Quantitative Finance

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Module – 01
Lecture - 04

So, as we were discussing the Markowitz's model, adding Markowitz in 1952 again for the benefit of the students, came up with the very simple model, where you wants to optimize some objective function, based on some constraints. Now, if you remember when we were discussing the initial the utility theory, we did mention that if they returns are normal, then the utility function can be considered quadratic, and that quadratic function is very heavily utilized in all investment purposes. So, with that assumption that, utilities of human beings are basically quadratic in nature. We can safely assume, even though that is not true in practical sense, the returns are normal, and if returns are normal, then the best parameters based on which you can find out the characteristics of the human being, and try to optimize would be; point number one, would be the mean value of the portfolio, which is a combination on the mean values of the stocks. And point two is basically the variance of the portfolio, which is a combination of all the variance and the covariance of all those stock. by the word stocks I mean; all the n number of assets or the stocks which are there in the basket which you have.

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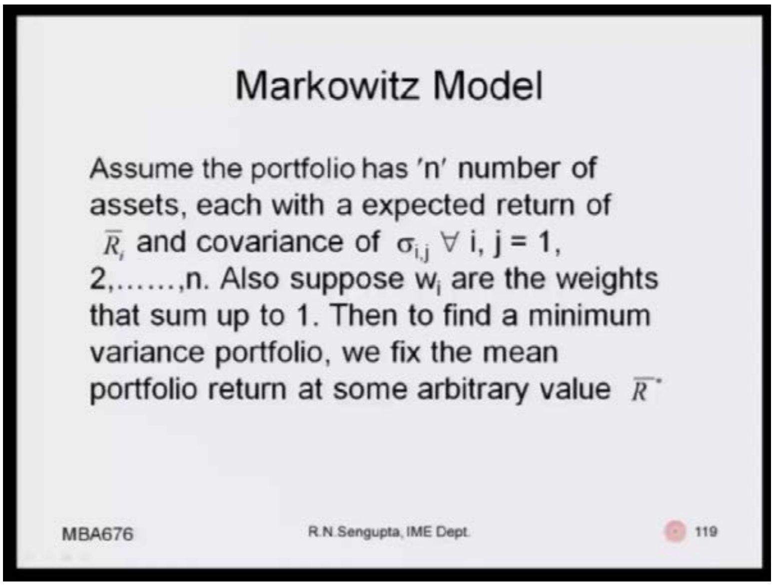

So, assume the portfolio again for the benefit of the students. It is n number of assets, each with the expected return of r. again R and r, I am using interchangeably, covariance of sigma i j. i and j running from 1 to n, and suppose the weights we are given for each and every assets. So, weights for the time being we will consider they are between zero and one, which means they are no short selling. So, you will switch over from sort selling mean they are not sort selling, not been this as that the optimization problem in it is overall contacts if you see, would not look much different, but when you solve the problem, they would be some different, not difficulty, the methodology of how you solve the problem could be different. Now let us bring for the first time, that any investor, if he or she is nonciliated; that means, more you want is, due to some reason once the overall return on the portfolio to be minimum, or greater than or equal to some r star value, which is known to us, or known to the investor, or known to the broker.

Another point would be, basically the level of risk for that portfolio, which is also known to the investor. So, you can basically two approaches; have the portfolio in such a way that the portfolio determine would be greater than equal to; that r star value, or have the oral portfolio and the ways in such a value, that the risk of the portfolio would be less than equal to the so called sigma star value which we have for n time. Again you may ask how what remaining both of them; yes, it is possible, where you can basically try to combine both the risk and return in such a way, that combination would result in either the objective function, or in such that constraint, that these two points are important which is, return being greater than or equal to r star, and risk being less than equal to sigma square star; whatever it is, values wise would be considered in the problem accordingly.

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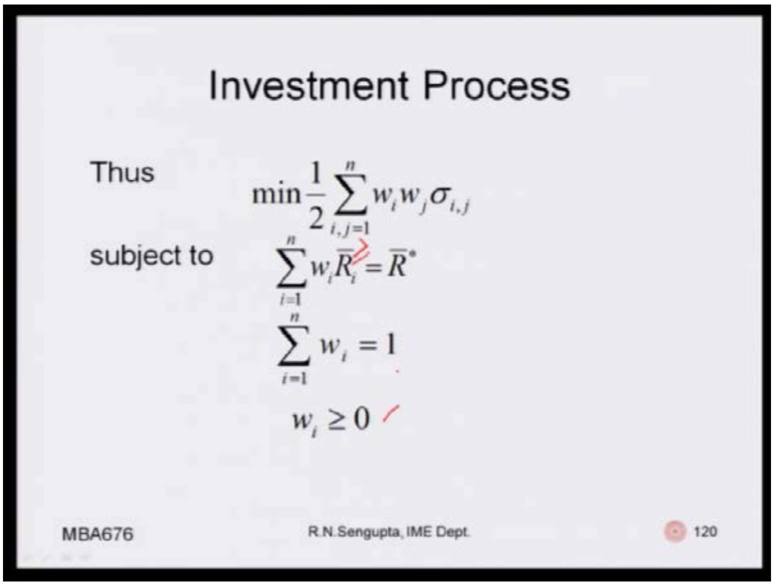

