Quantitative Finance

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Module – 01

Lecture – 02

So, now, continuing with the discussion about the utility where a person can be analyzed as a risk-cover person, risk-seeker person and risk in-different person; and, based on that, how we basically analyze the problem in the case of marginal rates and what are the… That is basically the first derivative and, what are the second derivatives; how can it be utilized with respect to a utility function also, with respect to a decision perspective. Also, as well as with respect to a case when we generally analyze the optimization problem. We will consider with these very simple cases.

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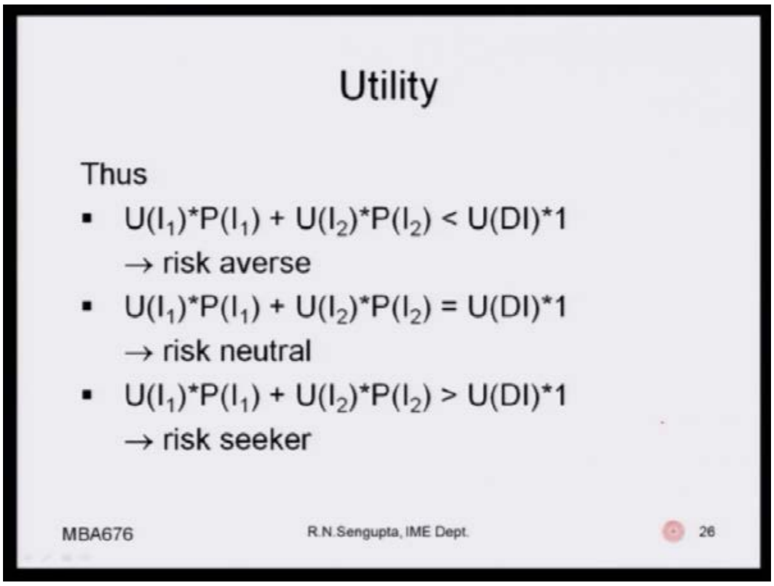

Consider I 1 and I 2 are the investment. In a very general sense, they are money. It is basically in Indian rupees, in dollars, whatever it is. And, a risk averse person would be one where he or she would basically be trying to take the decision, where the probabilistic outcome is not the case. So, if we see the first equation and see the right-hand side, what you have is basically the case where the probability is one. So, given that case, the person always thinks that the overall outcome is always fixed is no uncertainty in that. So, he or she would be tempted to take that one. If you consider the second case, which is a risk neutral person; he is basically… he or she basically trying to analyze the situation, trying to see what is the average outcome of that. So, if you see the expected values on both the sides; if they are equal; obviously, he or she takes that decision into consideration, where both the outcomes are of same expected value and, we consider a risk seeker.

The risk seeker is not seeing the oral outcome on a general sense. He or she is basically checking that, given the choices, if the risk case, which has some probability; which in the initial case, we saw probabilities being half and half. If there is a probability of P 1 and P 2 whatever it is; if the value is coming out to be higher; then the person is always having a look at that value as if the overall outcome in general sense is always positive to him or her. So, obviously, that person would basically be a risk seeker. By having said that, it should be remembered that, the human being or a person's risk attitude changes based on his or her income, based on different type of situation, based on his or her age. We will consider only the wealth perspective in our problems. Hence, the concept of risk changing depending on the attitude would only be dictated by the prices of the stock as well as the total amount of wealth or a person has in order to invest.

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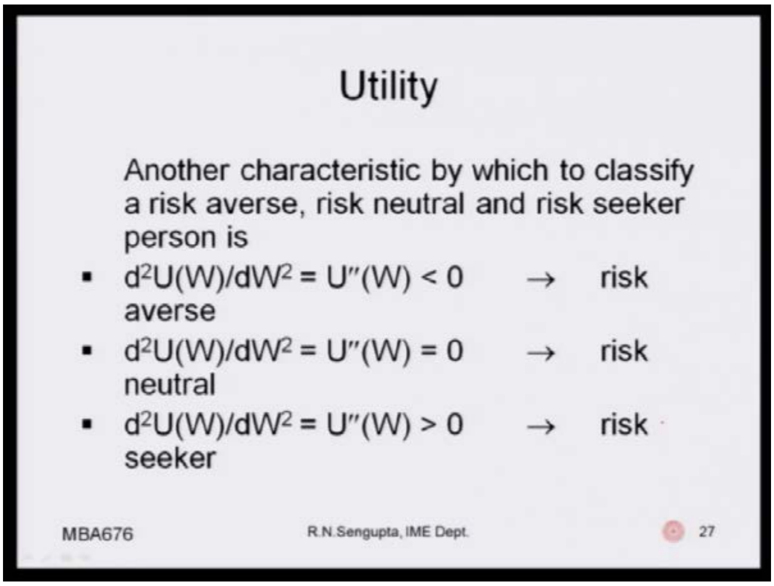

Now, consider that how a person can be analyzed as a risk seeker, risk neutral and risk averse person. So, mathematically, you will basically have two concepts: one is where which we mentioned is basically the first derivative; that is, more you want, more is given to you and obviously you are non ((Refer Time: 02:57)); that means, you would like to have more and more. And, the second one would basically be the second derivative of the utility function, which we will consider later on as using some concept of absolute utility and relative utility. We will see that later on, where absolute utility would be given by the symbol of a w and relative utility will be given by the symbol of r w. Like utilities u w; so comes corresponding d of a and w and r w for the absolute case and the relative case. So, if we see a person without going to the theoretical proofs, you will see the risk averse, risk neutral and risk seeker person can be analyzed depending on the second derivative of the utility function accordingly.

