discussed was basically are number of data points.

And, the second n, which we are talking or let us consider as k is number of variables which we have. So, for that matrix, when you solve; if there are k number of such equations and k number of variables, they should be independent in the sense that all the k variables, which you find out would be unique. In case they are not; obviously, it would mean the rank is not case such that there may be some equations, where that independent characteristic may not hold true. And, that we will see would basically give us some information whether the concept of multicollinearity, homoscedasticity, heteroscedasticity would hold or not. So, we will later use these terms under time series concept. So, this relationship between the error should be such that they are not dependent from one to other. So, if they are dependent; obviously, the whole structure of the problem would be basically collapse. So, we will consider this case, where both the covariance is between the so-called market along with the white noise as well as the… The covariance is between the white noise from time to time would definitely be independent such that the values are 0.

Now, if you find… Let me go back again. If you find… If we find out the expected value; so, what we will have? We will use the expected value of this is equal to the expected value alpha, which is a constant; alpha is a constant plus beta again will consider as a constant, which is the tan of that slope; tan of the theta, which is the slope. Then, R m is a random variable; obviously, it will have a expected value. And, the last term, which is the white noise basically be 0 according to the first assumption.

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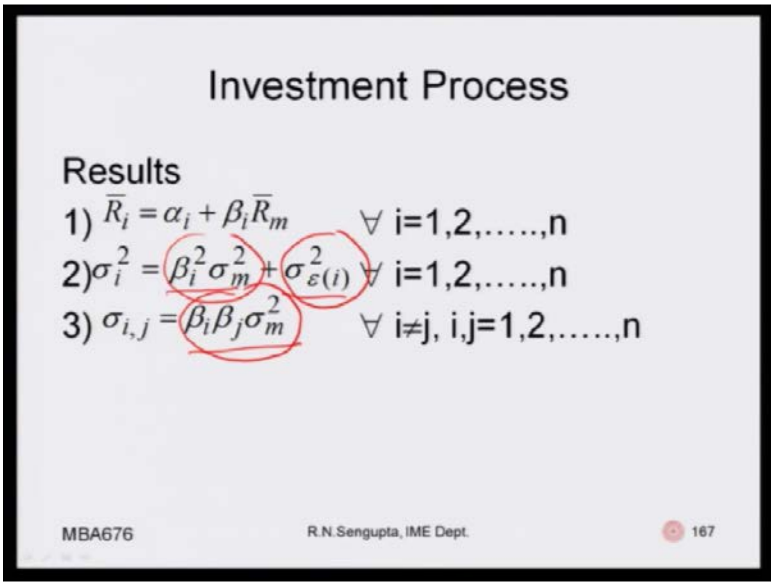

So, if we find out the first expected value, you have this equation when R bar or expected value of R whether it is stock 1, stock 2, stock 3; and, it is a question – is equal to alpha with a suffix depending on the stock number plus beta suffix stock number into R bar m. Now, if you want to find out the risk; if we go to the first equation again back; so, they would be on the left-hand side would be the risk of the variance of the stock; and, on the right-hand side, technically, we have so-called random variables even though we will see that one of them are technically deterministic. But, let us consider whether they are random or not. Alpha is not random; so, it is a deterministic one. R m is a random one; so, we need to find out the variance or the covariance. Epsilon is also random. We need to find out its variance or covariance.

Now, if there are three such variables; one of them being deterministic, another two being random. So, basically, we have a 3 by 3 matrix. So, the first element, which is the variance of alpha would be 0 along the principle diagonal. The second element would basically be the corresponding variance of beta i into R m. And, the third one would be the variance of white noise. While of the diagonal element, all would be 0 because we have considered two things; covariance of the white noises are 0 and covariance of these also are 0 as well as the covariances between the R i's and the epsilon. So, once we consider that, the second equation would basically give us the variance of the particular stock, which is consist of two terms: one is beta 1 square into sigma square n suffix m plus sigma square of epsilon.

So, now, we basically should make a note of these two; about which we will discuss in few moments. While the covariance is existing between the two stocks; say for example, in BSE, you are taking Tata Motors and you are taking Reliance; so, obviously, the covariance existing between them would be given by the values of basically betas multiplied for both of them into the market risk. So, what we are trying to do is that, if you see that we are trying to replace overall risk of any two different stocks or any particular stock by a simple concept of market variance only on the market risk. And, if it is only a single stock, we see that it will basically consist of two terms: one is the risk coming from the market and one is the white noise, which you cannot control. And, when you want to find out the covariances of the risk of one particular stock with the other, which is the covariance; you will find it is basically being multiplied by the variance on the market and the two different values of beta, which you have.

