Quantitative Finance

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Module – 01

Lecture - 03

Continuing our discussion about portfolio management and how we can have the efficient frontier based on which we should find out the weights of the stocks, which are there in the portfolio. So, what we actually have there; set of informations or the random variables which are the return of the prices.

And, as you know returns are basically given by the; in the numerator, you basically have the difference of the investment for time period one minus time period zero divided by the time period zero, which is i naught. And then, you have; you find out the average return of the sample, which is given by R bars suffix one, two, three, four, till the number of stocks which you have, plus you have the variances and also the covariances, which basically can be found out, given the standard deviations on the correlation coefficient.

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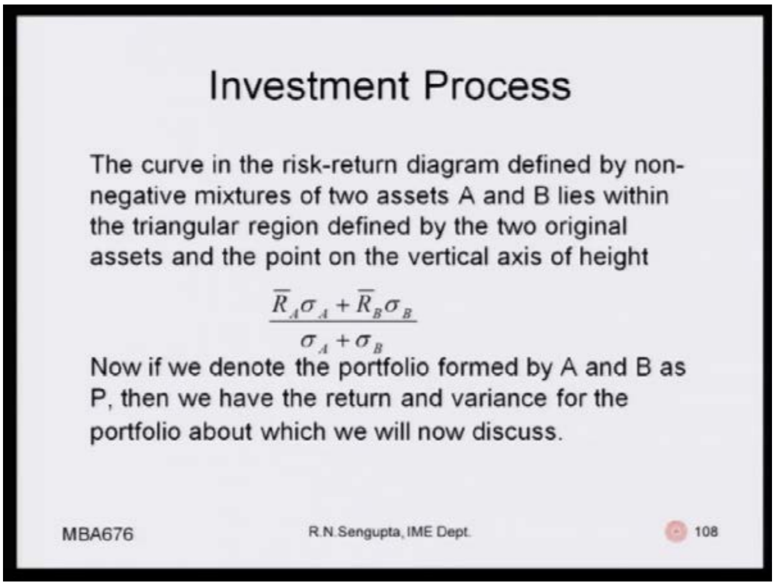

Now, our main concern and our emphasis is how do we find the weights. So, is there any mathematical method? Answer is yes. As I mention that Markowitz in 1952 in a seminal paper, he discussed about trying to find out the optimum portfolio based on some optimization techniques and some rule which will come to that in within few minutes as we discuss the problem. So, what we were discussing that you have two different stocks.

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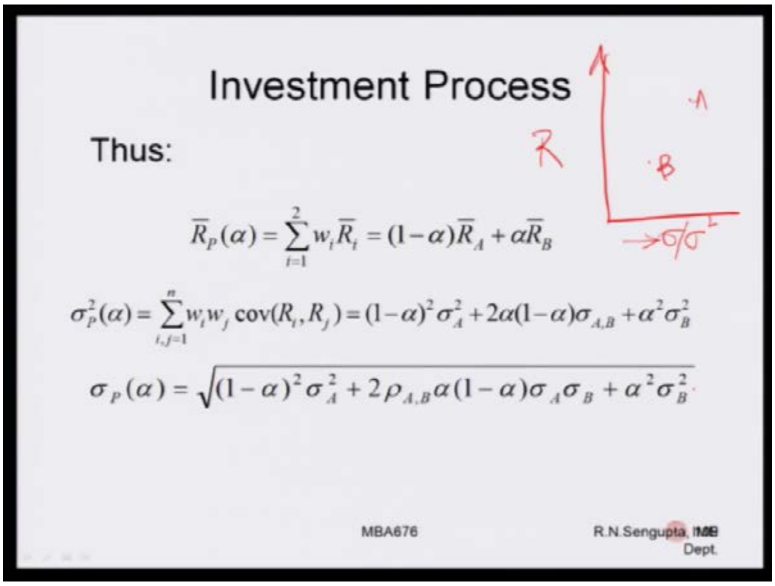

So, along the x axis you have the returns, which I will just write as R. Now, this capital R or a small r are interchangeable. So, consider that whether it is a capital R or small r, but won't make any fundamental difference in the way we discuss. And, along the x axis you will basically have the risk. The risk can be either standard deviation, which is the square root of the variance, all the variance or whatever is basically you are bringing into the picture.