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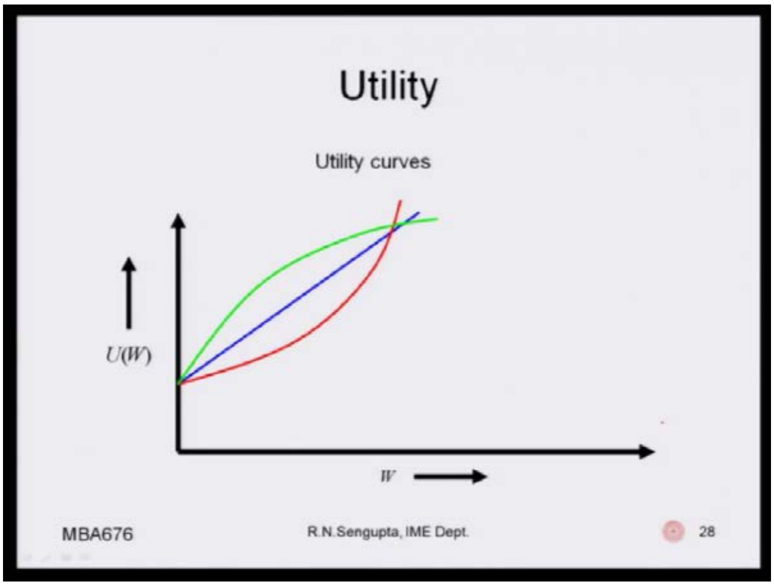

And, how they are utilized? I am going to come to that later on. So, if you see the curves; the green one is where as… For all the three cases are as wealth increases, the utility value also increases. But, if you see carefully for the green, the blue and the red, the rate of change on the function, which is u w divided by w is different for the different cases. In the green one, it increases, but increases at a decreasing rate; for the blue one, it increases and increasing at a constant rate; for the red one, it increases, but increasing at an increasing rate. So, obviously, that would basically be able to characterize the person accordingly whether the risk seeker, risk neutral or risk hater person or who does not like the risk.

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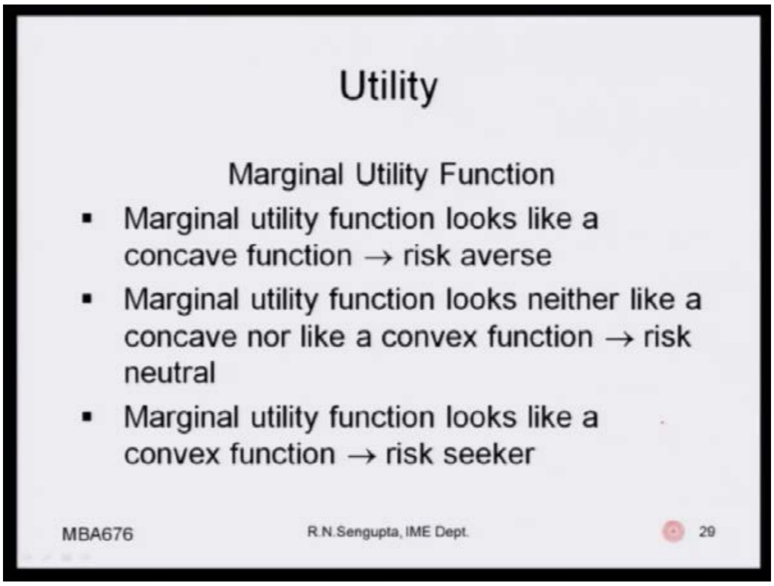

So, if we consider the marginal rates, margin rate function looks like a concave curve for the risk averse person. Similarly, for the person who is risk neutral, it would be a not a concave or a convex curve, which is basically the straight line, which is the blue one. And, the marginal rate looks like a convex curve is basically risk seeker person; that means, as the wealth increases, the overall risk perspective also changes accordingly and on the positive relation. Now, if we see, the marginal rates in… If you see the curves, which I mentioned; for the risk averse, the marginal rate is increasing and decreasing rate. Marginal rate is increasing at a constant rate for the middle person who is not a risk seeker, not a risk hater – in-different person. And, for the third case is marginal rate is increasing at an increasing rate for a risk seeker.

