Quantitative Finance

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Module – 01

Lecture – 06

So, continuing our discussion for finding of the efficient frontier, till now we have discussed that, n number of risky assets are there. Then, the next case we consider n number of risky assets plus the n-th one is the risk-free one is there. And, you can basically now understand that, given these two combination, you would also be interested to find out that, what is the case for finding out the… for these two instances, where both short selling is allowed and not allowed. So, very simply, our conceptually would be for the curve, it will be an extension with dotted lines; which we have already discussed would be for the short-selling case. And, for the straight line, it will be extended where you will basically borrow from the bank and try to basically invest that amount in trying to buy different above assets.

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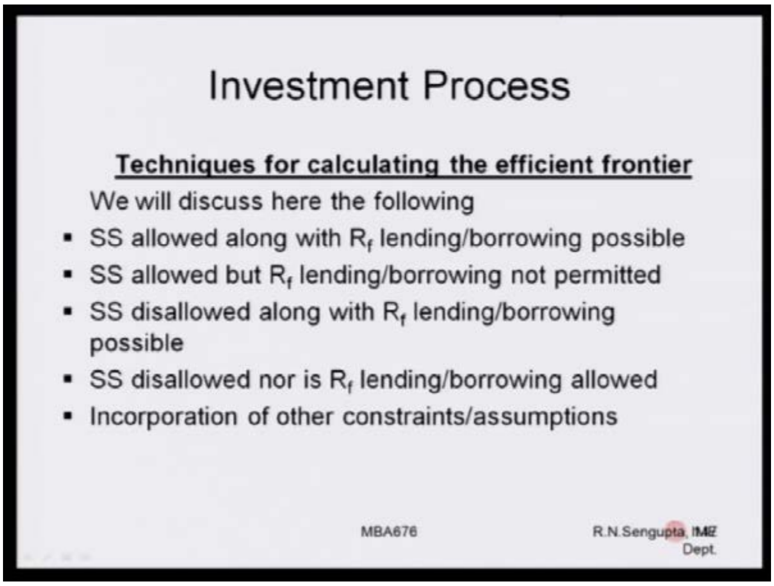

So, the borrow from the bank is basically you are borrowing from f or R f and trying to invest in point q. So, you will very briefly discuss here the concepts, which are that, short selling allowed along with riskless lending, borrowing possible short selling allowed, but riskless lending, borrowing not possible. And, in the next two instances would be short selling would be disallowed that, the short selling is not allowed with riskless lending borrowing is possible in one case and not being possible in the other case. Now, what we mean by short selling and riskless lending, borrowing? Let us make the distinction. Short selling – when I mean the word – short selling, it is respect to the risky assets only – point 1. Point 2 – when we mean my riskless lending, borrowing being possible and riskless lending, borrowing being not possible. It is a different way of saying that, short selling is being allowed for the risk-free interest rate. So, if we are allowed to lend, it means that, we are taking money from the bank, utilizing them in our basically n number of risky assets; which means in a way, short selling is being allowed for the risk-free interest rate. So, basically this is a combination of short selling being allowed. But, we are making a distinction that, for the R f and for the n number of risky assets, the overall conceptual frame work – how we analyze the problem would be different. And, the last one being, where we incorporated different type of constraints and assumptions in our problem.

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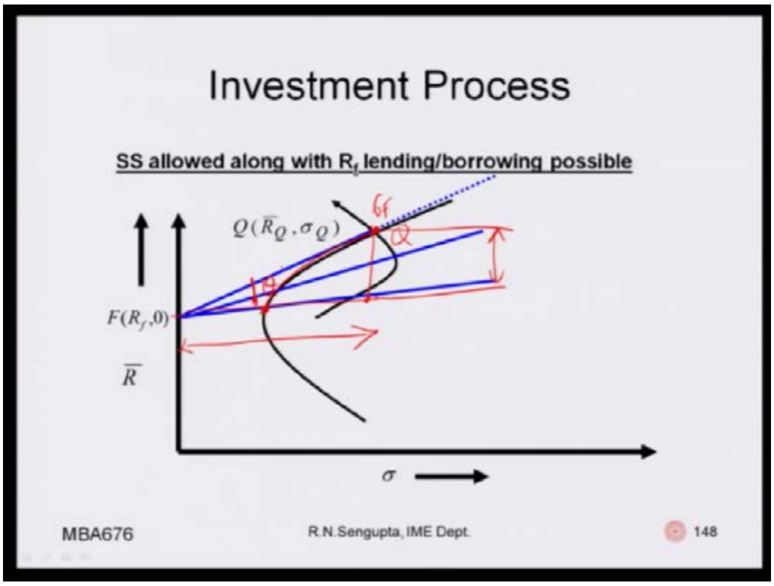