So, our first problem for the Markowitz model; very simple would be, minimize the variance. this half values has no significance is basically if you see the principle diagonal and off the diagonal element, had been multiplied by two. So, these basically being divided by half in such which is just a constant, it does not have any much consequence. subject to this constant, now the constant have written as equal to, it can be greater than equal to r. So, that would not matter much in the conceptual framework, but; obviously, the solutions would be different, which can be greater than also. the next constant is very simply is, the sum of the ways definitely one, because the ways have to one. And the third constant is, the weights of the stocks are limited between zero and one, which means there is not short selling. So, in case if we had short selling; obviously, these equation would change, the other would remains the same according to how we have formulate the portfolio given the overall basic philosophy is this; minimize the risk, subject to some constant where the return to the portfolio is greater than r star value; where r star is again know, and the sum of the weights is one, but the last one basically being, where the weights can be negative also.

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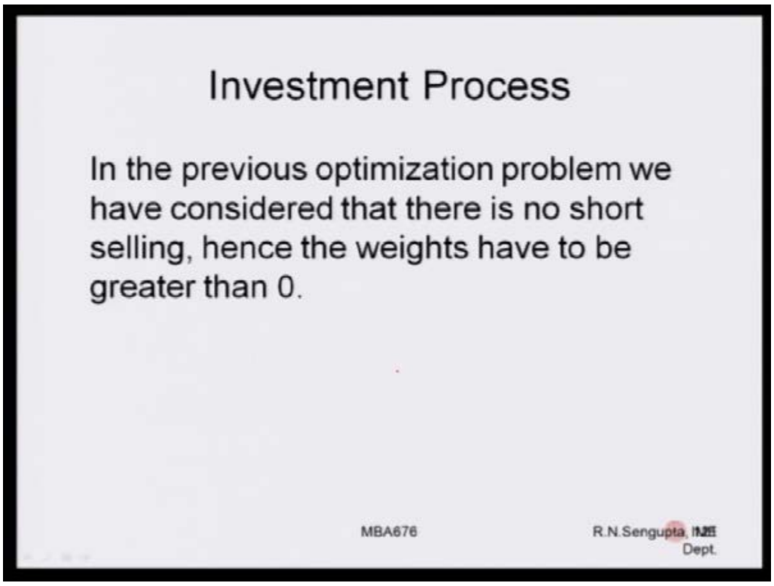

So, in the previous optimization problem as we have said, we have not considered the short selling; hence the weights have to be greater than zero. In case if their short selling, weights can be less than zero also.

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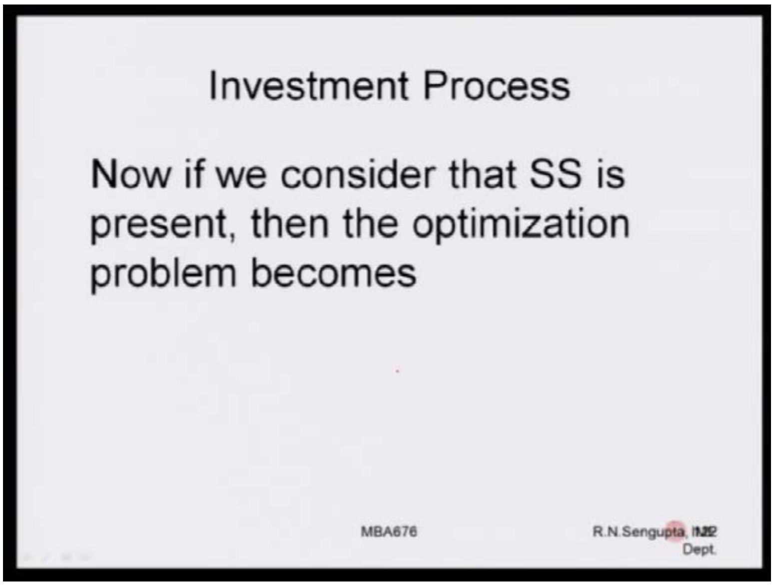

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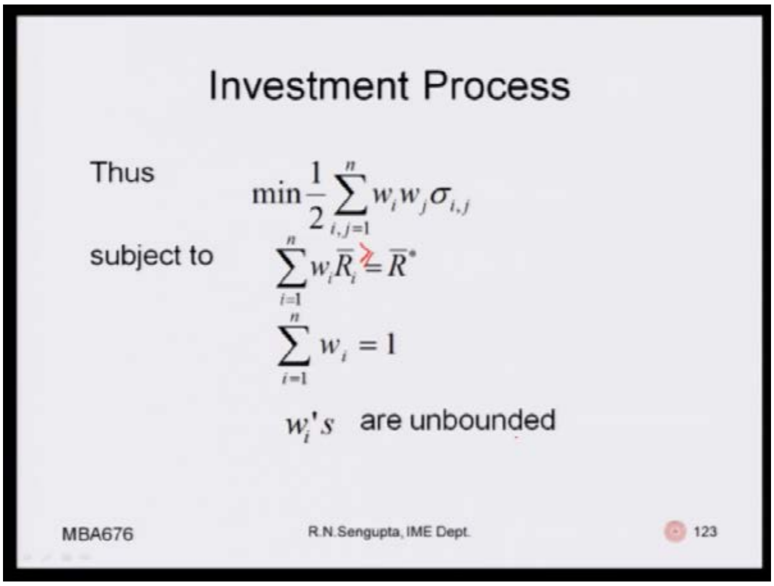

Now, if you consider short selling is present, then optimization problem becomes this. Again same thing; minimize the risk, subject to this constants with their greater than equal to, or equal to, the sums are one, and the w wise are unbounded or unconstraint.