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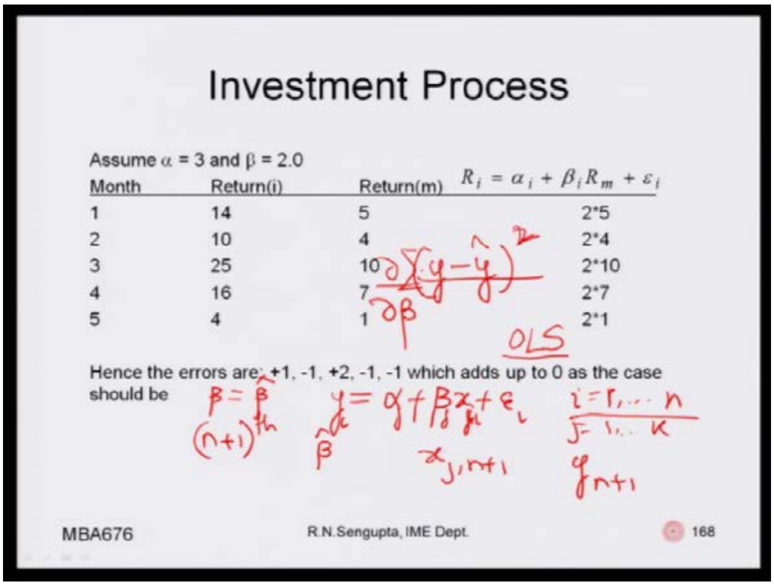

So, consider this very simple example. The months are given on the first column. The returns are given on the second; and the returns on the market are given in the third and you basically utilize the equation, which is equal to R i is equal to alpha i plus beta i into R m plus epsilon. So, you consider alpha is 3; constant beta is 2. You find out these values. And, the reason I am trying to highlight with this simple equation is that, even though it is a theoretical one, the important point to remember is the sum of the error should definitely be 0.

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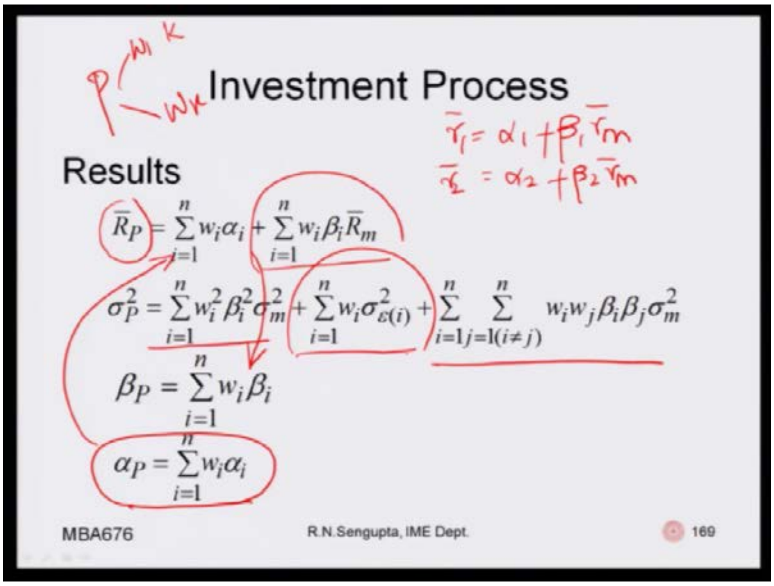

Now, having said that, what I want to find out later on also is that, what are the different types of methods we use. So, first, let us very briefly go through the concept based on which we are trying to solve. Now, in linear equation, what we have; if you want to go into the depth of this; is basically given any equation; y is equal to alpha plus say for example, beta x plus epsilon. So, we have basically n number of equations. So, basically it is y i; then, it is x i and x j. Let me denote it x j. This is basically j. This is j. This is being comma i. And, j is basically for the number of stocks, which is 1 to k. Now, what you actually do is that, given this n number of data points, you find out the values of beta and then basically replace beta which a beta had, which is the estimated value. And, for an n-th stock, which you want to find out – n-th plus 1, you use this beta values – beta hat values along with the value of x for the j-th stock for the n-th plus 1 time in order to basically predict y, which is basically for the n-th plus 1 time.

So, what we actually do is find out the difference of the errors between the real value and its predicted value, which is y hat. And, you basically take the square of that and minimize that. So, once you minimize that, your actual equation would be differentiating with respect to all the beta value, which you have. So, if you expand this equation; the first y would be the actual value and the y hand value within itself would basically have the two unknowns which are basically alpha and beta. So, what you do is that differentiate with respect to alpha or beta or whatever it is and equate it to 0 and then find out the second differential and basically try to find out the minimization one.

Now, if you remember that why we are taking the square; if you remember the concept of a simple variance, what we are trying to do? We are trying to basically minimize the variance to the maximum possible extent. So, this is basically known to the concept of ordinary least square method. And, here in the ordinary least square method, we basically try to find out the errors in such a way, which the errors are the square of the differences of y and y hat; sum them up. So, this what I am using is that, this is basically a vector. So, if I basically have the scalar points for each and every dataset; it will basically be coming up all the errors which you have for all the number of data points we are going to take. And then, sum them up; differentiate with respect to all the variables, which are unknown to us and equate to 0 and find out those alphas, betas and so on and so forth. So, once those alphas and betas are found out, you find it for the n-th plus 1 and then proceed in such a way that you are able to plot the point; keeping in mind, your main focus was trying to basically minimize the variances.