Now, our main concern is to find out what is the overall shape. If you remember what we have discussed is that if there were two different stocks, where point A and point B were. If the variance of the portfolio is given by sigma square suffix p as in this formula, in the last equation which is given in the slide. And, we consider that if alpha is the overall weightages out of one, which is out of hundred amount of rupees, which we have or two hundred, whatever we have. If alpha is the weight for one stock, one minus alpha for the second stock. If the variances are given for each and every stock or two stock as sigma one squared and sigma two squared, the overall variance for the whole portfolio is given by this formula. And, the square root of that will give you the standard deviation. And, correspondingly the weights can be found out about based on the optimization problem, which I just mentioned few minutes back. We will come to that. And, our other concern is basically to find the return.

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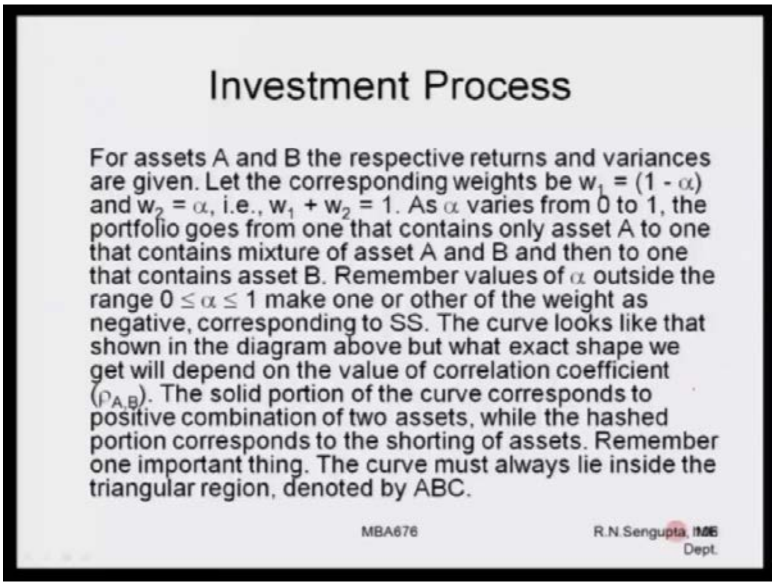

So here is, in the slides which we have. We have basically the return, which is given in by the first equation. So, if you plot them, our main question is two things. Find out the weights, given the condition that all the input values are given which are, again I am repeating; the returns, the variances, the covariances. Inside the covariances you have the standard deviations for one and two plus the correlation coefficient.

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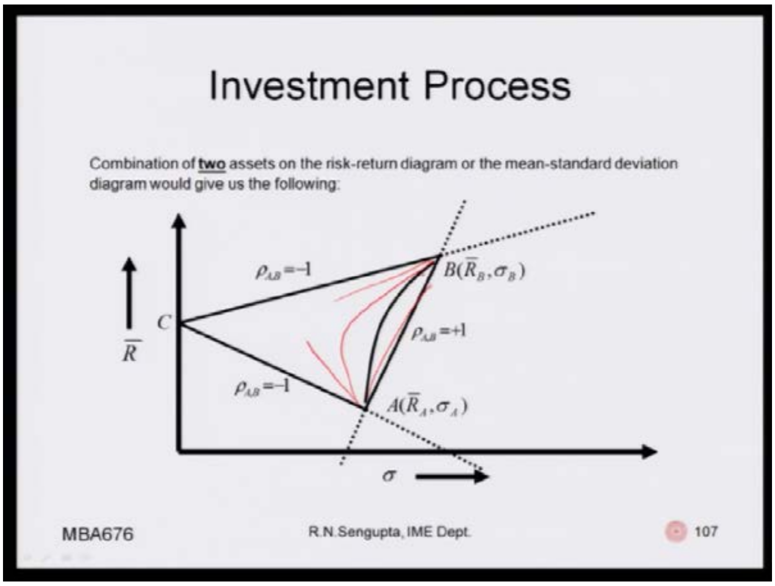