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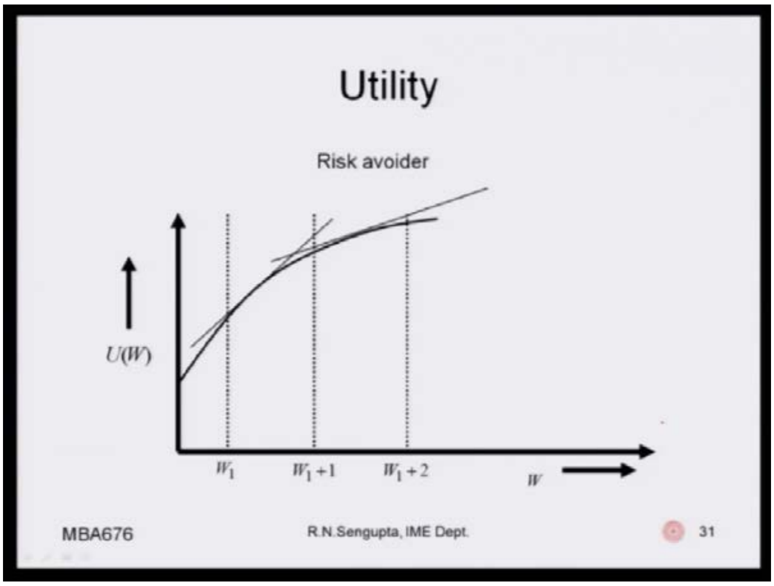

So, with… This is as you see the dy dx or du dw changes its slope. So, that means it is basically becoming more and more towards 0; becoming almost horizontal to the x-axis, which is w.

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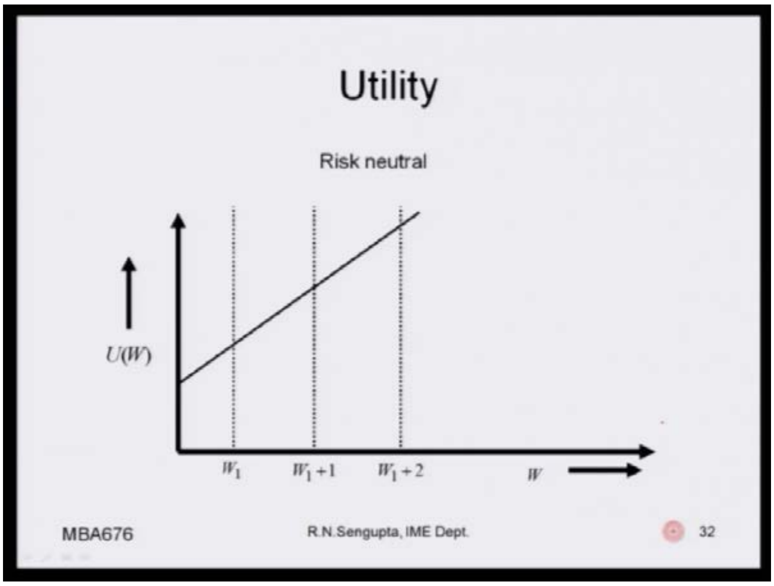

If you see the risk neutral person, again it is constant.

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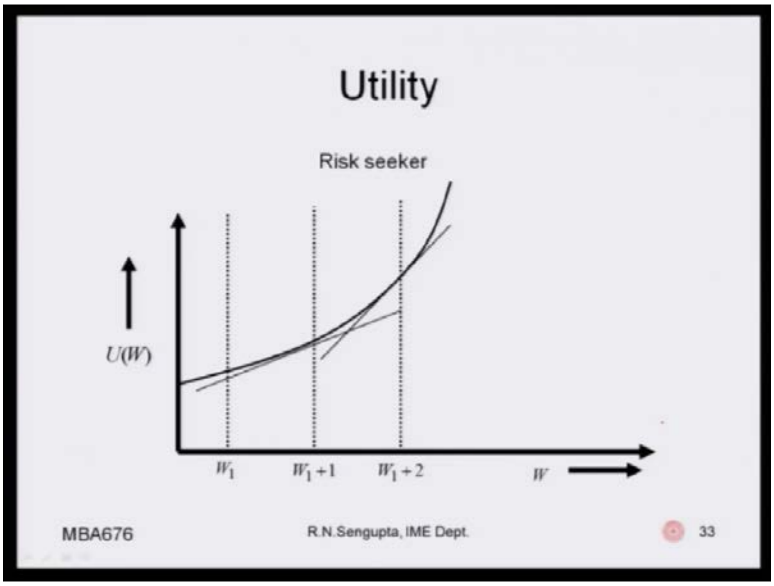

And, if you seek a risk seeker person, it is increasing and increasing rate; so, that means, it is basically becomes 90 degrees – dy dx, function which is basically du dw.

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Utility

Few other important concepts

Condition	Definition	Implication
Risk aversion	Reject a fair gamble	$U''(W) < 0$
Risk neutrality	Indifference to a fair gamble	$U''(W) = 0$
Risk seeking	Select a fair gamble	$U''(W) > 0$

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Now, few other important concepts, which we utilize later on also, would be a risk person, risk averse person rejects a gamble; gamble means basically an outcome, which is two probabilistic trees or arms. So, implication is the double derivative is less than 0 for a risk neutral person; double derivative is 0. And, for a risk seeking person, double derivative is greater than 0; that means, a person rejects a fair gamble is indifferent between a fair gamble. And, the other third case is risk seeking is basically select a fair gamble.