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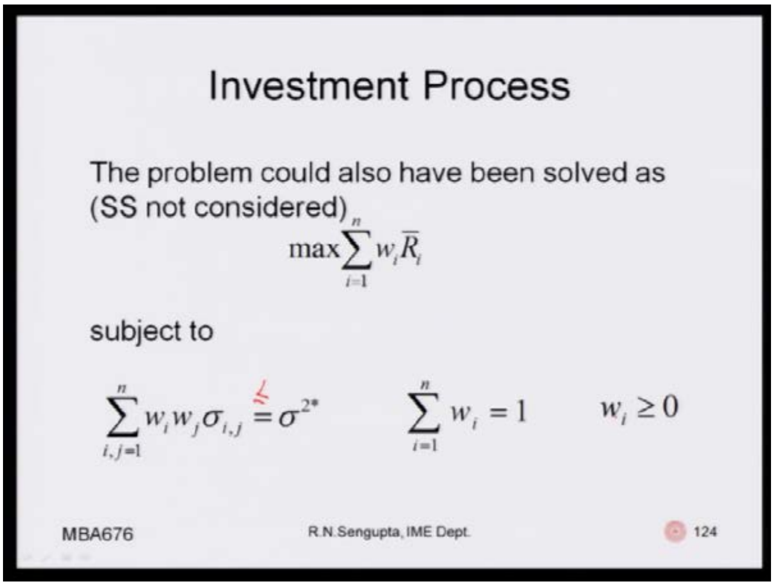

The problem could also have been solved in other way; like initially with the first two problems what, where we are trying to minimize the risk. How about trying to basically maximize the return? So, now, your problem is, maximize the return, subject to constraints where, the risk of the portfolio is equal to sum sigma star, which we were talking about or less than equal to sigma star. Weights again equal to one, which is true. Weights is greater than zero, for the case when there is no short selling involved.

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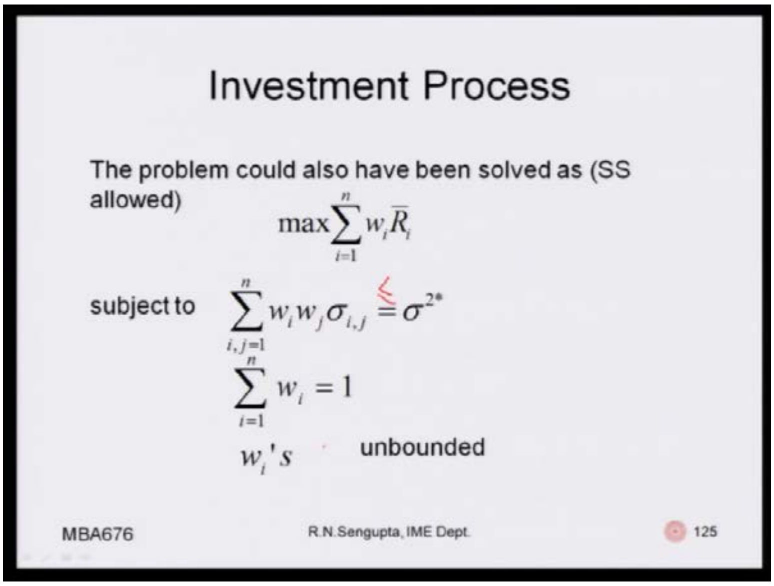

If short selling is involved, exactly the same thing; maximize the return, subject to constant where the risk of the portfolio is less than equal to sum sigma square star, which is known value. Weights is one, and w wise are unbounded.

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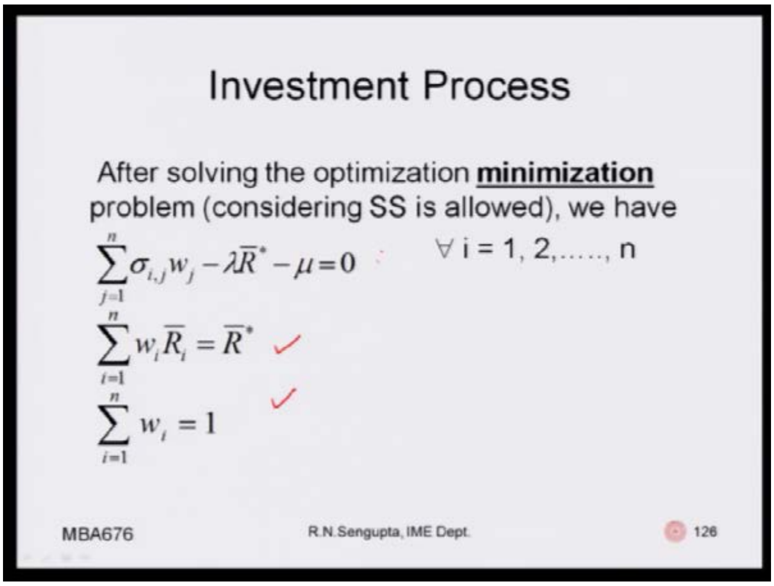