Now, if you remember; let me go back to the two slides before. If the correlation coefficient is between minus one and plus one, then obviously the overall area between which the curves will lie is basically the triangle A, B, C. And, if you remember that if you extend this dotted lines along A B and along C A or C B, it basically gives you; the dotted lines gives you the amount of short selling which is being done for each and every stock. Now, any correlation coefficient in between minus one and plus one would always be a curve. If you remember, I had drawn it like this. So see for example, this is for zero, this is minus point nine five and so on and henceforth. So, if you look at these figures or the shapes of the curves, the concave properties and the convex properties will be utilized later on.

So, basically we will consider them with such properties as we are trying to find out the optimization problem. In its true sense, according to that assumption would be much easier. So, now if we solve these equations which are there so, the first equation differentiate with alpha and then put to zero and then doing the second differential will give you the maximum deviation with respect to alpha and then find out that the second differentiation will give the minimum amount of this. That is what a main concept is try to increase; there is an increase by return and decrease the risk.

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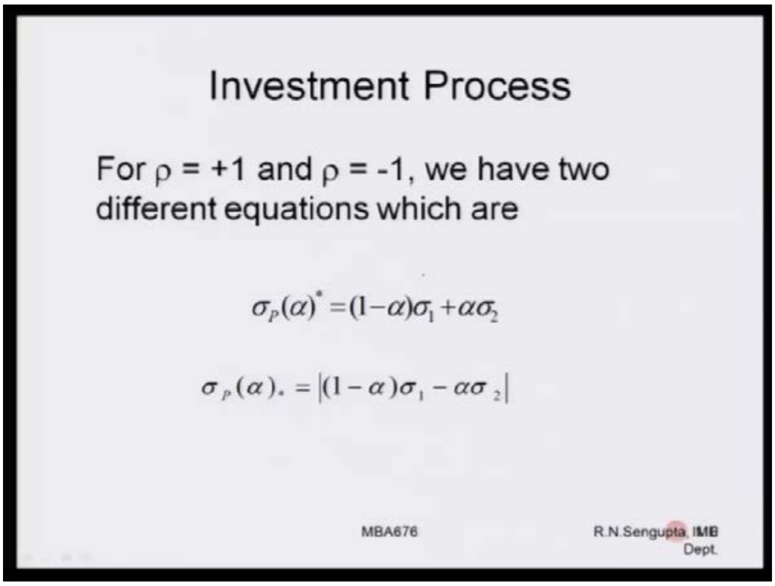

Now, if we solve this equation, the two extreme points would be for the case when alpha is plus one and one alpha is minus one. And if you plot this graphs, it will be; basically these two lines which you have, which is basically A C B for value of minus one and A B for the value of plus one. Now, let us see how that this concept can be extended. So, the graphs which I have drawn are all inside the triangle A B C. And, that would be the fundamental truth as we proceed for more than two different stocks to three to four and so on. So, now for plus one and minus one we have these two differential equations. So, the first one would basically be given by the line A B and for the mod one, which is the second one would be given for the point which is basically A B and C.