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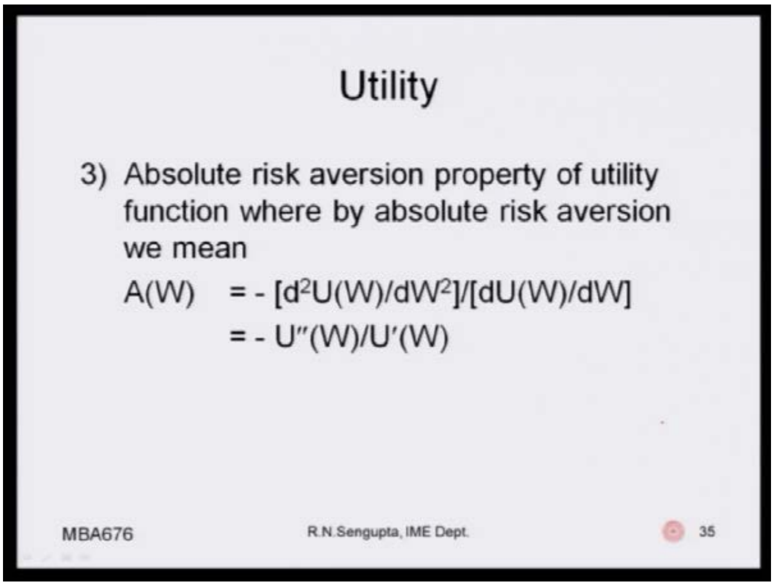

Now, there are two terms, which we will consider is as I mention is the absolute risk aversion and the relative risk aversion. So, without going to the proofs, I just state the formulas as it is. So, A W would be the ratios or double derivative of U W divided by U W's first derivatives and how its implication; we will come to that later on.

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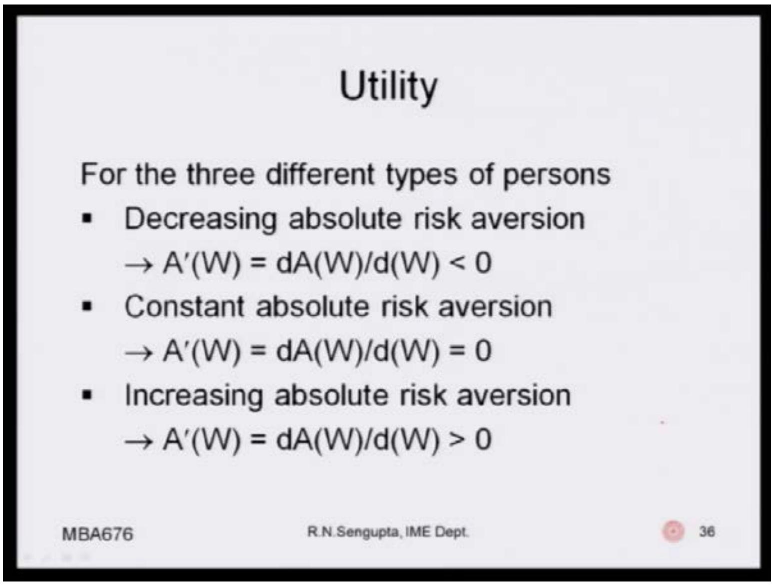

So, for the three different types of persons; if the first derivative of A W; that means, the equation, which you have is minus of that, that the ratios. If you take the derivative of that and the values are less than 0, equal to 0 and greater than 0, the person can be clubbed as risk aversion, a constant risk; that means in different and the third case would be basically person who has an absolute risk aversion, which is increasing on the general trend.

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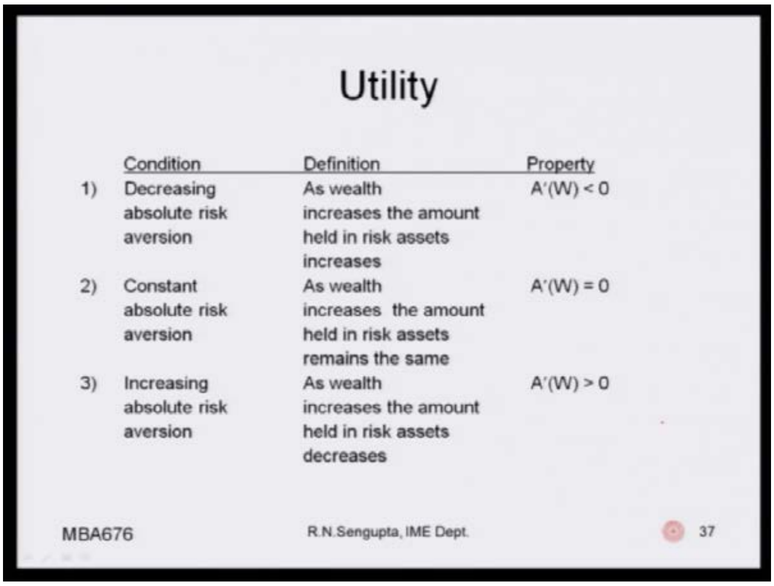

So, now, if you have A bar, which is first derivative and all the derivatives would be taken with respect to W. So, if A dash or A bar is basically less than 0, which you see in the last column, is equal to 0 and basically greater than 0. So, it is obviously… It will mean… On the absolute sense, as the wealth increases, the amount held in the risky assets increases, which are stocks is basically increasing for the person whose absolute risk aversion is decreasing. Similarly, for the other two persons, it would basically be equal to 0 and greater than 0 would basically quantify that person as constant absolute risk aversion and increasing risk aversion.