Now for the case when considering s s is allowed, which is short selling is allowed. We use a simple concept of Lagrangian multiplication. Lagrangian multiplication is very simple, where we have basically sum Lagrangian multiplied being multiplied to the objective function to all the constraints, and then we basically find the combination of the objective function, and the constraints, and try to differentiate that combination on the function with respect to the Lagrangian at each in every step. If they are three Lagrangian lambda 1, lambda 2, lambda 3, then it will be partially differentiated with respect to lambda 1 put to zero, partial differentiate with respect to lambda 2 put to zero, partial differentiated with respect to lambda 3 and put to zero, then we find out the solutions. So, now, if you consider the short selling is allowed, after the Lagrangian is formed; the Lagrangian equations like this; the first one as 2 Lagrangian which is lambda and mu, and the second one is basically, would basically bring us back to the original such that constraint, which would be basically going one circle. So; obviously, they would not be utilized. What would be utilized later, actually for the Lagrangian, is the first one which is given with Q2 Lagrangian multiplies lambda and mu. Now once we differentiate that, and basically find out the lambdas and u values, will get two extremes; one case when lambda is zero mu 1 and mu zero lambda 1, that will give us all the values which I needed in order to basically draw the portfolio which is required for us.

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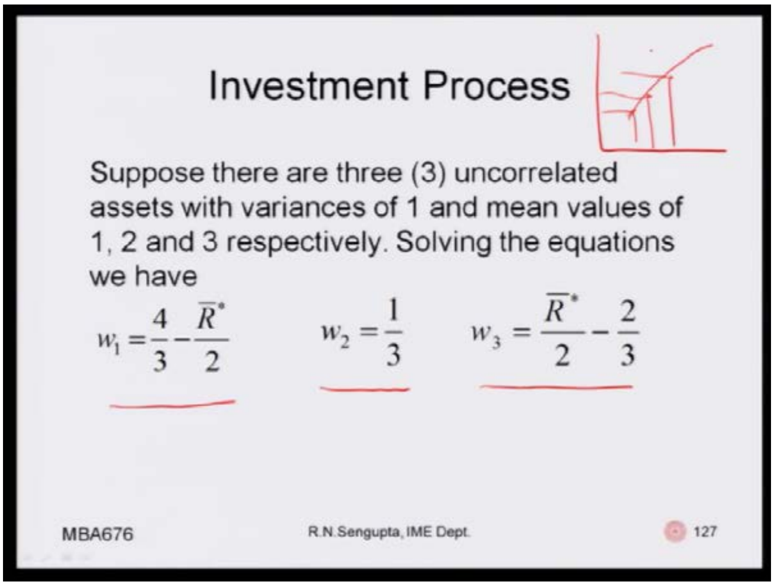

Suppose we have three uncorrelated asset with variances one, and mean value is a 1 to 3. So, if you solve the problem using the Lagrangian, the weights comes out to be as given w 1 w 2 and w 3. now notice one thing, if you end up all of them, the values is one. Now if see for example, r star value is such that r star, but divided by 2, is less than equal to minus two third for the third equation. then obviously, it will mean that the weights is negative, which means in a way that the actual original condition which you have considered, which is, there is short selling, is being again collaborated in the fact that depending on the r star value. We can get some values or weights, which in this case as I pointed out is w 3, then it would be negative w 2 is; obviously, one third which is positive. And the value of a r r star by 2, if we say basically less than two third; obviously, this value would not be negative in the sense, and it can be changed accordingly.

So, one of the w 1 and w 3 can be made negative and positive, depending and how the problem which we solved. Now if you see your question would be how do we draw the portfolio. Now look at it very carefully, in our initial problem we considered that such constants are constrains are, that either the subject to conditions has, the returns greater than equal to r star, or the risk being less than equal to sigma star. So, these are the values which we have. Now what we do is, simply like this. Keep changing r star. So, what if we have this overall Cartesian coordinate where it trying to draw. As you keep changing r star you will get different values of w 1 w 2 w 3 as a combination. So, given this values of combination of w 1 w 2 w 3, what you do is that, find out the returns for the portfolio corresponding to that value of w 1 w 2 w 3; point one. And point two is that find out the correlation coefficient between this stocks, and based on that particular coordination coefficient, you find out the variance of the portfolio.

So, what we are doing is that, as you keep changing r star value, you will get different combinations support of the weight; such that they will lead two different points, which are with each other as in the Cartesian coordinate; one point x one point y; x would be along the x axis where you are trying to find out the risk. And y would be basically those value corresponding the returns. So as you find out you would basically have different combinations of r and sigma. So, change r star you will have different values of sigma. So, once we are able to draw that, you can get a curves such that overall risk return profile, would be exactly equal to the risk return profile, which we were tried to explain in a very simple quality differences.