Short selling being allowed along with riskless lending, borrowing possible. So, let us analyze this problem very simply. We have already done the techniques, but will try to basically give a conceptual frame work. Consider the curve, which we have been in front of us is what? Is the feasible set. Now, once we basically use the concept of trying to find out the efficient frontier, the efficient frontier would be – let me change the color – would be from the minimum variance point till the value of Q; and then, extended depending on whether riskless lending, borrowing is there or not – short selling is there or not. Now, what we do is arbitrarily take the risk-free interest rate as given and then turn the tangent point, which we want to achieve in such a way that it turns counter clockwise. So, as you are turning it, at one point of time, it will just touch the curve, which is the efficient frontier, which you have. And, that point would be the point of Q, which we can find out using the Lagrangian. So, what we are doing is that, if short selling is allowed, then it will be just an extension on the tangent point towards infinity, which are the dotted lines. And, if lending and borrowing is possible; obviously, it will also mean the same case, where lending and borrowing is possible for the risk-free interest rate. So, which means that, the point F and Q when extended till infinity would basically consider all the cases of lending and borrowing possible and short selling being allowed for the case. So, what we will do is that, first, we may solve the curve, find out the point of Q and then join Q and F and extend it to infinity to get all the combinations of points. So, one – I mean the combination point is basically the combination the way, which you want to find; which is of more importance to us.

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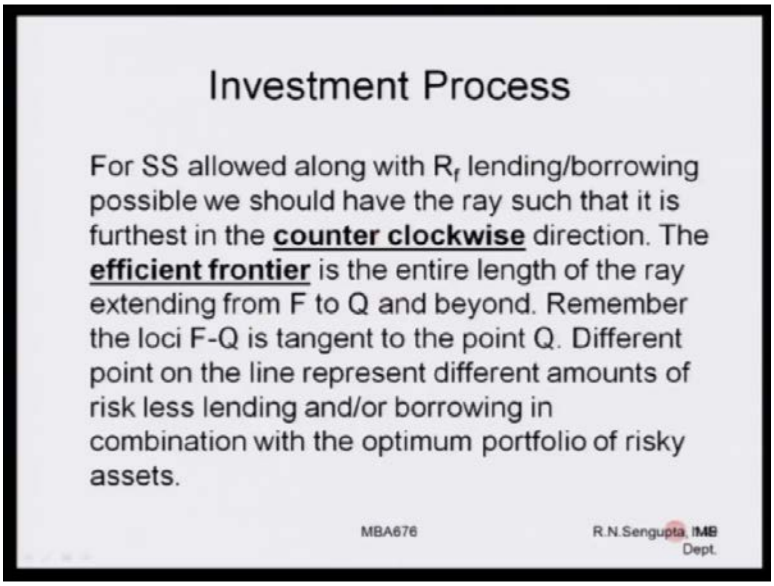

For short selling allowed along with riskless lending borrowing possible; turn it counter clockwise; efficient frontier is the entire length of the re-extending from F to Q. Remember the locii FQ is the tangent to the point of Q, which you have already found out. Different point on the line is basically denoting different combinations of the weights for this n plus 1 number of assets, which we have.