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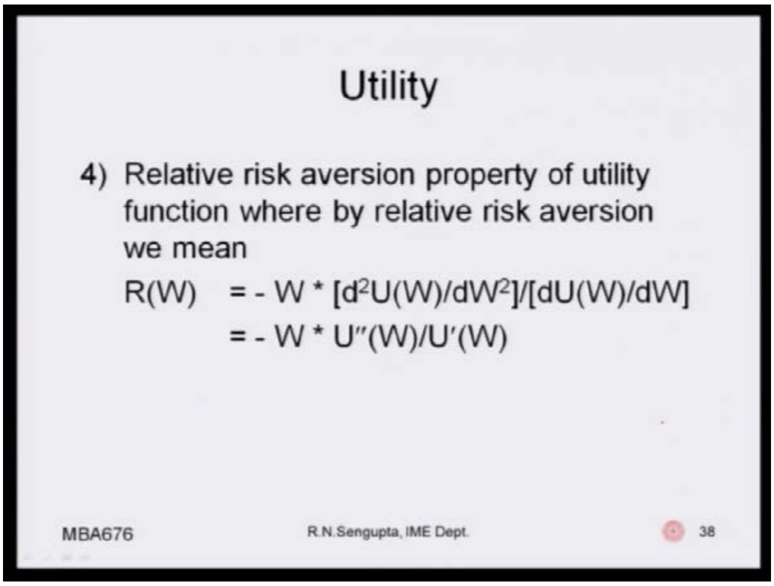

Now, another property, which will also be relevant is the relative risk aversion. So, if you see the relative risk aversion, it is basically minus W into the terms, where the ratios are exactly equal to the A W part. So, what you are doing is that, you are multiplying the A W with the value of W.

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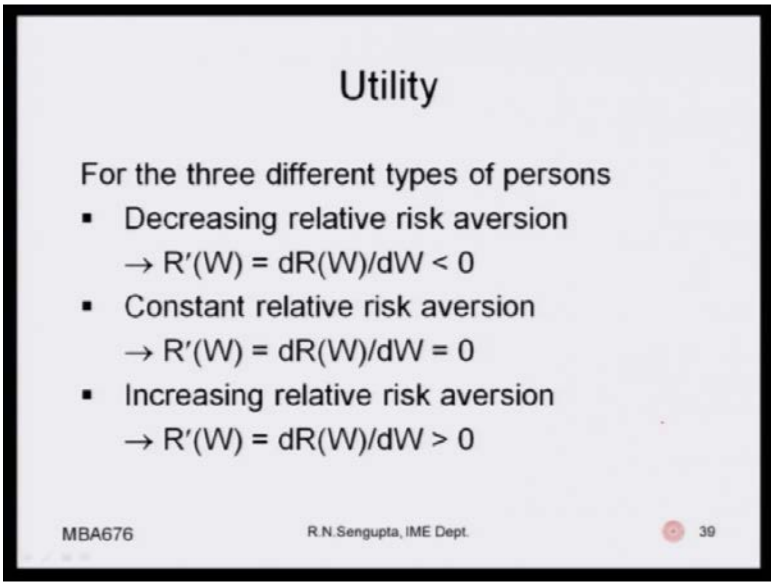

So, if you take again that, a derivative R W with respect to W; which is where W is basically wealth. So, again you can analyze the person as in having risk aversion property, where R bar or R dash is less than 0. Then, if it is equal to 0 and greater than 0; obviously, you have constant. And, another is increasing utility risk aversion properties.

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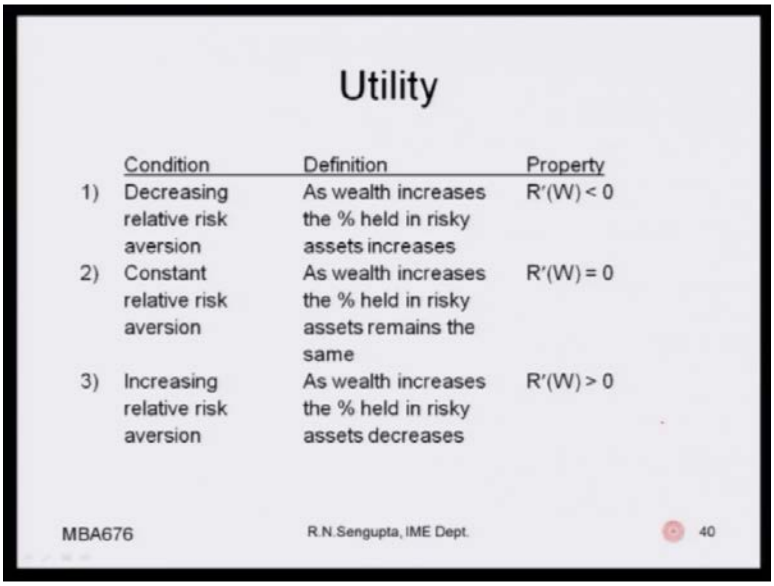