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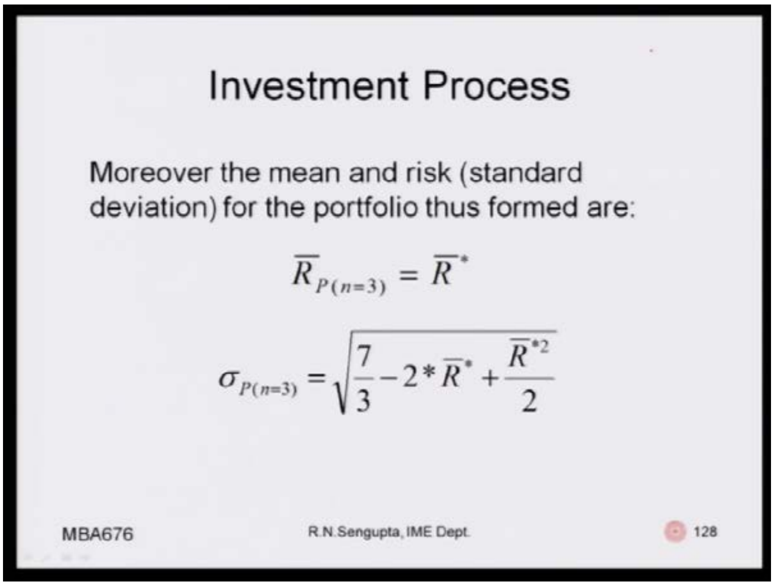

Moreover, now we want to find out the minimum risk; mean and the standard deviations. If you find out the returns for n stock, it will basically r star, and the sigma would be given by these values. So, keep changing r star, you will get different values of R P for the portfolio as well as for the sigma.

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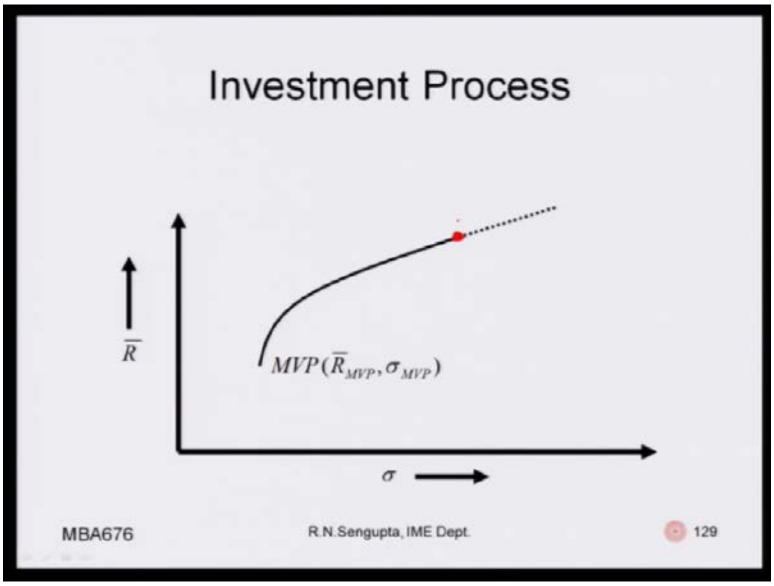

So, once we have that if you are able to draw, again I am coming back to the case, where you are able to draw the efficient frontier, with the minimum variance point and the maximum point which is there, and any extension obviously you will do, because there is no short selling, you will get the dotted lines which is there, case were short selling is allowed.

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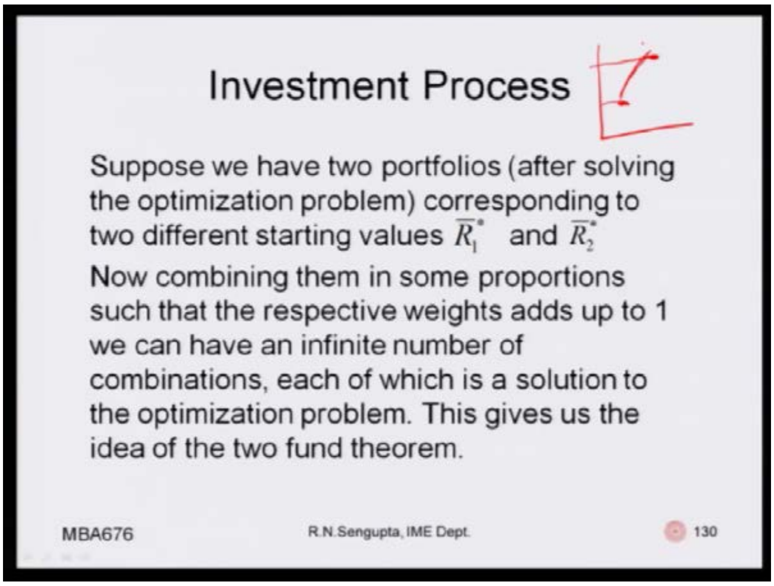