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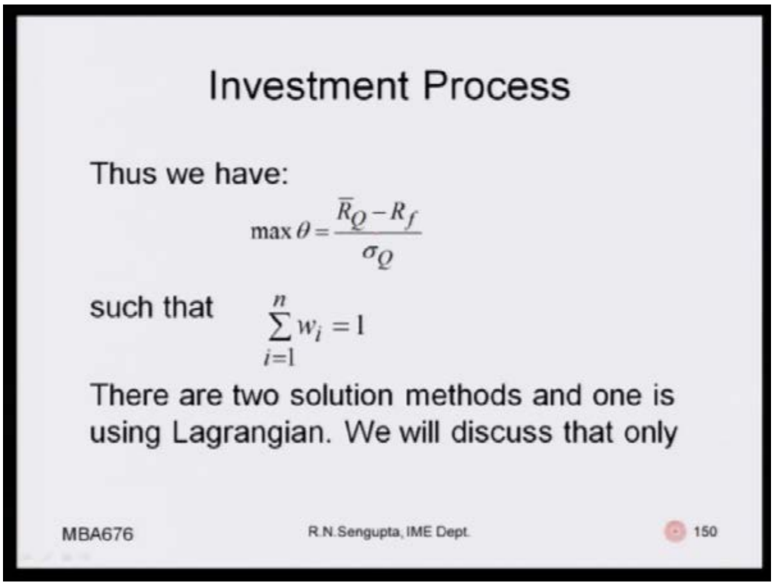

Thus what we are trying to do is maximize the tangent values. If you see the numerator, it is basically the height, which is R bar Q suffix Q, which is the return for the Q point minus R f; that means, we go back to the slides. It is this height. And, what is the base? Base would be simply the Q – sigma Q point, which you have. So, this is this point. And, what we want to find out is the tangent theta. So, such that the weights is 1. There are two solutions; we have already discussed: one is the Lagrangian.

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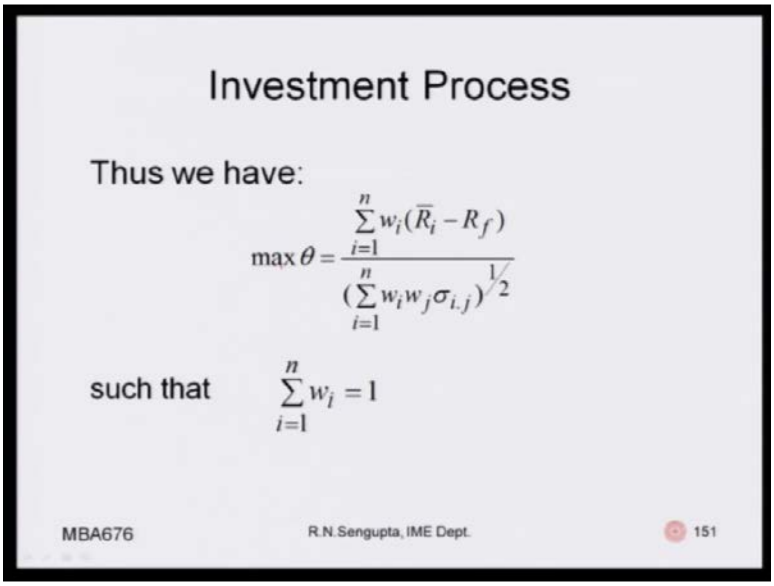

We will discuss that only for the time being as the other procedures may be little bit more cumbersome; based on which we will try to discuss that later on once 1 – the fundamentals have mean built up. Now, if you basically expand this equation, you will feel the numerator has basically the difference of the R i minus R f. So, this is very simple, because the first equation was basically point in the numerator was R Q. So, make R Q as summation or w i's into R i's. And then, you basically get the point, which is R q. And, in the denominator, what you have is basically the variance of Q point which is sigma suffix Q. So, what you have is basically summation or double summation or w i w j into sigma i j. So, and basically, you try to find out the square root of that.