So, on the percentage of wealth, this is just a different characteristic how you analyze the person. If the wealth percentage increases held in the risky assets increases; if it is basically decreasing on its constant, again you can basically analyze a human being on an investor in the three different categories, which we have already discussed. Now, in general sense, we consider utility function having different characteristics. So, a utility function in general if you see any economic literature or some, you analyze different type of economic decisions, it would basically a quadratic one. And, while mentioning the quadratic term first is that, it has huge amount of implication in the area – finance.

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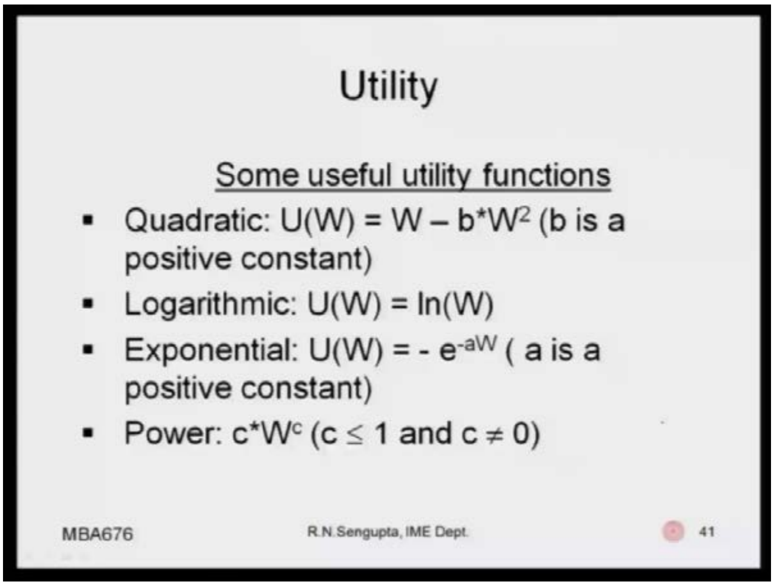

If the logarithmic one; if we have considered those examples which was ln of p; that was just a theoretical example, but it has some implications later on also. We can have the third class as the exponential one and the fourth one as the power functions. So, this a, b, c's are of whatever parameters. So, in general, finance perspective would not be that much interested in finding out or estimate the parameters. But, what we will be interested is to find out the general characteristics of this utility function.

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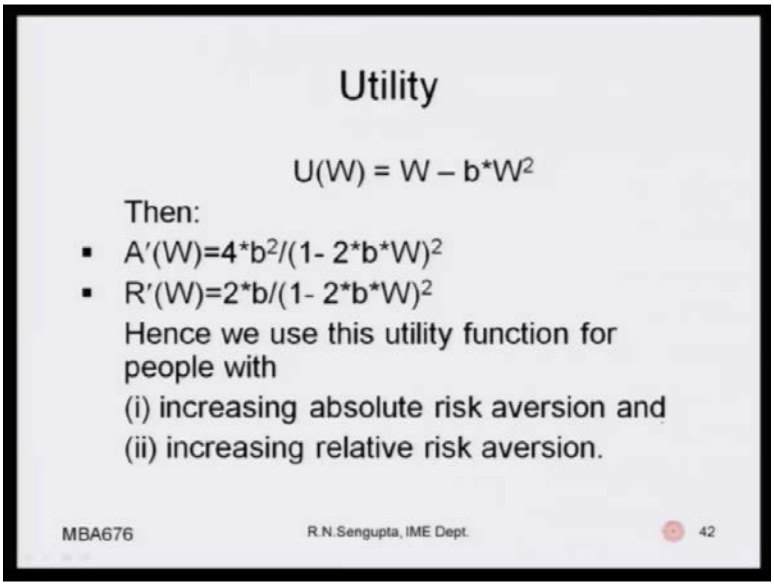

Now, if you consider the quadratic utility function and do some simple calculations, you can find out the values of R A – R bar and A bar. And, they would be given by the values; if you see as I was given in this bullet points; which would mean that, if we have a quadratic utility function, it will be having an increasing absolute risk aversion property and increasing relative risk averse aversion property.