So, suppose we have two portfolios. after solving the optimization problem corresponding to different sets of returns; one is r 1 star and another is r 2 star. So, I am not using the word bar, bar is basically the average . So, r 1 bar star and r 2 bar star which is the average value. Combining them in some proportions such that the respective weights add up to one, we can have an infinite number of such combinations between these r 1 star bar and r 2 star bar, which would give all the infinite points; such that you were able to find out the variance, and if the variances are found out, considering the fact that we had already assume some r star value between r 1 star r 2 star, we can definitely get all sets of feasible region, which is basically the efficient frontier. Remember they are not. I am using the word feasible region; that means, that you are taking the best combinations such that you find out the efficient set of points in the efficient frontier, which means that if you have r 1 star here, and r 2 star here, what you are doing is that, you are basically they met in different points, and this different points will give you the set of the efficient frontier, which is part and parcel of the efficient frontier, which we have already discussed.

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Now this reset to the fact of two font theorem. So, for example, two efficient fronts of portfolio can may establish such that in any efficient portfolio can duplicated in terms of the mean of the variants, as the combination of these two. In other words all investments, investor seeking to basically have a chunk of the portfolio such that they are always efficient, they would be basically take these r 1 star bar and r 2 star bar in some proportions, depending on the risk return ((13:00)) such that they are able to formulate the portfolio which basically meets their criteria to the maximum possible extent.

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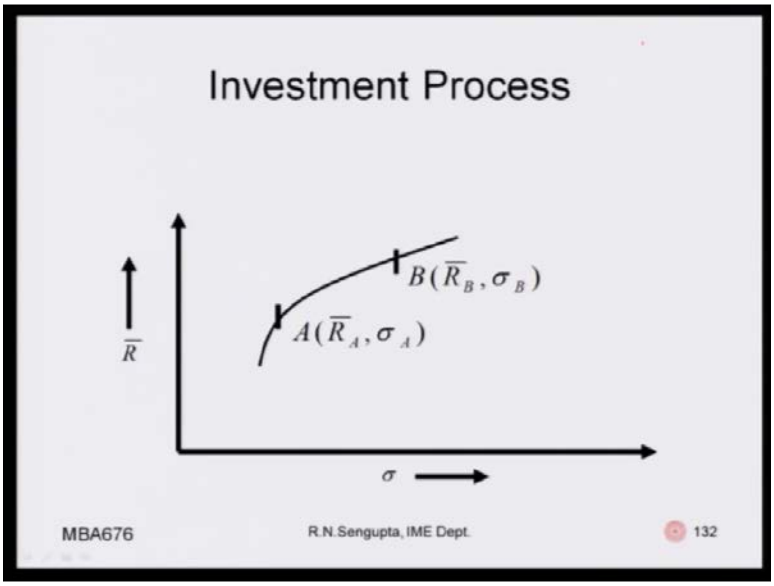

So, this is your point which you have drawn; a is basically r bar star 1, b is r bar star b, which we have basically denoted as r bar a and r bar b. Given this weights which you have found out, you can find out the sigma a and sigma b, and basically plot all the points can between them.

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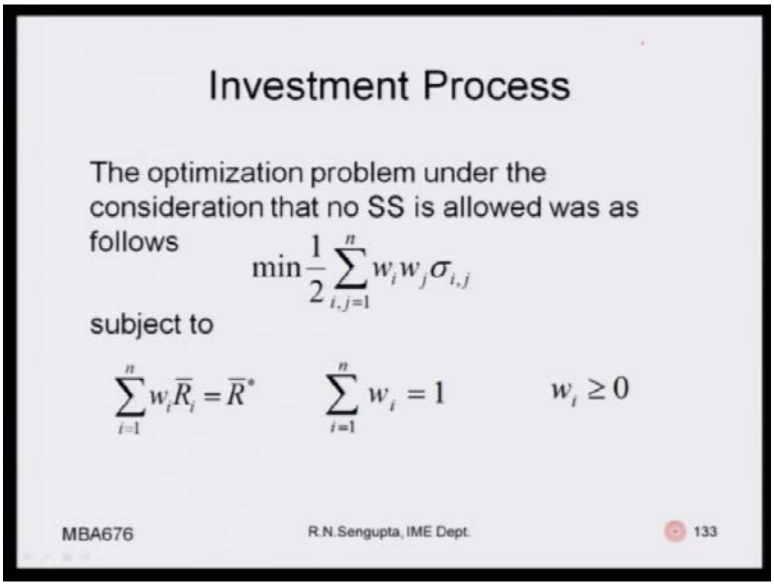

The optimization problem under consideration that no s s is allowed was as follows, which we have already discussed that.