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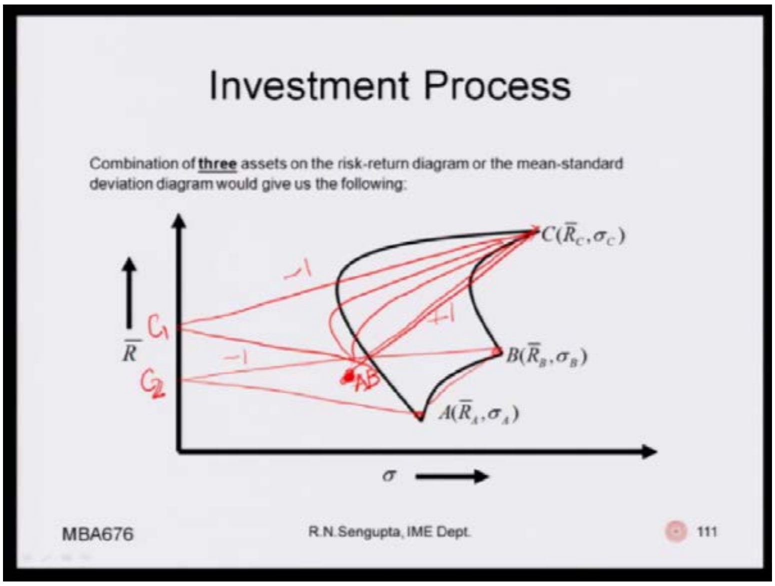

Now, let us extend this for the case when you have three assets. Consider you already have an asset of assets A and B. So, their combined shape is see for example, this triangle. This is perfectly fine. We have already discussed that. Now, let us consider the second stage. We are bringing the third investment which is an asset, which is C. So, now our main question is that how should you combine them.

So, our methodology of combining them and finding out the overall shape and the size of the curve is very simple. First combine A and B, find out a point which is basically the combination of A and B depending on some proportions and depending on some correlation coefficient existing with A and B. consider this is this point, which is the combination A B. Next, consider B and A are out of the situation; because the overall character is basically this point which we mark as A B. Now, consider there are only two points. One is C; one is A B. Again if you combine them using this same formula, now you will basically have a curve which is basically C one, A B and C. And, the initial graph which we had was basically A B and C 2. So, with this I am making a difference in C 2.

Now, if I formulate now on your portfolio which has now basically three stocks; A, B combined as A B and C. Then, again we will see the straight line joining A B would be the case for with the correlation coefficient existing between two stocks; where the first one is A B and the second one is C. Here it will be plus one and the minus one would be the line joining A B to C 1 and from C 1 to C.

So, again it can be easily seen and proved that whatever the combinations between three stocks would be it will always be curves inside this triangle; A B C one C. Continue to the next stage. Now, I have the fourth stock a D 1. So, what we will do is that combine A B and C and get a new point A B C and then in the next step combine A B C, which is one point with D. And, continue in doing this, in such a way that slowly as we bring all the n number of assets in the portfolio, we slowly get the actual frontier based on which we can find out what is the best combination.

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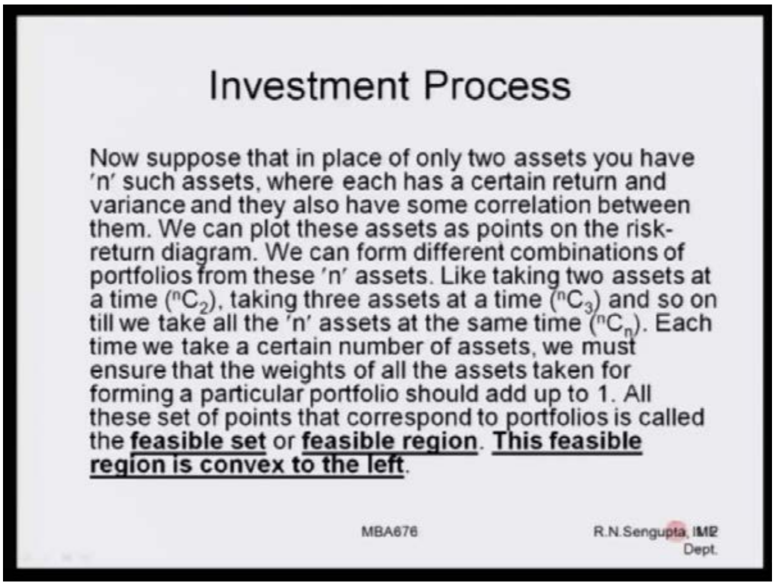

So, now suppose in place of two assets, as I mentioned we form portfolio of three assets, four assets, five assets, so on and so forth. Consider, now we have n number of assets. So, this n is the number, which means that the i value index which we had suppose will change from i one, two, three, four till the last value which is n. So, for each of this n number assets we will have what? We will have their standard deviations of the variances, which is sigma one square till sigma n square. And, after the triangle element would be the covariance of the i th and j th one, where again i and j suffixes are basically changing from one to n. And, if i is two to j, then obviously have the principle diagram.