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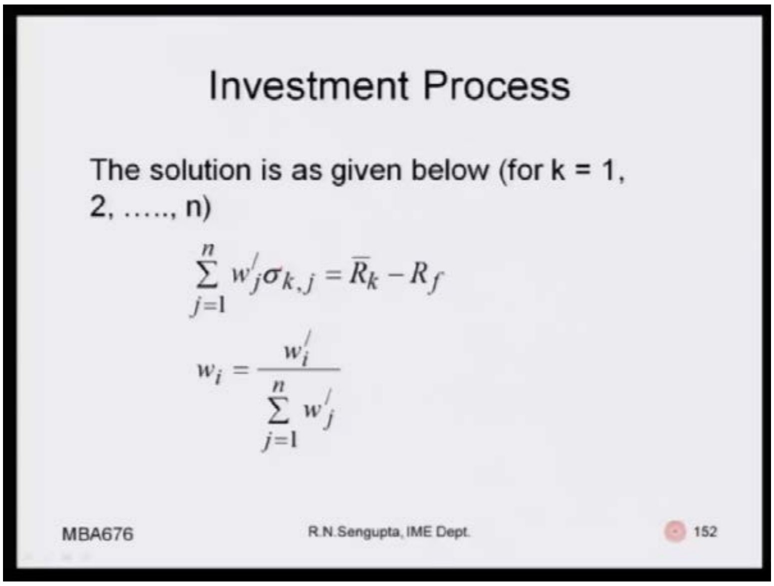

A solution is given if this in a very simple way. When you formulate as a Lagrangian… means again solve it, it will give you basically very simply n number of equations. Now, you may be asking the question that, from where do we get the n number of equations. So, solving each of them with respect to the assets would result in n number of equations. So, what we have is basically n number of equations with n unknowns. And, if they can be solved, immediately you can find out all the weights according. So, let us basically give the solution technique as such, where we have for k is equal to 1 to n, this is the number of assets, which we have; you have this equation. Now, there are two important points to be remembered. As you change k 1 to 1 – n; in each equation, you will basically have n number of terms. So, the first equation would basically have the first term, which is stock 1 to stock 1 itself. Second term will be stock 1 to stock 2; third term will be stock 1 to stock 3, so on and so forth.

Second equation would be stock 2 to 1, 2 to 2, 2 to 3, so on and so forth. And, the last one would be stock n to 1, n to 2, n to 3 and the last one would be n n. So, if you visualize, what you are doing is that, you are trying to utilize the covariance-variance matrix in order to find out the best feasible point. But, remember one thing – if you remember the concept of normalization, here the weights, which we found are non-normalized. Hence, we need to basically convert them in the normalization case as the weights add up to 1. And, based on that, we can find out the overall portfolios – weights and the portfolios return and the risk corresponding to these weights.

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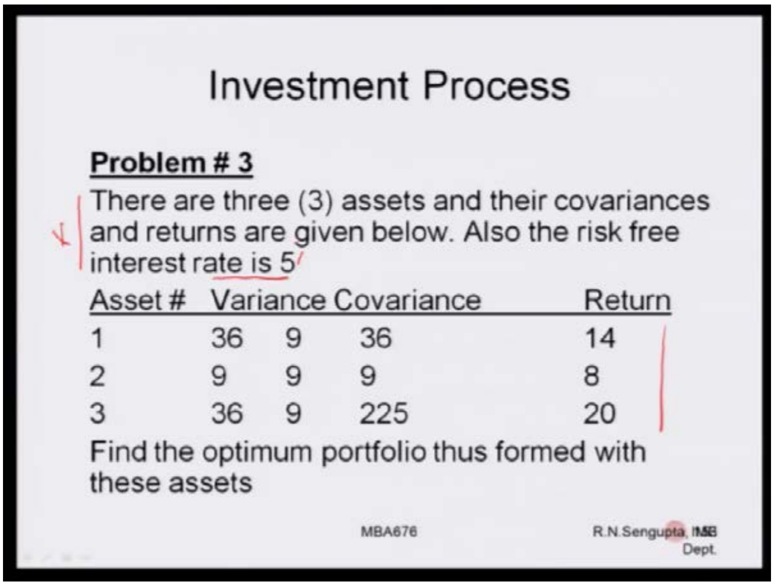

So, let us comes consider again a problem. Here the variance covariance matrix as given; the returns given as 14, 8 and 20. Find the optimum… Our main task is to find out the optimum portfolio. But, now, remember – this risk-free interest rate is also there. So, if this was not given; so, say for example, the first set was not given; then, obviously, you could have solved it easing the first technique, where you basically use the Lagrangian; find out minimum variance point; find out Q; and, basically join them depending on what is the correlation coefficient using the concept of two front theorem. Now, when in the case when you have basically the risk-free interest rate, you basically use again the Lagrangain, but with the added information that, the risk-free interest rate is already there.