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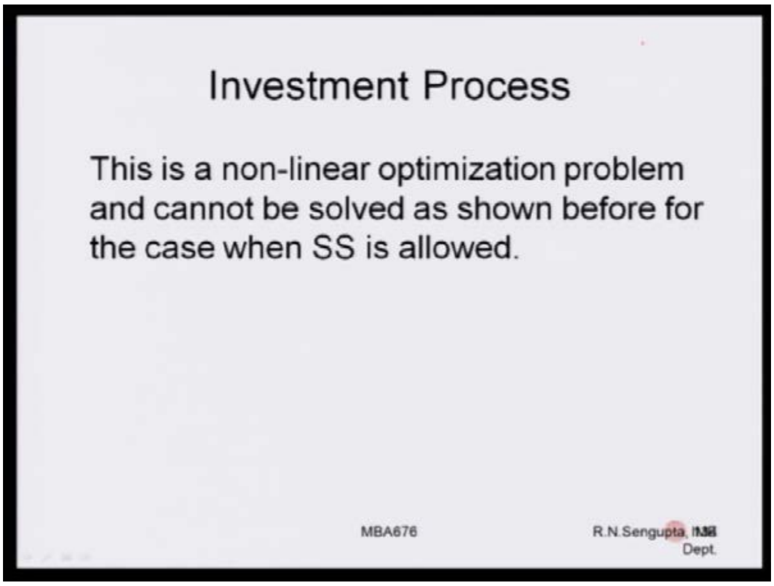

So, there is no non-linear optimization problem, and cannot be solved as soon before, as we have done for this s s case for the short selling case, but will see that how we can solved it in indirect method or solve later on. Obviously, there are different type of techniques to solve it, but we will simply solve them in a different method regime, either within another in the next to next set of classes, or say for example, by the next class which we will continue with the case of optimization in the area of portfolio management.

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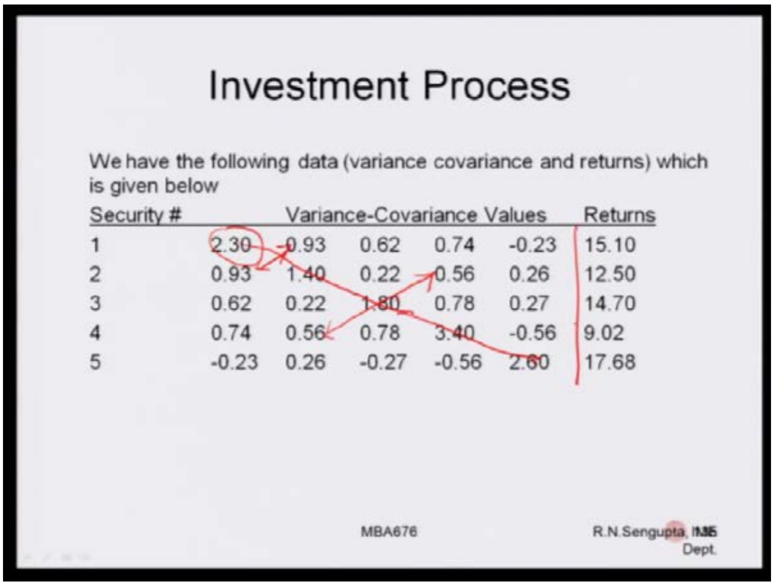

So, we are giving the falling data whether the first column or the security, second columns or the variance covariance matrix. So, if you see, the variance covariance matrix the principle diagonal which you have here. the value 2.30 is the variances, the first one simply for the second third fourth and the fifth one. So, these are the variance values, and off the diagonal in mean this one, is the covariance between first and second or second to first.` If you see these values, which is 0.56 is basically the covariance's existing in between the fourth and the second or the second or the fourth, and the last column which you have is basically the returns. So, these are all in percentage sense, but I just taking their actual value, as it is in order to basically illustrate what we have been discussing for trying to draw the efficient frontier.

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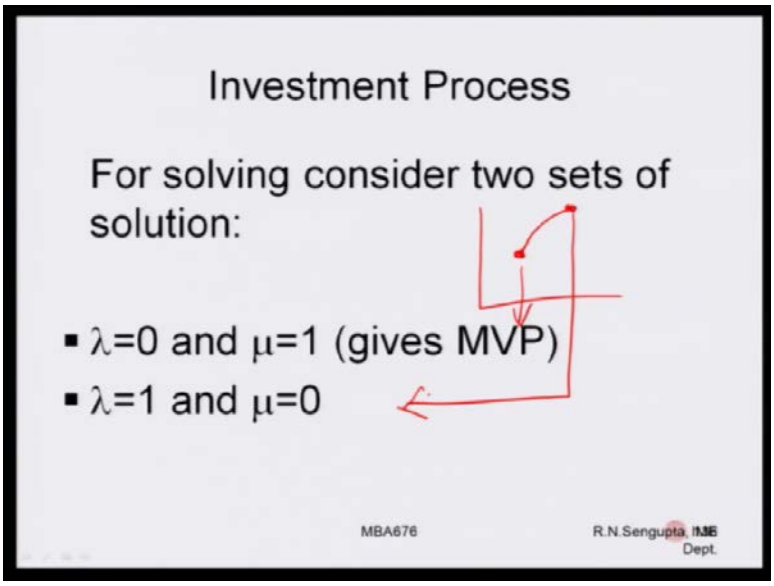

Now for solving the two sets of solution which I, if you remember initially the Lagrangian we are taken, and I mention the Lagrangian lambda and mu. It can be proved that where lambda is zero and mu is 1 if always giving the minimum variance point, and when lambda is 1 is mu is zero it will give you the other extreme point which is the for the maximum case. So, what we mean is, basically this is the minimum variance point which is here, and this point which for, which you give the maximum value, would be given by the case when lambda is 1 and mu is zero.