Now, if we can form a different combinations of portfolios from this n assets, likewise taking two assets at a time will have n C 2 combinations, taking three assets at a time n C 3 combinations and so on and so forth till we have the last one, which is n C n combinations. So, all this sets of points which we get at the last stage when we combine those all n number of assets one at a time, the first two, then consider the combination of first two into the third, then combine the combination of first one, two, three into the fourth and so on and so forth.

So, you will basically get the feasible set or the feasible region. And, these feasible region would be convex with respect to the optimization problem, which we are going to solve. Now, what would you mean by the feasible set and the feasible region? When we solve any optimizing problem, there are some properties that the overall space of search which will do should basically have a convex property plus one. And, number two is that all the set of points which will be there inside that overall region, inside the triangle if you remember that triangle A B C or A B C C one and so on and so forth would be such that the overall search space inside the triangle would always be convex such that trying to utilize any simple optimization problem. That would always give you the results.

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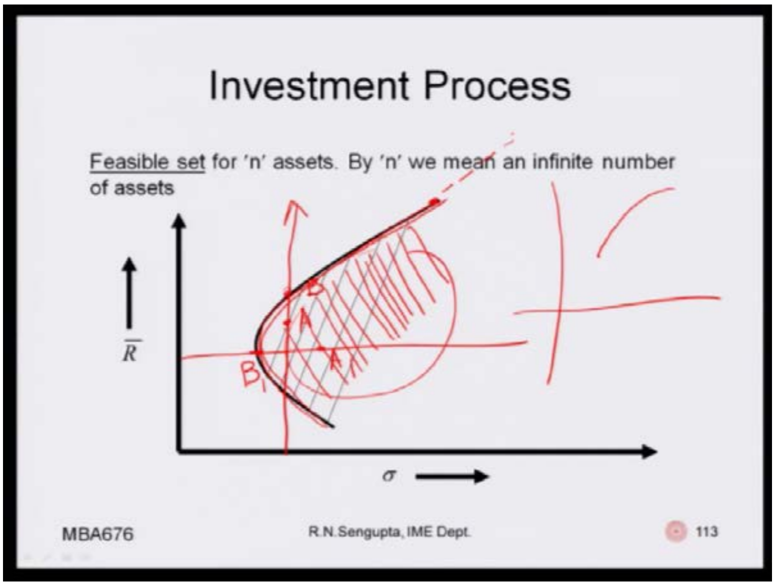

So, now hypothetically consider and it will turn out to be true that the feasible set for this n's assets would be this region. And, all the points which I have marked here, which I am marking with the red in this pointer or the black one is which is already given is the set of all contending points, which are to be considered in order to find out what is the best possible combinations of those which will give you the optimum solution based on your criteria as the investor.