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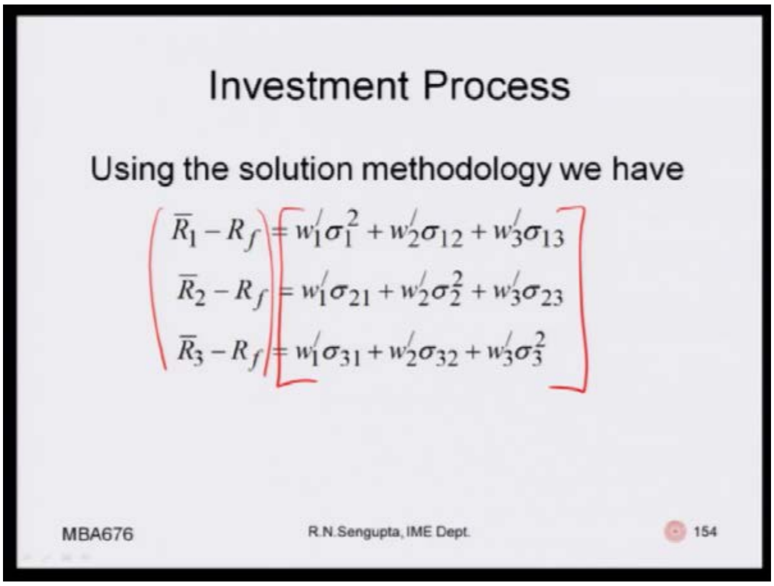

If you solve this equation using the n number of equations, which are there each and with n number of variables; then, the actual equation, which we will get is this. First equation is R 1 minus R f multiplied by the normalized weights of the first into sigma 1 square; normalized and unnormalized weight of the second into sigma 1 to which is the covariance of first two. See if you consider this graph or the matrix; if you for the time being, suppose is w 1's, w 2's and w 3's are not there, what is actual the covariance-variance matrix, which I just mentioned. So, what you are doing is that, multiplying the non-normalize weights with their covariance-variance matrix and basically equating to the left-hand side, which is basically simply a difference of the particular returns and the risk-free interest rates. So, given all the information, which is already available with us, is this R 1, R 2 till R n, R f, sigma 1, sigma 2 till sigma n n. And, all the combination – the covariances – you can immediately find out – w 1 bar to w n bar.

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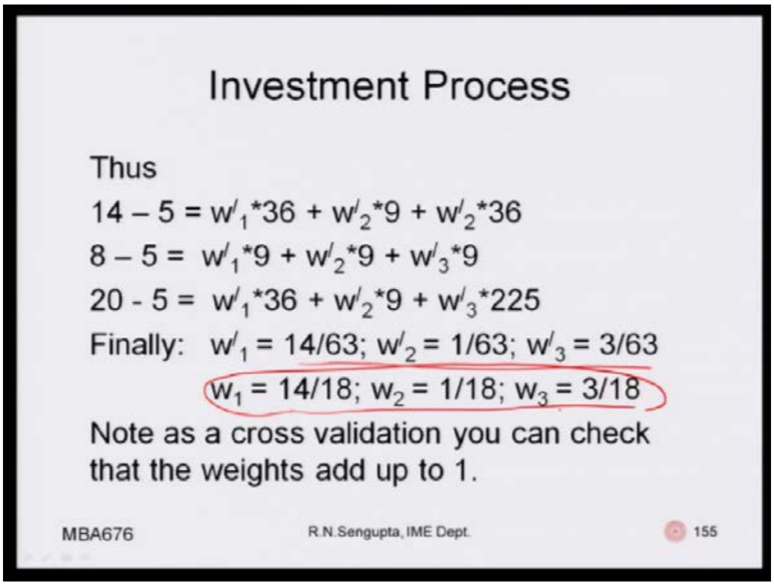

So, if you solve them, the w 1 bars weights comes out to be like this. Verify them; they do not add up to 1. So, again we normalize them and then we find out the weights, which they add up to 1. Check it for yourself. So, once the weights are found out, you multiply it by the corresponding returns of the each and every asset. And, the value which you get is basically the value of the portfolios. Similarly, we can find out the variance of the portfolio also.