Now, consider… So, given this, what we have is basically; there is the i-th stock. So, such i-th stocks are there in the market. And, using this i-th stocks in different proportions for say for example 10 of them; we want to basically formulate the portfolio. So, if you consider the BSE; 30 or say for example, NSE or CACS or DAX whatever it is; there are different type of stocks, which are there inside those conglomeration of the market index, which you have. And, when you combine them, what you want to do is that, find out the overall portfolio, which you are trying to formulate with respect to the risk of each and every individual index, which is there in that portfolio with respect to the market. And also, find out the risk of the total portfolio, which is being formulated with respect to the variance as well as the beta values, which you have for each and every stock.

So, if you have the equation for the first – the index using the single index model, what you actually have is this – R 1 is equal to alpha 1 is equal to beta 1 is equal to R m. So, these are all bars. And, these alpha 1 and beta 1 are constant. Similarly, you will have basically R 2 is equal to alpha 2 is equal to beta 2 into R m – again, bar. Now, if you consider all the particular different type of stocks which you have in the portfolio; if you remember, the portfolio consists of say for example, k number of stocks; and, each is being invested in proportions of w 1 to w k. So, when you multiplied the corresponding alpha values; multiply the corresponding values of beta’s R m; they would be multiplied by the corresponding weights which you have. So, when we have the return on the portfolio with respect to the single index model, you will find the overall alpha for the portfolio, which I am hand writing now, will be equal to the weights for each of the stock in that portfolio multiplied by the corresponding alpha i's which you have for that particular stock. y in the overall returns which is coming from the market, which is going to effect your portfolio would be equal to w i beta i into R m bar; where, i changes for all the stocks.

And, if you consider this and expand that; what you will have is basically the beta p. So, what we are trying to replace – the alpha i’s and the weights had been replaced by alpha p, which is the portfolio alpha and the betas and the w i’s and the R m’s which you have formulated; basically, that gets replaced by beta p multiplied by so-called the return of the market. So, what you have actually done is replaced any portfolio you can formulate by alpha p of that, which is in relationship with the market as well as equating it to basically beta p, which is the portfolio's beta with respect to the market. Similarly, if you find out the variance of the portfolio; it will basically be considering of these two terms. I am going to highlight these two terms. They are almost similar; because in the first term if you see; you are taking i as 1. as 2 and so on and so forth, which is the principle diagonal. And, the second terms you are considering is basically the very simple way is equal to the covariance – covariance concept, which is of the diagonal element.

And, the third one is basically, which is with respect to the white noise. So, what we have is basically a whole amount of risk, which is coming from the market. I am not using the word market result; I will come to the later. And, one would basically the white noise risk, which we want to basically… Over which we have no control, but we will try to see that how we can basically make a control over that.

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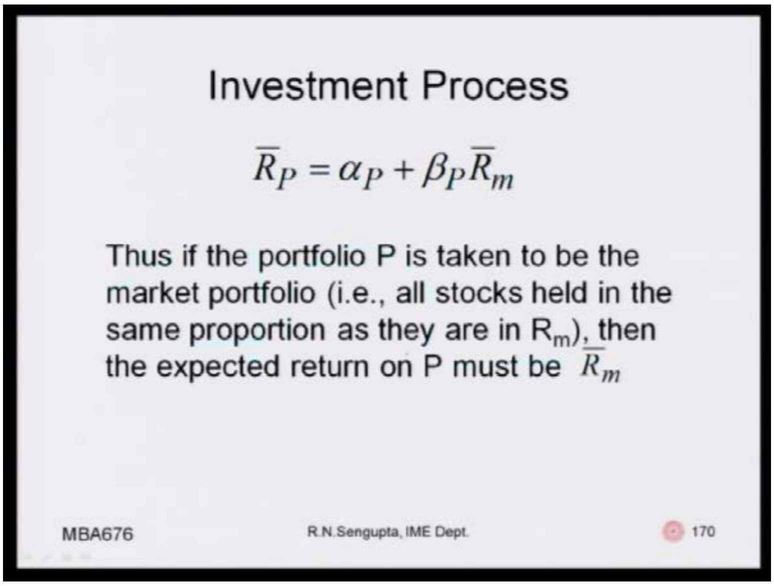

So, thus if the portfolio P is taken to be market portfolio; that is, all the stocks held in some proportions as in relationship with the market; then, the expected value of the portfolio if it is exactly equal to the market would be R m. Hence, if you are able to formulate a portfolio such that its beta is exactly going to the market; which means that, in the market to the maximum possible extent. But, in generally, that would not be true and we will try to formulate portfolio such that they are theoretical as close as possible to the market. But, in practical sense, there may be some difference; we will try to basically analyze that in detail.