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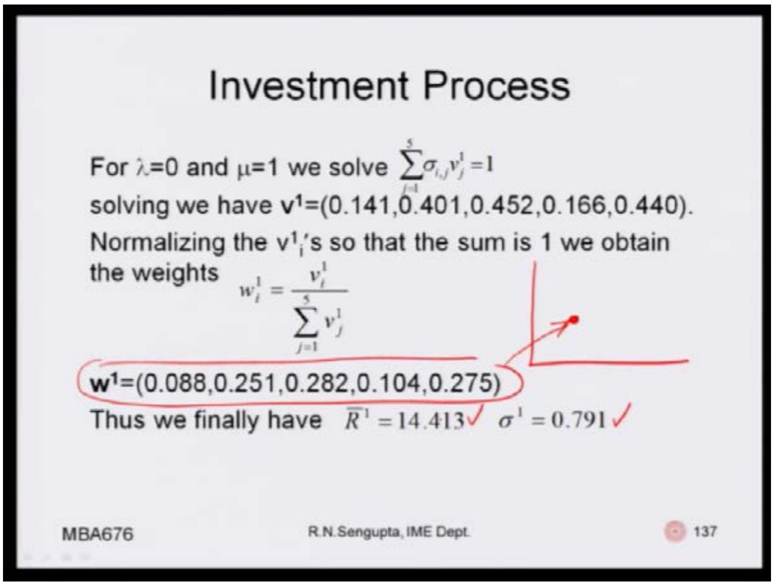

For lambda zero and mu 1, if you solve this equation, you get the set of weights which is given. Now let us pause, we have been always mentioning that the weights should added up to one. Let us add them to the add up to one. Answer is no. What we need to do? We need to normalize them. A simple concept on normalization would be, try to divide that particular value of v, which is basically again to weights divided by the sum of them. So, once we do that, if we have the weights as given for the first stock is about 8.8 percent. Second one is about 25.1 percent and so on and so forth till the last case which is 27.5 percent. Now the question do the weights added to the one. Answer is yes. Check it for yourself. And when wants we have found out this minimum variance point, what we need to do. Weights are as such are not sufficient. You need to go back in the calculation, find out the weights, multiply by the corresponding returns, and find out the returns of the portfolio. So, once we find out the return of the portfolio it comes out to be 14.13, and the standard deviation, or the square root of the variance come out to be 0.791; that is what you have done is, for the first step we have found on the minimum variance point between this five stocks which are there, where the weights are now given by this combination of vector; that means, if you invest in this proportions, the overall returns and risks are given by 14.13 and 0.791.

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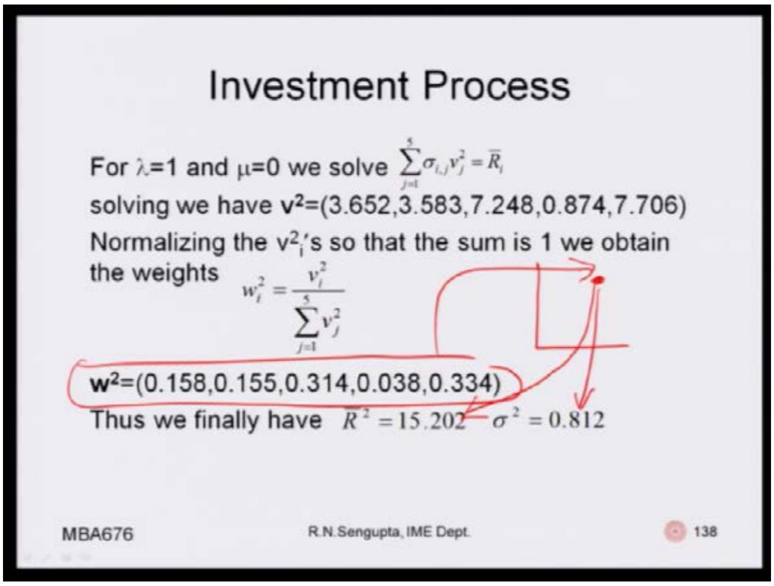

Second you want need to find out the another point, which is basically. Why we are need to doing it, because if you remember in few slides back, we discussed about the two front theorem; that means, given to financial portfolio which are there. A person can combine both of them in some proportions to meet his or her risk appetite or return appetite. So, considering that, if consider a lambda 1 and mu zero, we can solve it and find out the next vector which is v. V suffix 2, two is not a square is basically a suffix. If you add them up, what do you see the weights do not add up to one. And also very interesting thing that you find is weight are 300 also, the 65 percent also in the first case. So, again if you do the normalization, find out the weights, which is given by w suffix 2 . So, this vector is, when multiplied by the corresponding weights, the returns of the assets; obviously, will get the other extreme point; such that the other extreme point would basically have the return, as 15.20 and risk as this, where this return and risk has been formulated by combining this set of vectors which are there for us.

Thank you.