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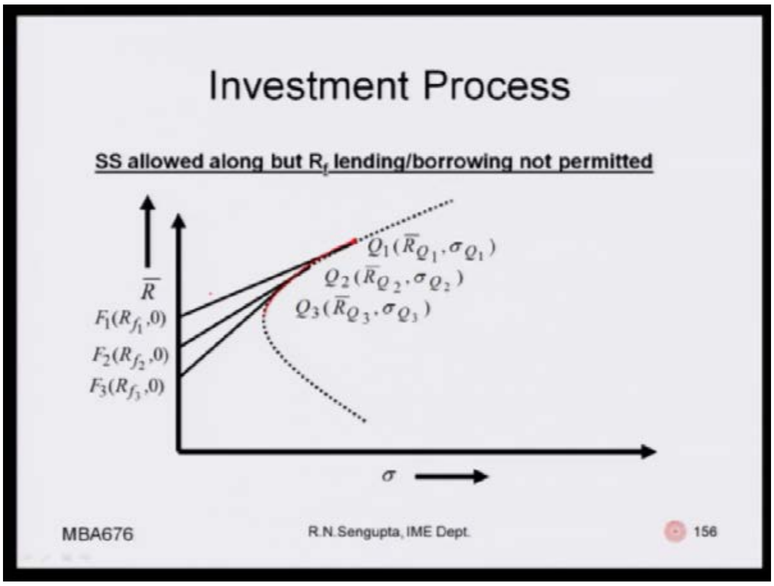

SS allowed along, but riskless lending borrowing not possible. Again, the answer is simple; you go in a round about way. First, you take the tangent point. Now, what you do is that, keep changing R f either on the upward scale or the downward scale. As you keep doing it, each of them would be tangent to say for example, different points like Q 1, Q 2, Q 3, Q 4, so on and so forth. Join Q 1, Q 2, Q 3, Q 4 – everyone. And then, ignore the R f's. So, what you have is basically the efficient frontier considering that, there is no lending and borrowing on the risk-free interest rate; that means, they basically know risk-free interest rate as such for you.

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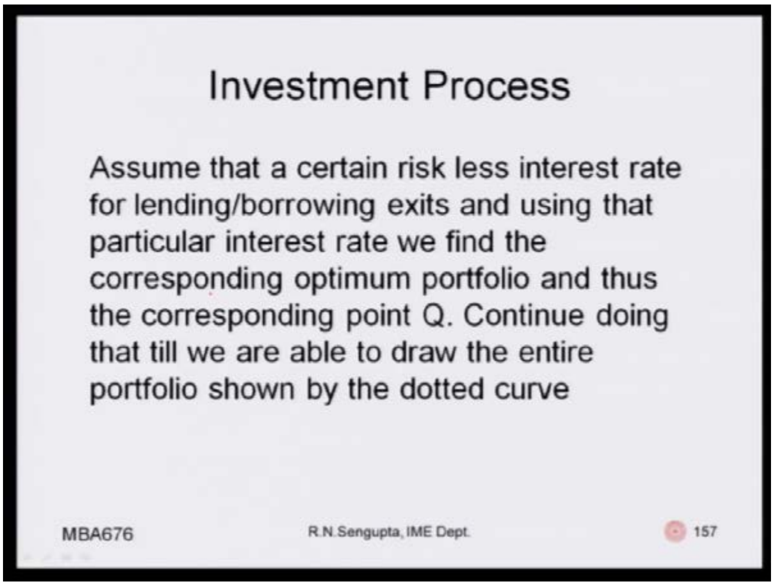

So, assume that, certain risk-free interest rate is there for lending, borrowing existing. We find out the corresponding optimum points for as we change the values of R f; continue doing so and join the points and we get the efficient frontier.