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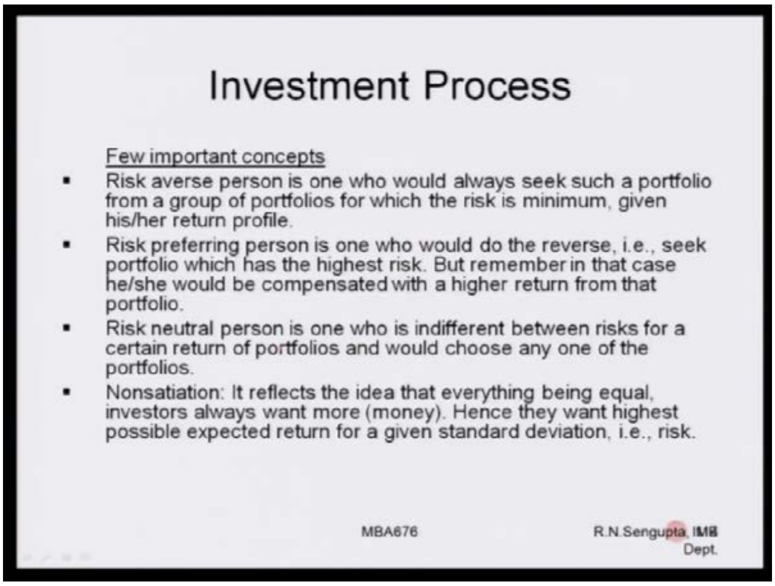

So, few important points. So, if you remember that initially when you were discussing the utility theory, we considered two important points. One was nonsatiation, that is, more I give you more you want. Another one is that every human being is basically risk averse. That means, even though people can be classified as risk lover, risk taker, then the risk in different person and risk hater will always consider the people or human beings are generally risk averse persons such that trying to find out the best possible set of points from the feasible region becomes much more logical and much more easy for us to in order to find out the optimum set of points.

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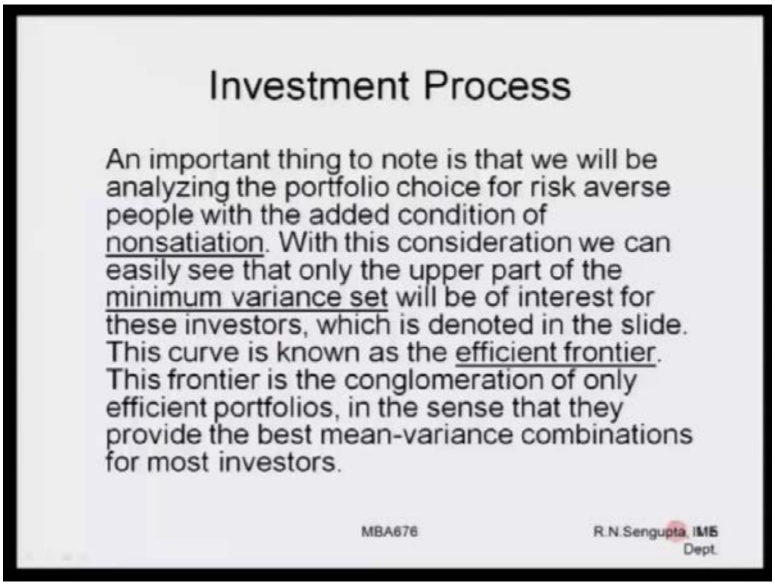

Now, an important thing to note is that we will be analyzing the portfolio choices for risk averse person with the added condition as I mentioned about nonsatiation. With this condition, we will try to find out what is the minimum variance point and what is the set of points which gives us the optimal zone. So, now what we have? Initially, we had the feasible set. Now utilizing this property, we will try to basically find out the actual efficient set of points, which gives us the best solution for our optimization technique which we are going to use in few minutes. So, this curve is known as after we have applied these properties of nonsatiation and basically risk averseness, this curve which we will get is the efficient frontier. And, main task is to find out the set of points which are there along the efficient frontier.