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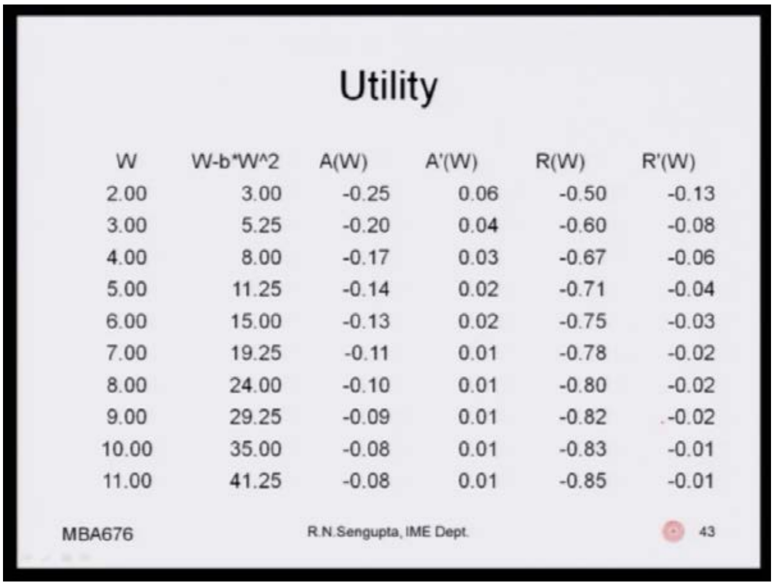

And, I have done a very very hypothetical example taking the values of W on the first column. And then, I have taken the values of a quadratic utility function and then tried to find out using very simple excel sheet; found out the value of A W, A bar, R W, R bar.

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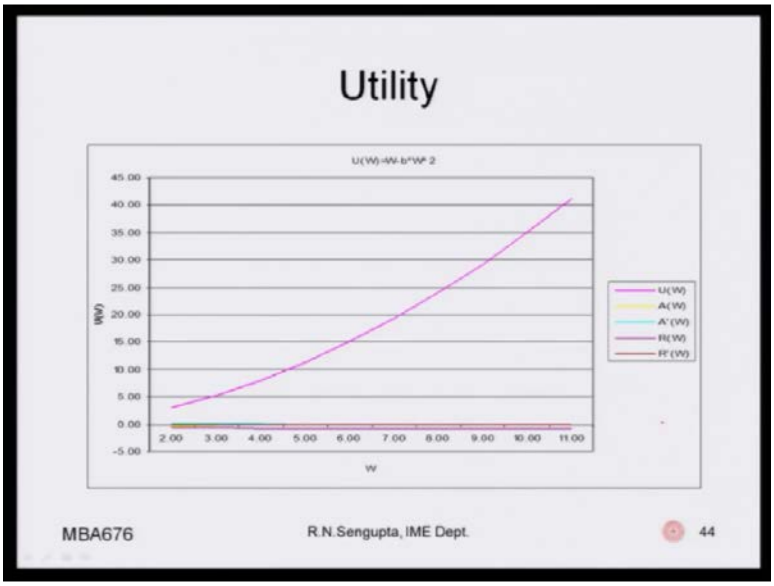

And, if you plot them, you see the values – the lower values are not that visible. But, if you see the values of U W, it looks like a quadratic function. And, the corresponding A bar or R bar would give you the characteristics of the utility function.

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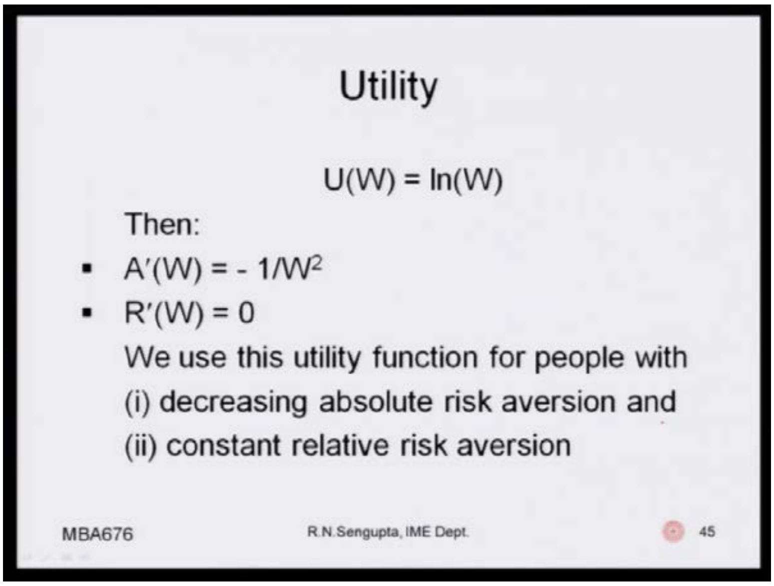

Similarly, if we go to the logarithmic utility function, again the A bar and R bar’s are values are given by minus 1 by W square and 0. And, again you can basically quantify the human being depending on whether he is risk aversion person or a risk lover person or risk hater person depending on the values or what are the characteristics of R bar and A bar.