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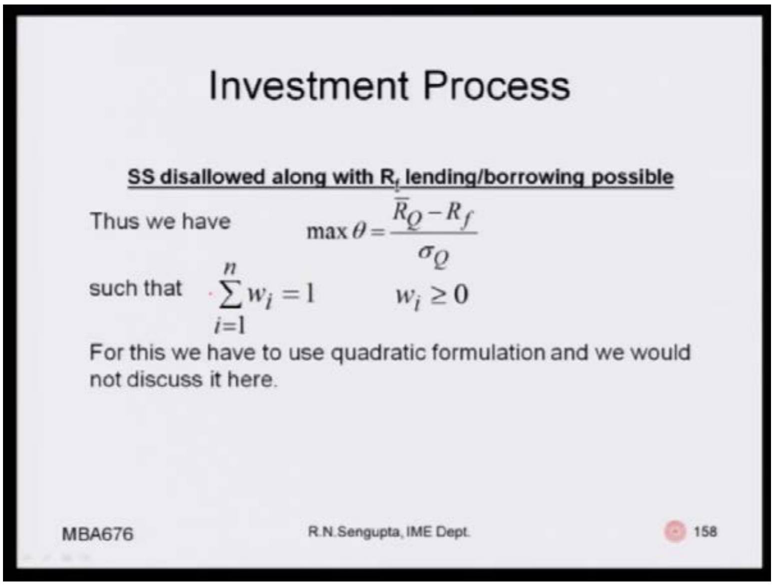

Now, the third case is riskless lending… This short selling disallowed along with riskless lending, borrowing possible, again go back to the same problem. For this, we will basically have maximization of the theta value, which is the ratio of the height and the base such that sum add up to 1; weights are basically greater than 0. For this, we have some quadratic formulation; and, we will not discuss it here, we will come back to that later on. But, in case if you want to solve them, again we can find out simple techniques – what we have already done; use that again in an extended way and you can find out the efficient frontier accordingly.

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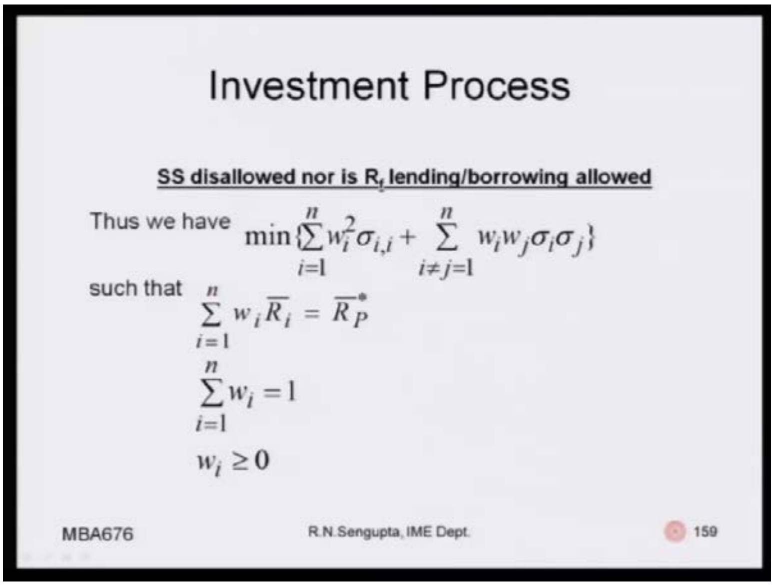

Third case is SS allowed is disallowed. Similarly, riskless lending borrowing is also not allowed. The problem takes us back to the same problem, which we have initially discussed – Markovic problem, where riskless lending, borrowing is not allowed. So, obviously, for that, we will have simple solution techniques using some optimization methodologies like simplex method and so on and so forth. And, we can easily solve them using the matlab code whatever it is and get all the answers. Another way would be to go in a around about way. Consider riskless lending and borrowing being there or riskless lending not being their or short selling being there, not there. And, basically eliminate those sets of points for which you basically… There would be no riskless lending, borrowing being allowed or neither short selling being allowed is there.