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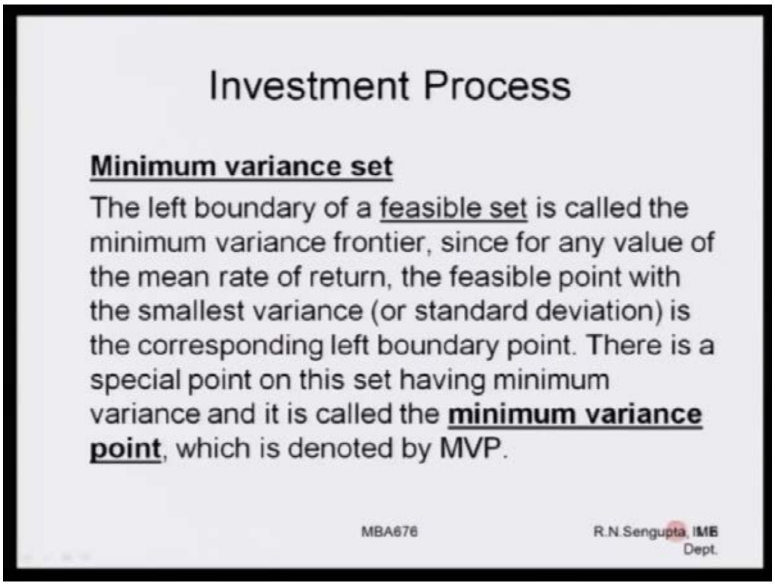

Now, what is the minimum variance set? The left boundary of a feasible set. Feasible set, let us consider that this is the overall region which you had. And, the left most curve is basically the one which will be satisfied by this two properties. How? Consider, I draw a vertical line. Now if I see the vertical line, if I go up; that means, the return is increasing; which means the more I give to a person, the more he wants which is in a way a property which is exactly mimicking nonsatiation; point one. Point two is that if I draw a horizontal line, any point to the left if I go, would always mean that the same level of return would give me a less amount of risk.

So, if I try to combine both of them, if I am at point a, see for example, if I am point at A and if I am, see for example at point B. So, for both A and B the risk is always the same. But, if I follow these two properties, it will mean that I will always, as an investor I will always try my overall portfolio to be at point a, which is one of the point in the curve. It means, technically any point below B would not be considered because at that point of B for the same level of risk we get the highest return.

Next, consider the horizontal one. Move left. So, consider this is basically A 1 and this is basically B 1. If you move left from A 1 to B 1, you will see that for the same level of return, which try to decrease the risk. That means, we are risk averse person. In a sense that we add the boundary, our level of overall risk is the least. So, if we combine both of them, the actual frontier what we will have, which is the efficient frontier would be just this curve. That means, starting from B one. If you say for example, some point here. And, obviously there would be extension in dotted lines. And, I will come to this dotted lines later on because that has something to do with the concept of short selling. So, the left boundary of a feasible set is called the minimum variance frontier. Since, for any value of the mean return of the rate of the return feasible point with this smallest variance of standard deviation is corresponding to the left boundary point. So, there is a special point for which the variance which is the least, that is known as the minimum variance point are denoted by MVP.

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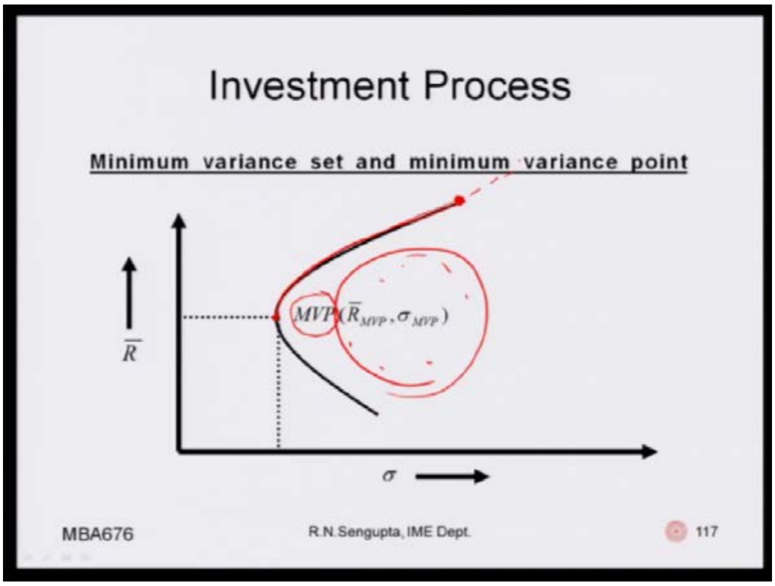