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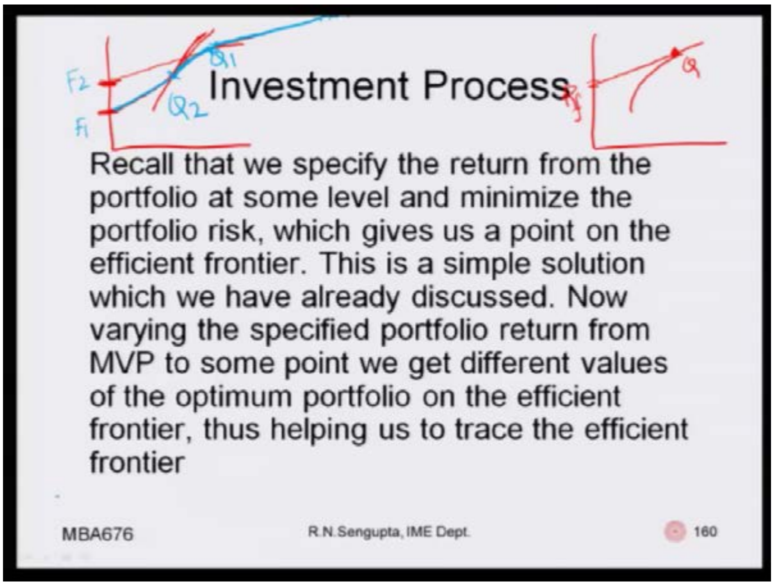

Now, before we return to the case of trying to consider other constraints in the problem, Remember one thing – important thing. And, for that, let us draw a simple diagram. Consider the riskless lending and borrowing is there and you have basically the efficient frontier for the case, where the riskless lending, borrowing is not being considered. So, initially, we had the curve; now we have basically the riskless lending, borrowing such that any tangent point will give you the efficient frontier. Now, the question is if this is Q, what you have is basically this is R f. Now, the question which many would be asking is that, if I go to the bank, borrow money; if I go to the bank and lend money, are the riskless lending and borrowing same? Answer is not so; it is definitely not the same. Say for example, if I go to the foreign exchange counter and basically I buy dollars or basically sell dollars; then, the rates are different for each of them. Similarly, if I go to the bank and I take a loan and deposit some amount as a fixed deposit or a recurring; obviously, the interest rates are different. So, if you want to find out the efficient frontier for the case with the riskless lending, borrowing are different for lending and borrowing; then, the curve can be drawn very simply like this.

For one set of points, which is R f 1 and another point is basically R f 2, which is… The higher value is for the riskless lending and borrowing when I go to the bank and ask money from them. The lower one is for the case when I go to the bank and deposit money from them. So, initially I assume that the curve is given. Now, what you mean to do is very simply like this – draw the tangent for the first case; draw the tangent for the second case. So, I am trying to use my skills artistic sense here. So, what you have is this. So, let me use a different color in order to make it understand. So, actually, what you have would be the frontier which goes straight here. So, this is f 1; this is f 2. Then, there is a small portion or the curve, which is curve; and, that will be basically the combination of two points. This is Q 1; this is Q 2; where, Q 1 and Q 2 can be found out from the case, where you had taken the Lagrangain multipliers differently for this frontier problems.

And then, if you want to consider riskless lending and borrowing being there, this will be a curve. So, which means first, a straight line, then a curve, then a extended straight line, which maybe dotted depending on whether you have riskless lending and borrowing being possible or whether short selling is allowed. So, with this, we have some lecture will now close for the case – for the simple cases, where you are considering four combinations. Riskless lending, borrowing being allowed, not allowed; and basically, short selling being allowed and not allowed. So, with this one, we will basically extend our problems for the other cases for the portfolio optimization cause later on such that we can consider whether one may assume different combinations of the objective function, different type of constraints such that it will be much easier to appreciate how optimization can be utilized in order to solve different portfolio optimization problems.

Thank you.