So, if we see the curve and now eliminate all the points, which are there on to the right; all these points. So, they were basically totally the feasible region. Now, I have only considered the curve. And, in the curve I should only consider the red one which is marked here; which means this is the best frontier which we have which is basically the efficient frontier. And, the point for which the risk is minimum is known as the MVP, which is the minimum variance point. The other extreme would be basically be the point for which the risk and return of the maximum. And, any extension would be coming from the fact that if you are doing short selling. So, if you remember in the triangle one, any extension for the line A B or C B or C A in the dotted concept would basically give us the concept of short selling. So, what we are doing here again is that we are trying to draw the concept of short selling again to the picture.

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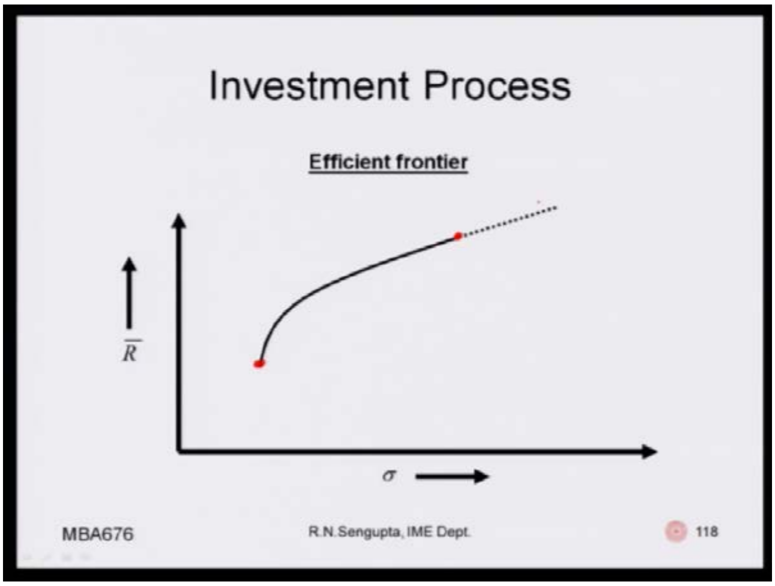

So, this is the efficient frontier as I have mentioned; the left most point. Again, for the benefit of the student I am denoting; this is the minimum variance point, this is the maximum point of risks can be drawn. And, any dotted lines as shown is basically the short selling case.

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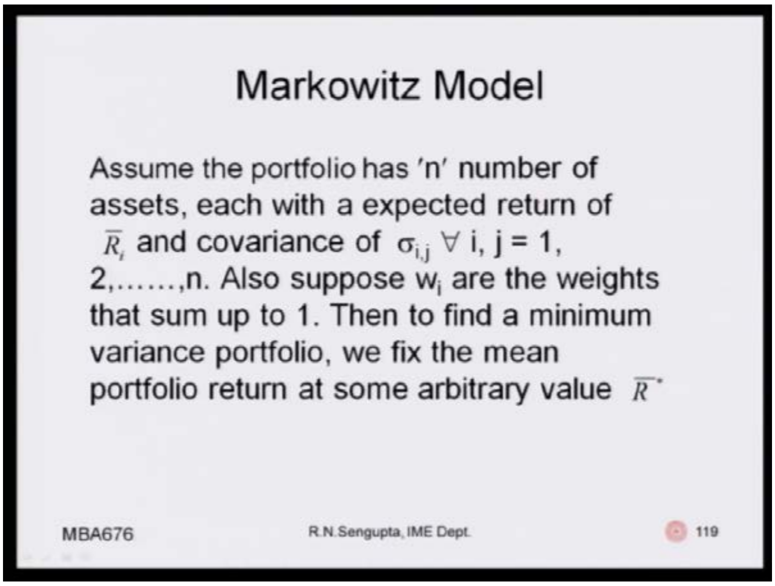

So, now for the first time if you remember we have mentioned about Markowitz, which we now start with the concept of Markowitz principle based on which we will consider the optimization problem and in the starting few slides now.