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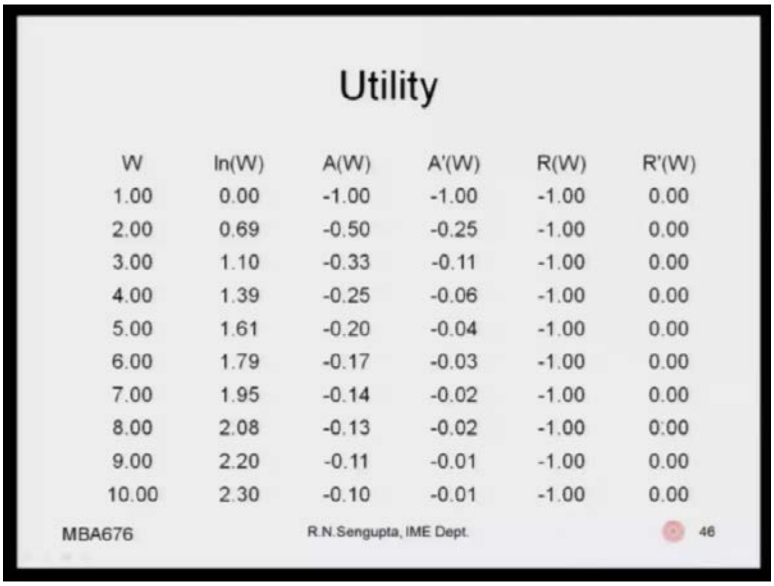

Utility

W	ln(W)	A(W)	A'(W)	R(W)	R'(W)
1.00	0.00	-1.00	-1.00	-1.00	0.00
2.00	0.69	-0.50	-0.25	-1.00	0.00
3.00	1.10	-0.33	-0.11	-1.00	0.00
4.00	1.39	-0.25	-0.06	-1.00	0.00
5.00	1.61	-0.20	-0.04	-1.00	0.00
6.00	1.79	-0.17	-0.03	-1.00	0.00
7.00	1.95	-0.14	-0.02	-1.00	0.00
8.00	2.08	-0.13	-0.02	-1.00	0.00
9.00	2.20	-0.11	-0.01	-1.00	0.00
10.00	2.30	-0.10	-0.01	-1.00	0.00

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Again, we have again the theoretical values given in the first column; the logarithmic of those in the second column. And, based on that, you can find out what is R bar and A bar.

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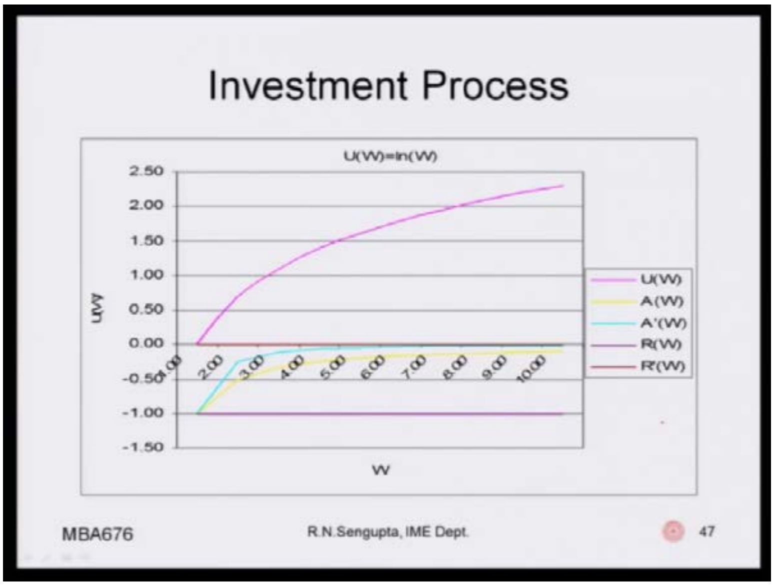

And then, again I draw the curves and they give you the general characteristics of the logarithmic utility function.

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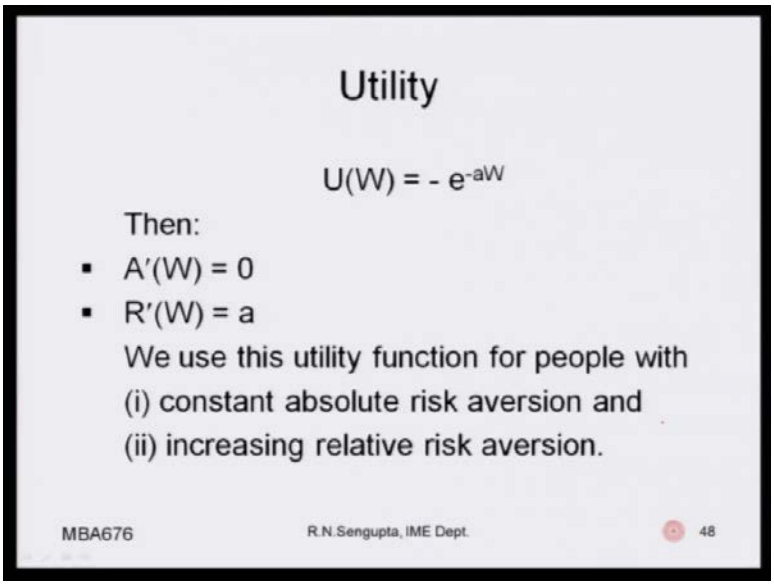

Similarly, when you go to the exponential one, the A is just a theoretical, which you have taken. And, if you find out both, the R bar is basically constant and A bar, which basically is the absolute risk aversion. The derivatives with respect to W is basically 0. Again you can basically classify human being if he or she has that characteristic has a constant absolute risk aversion property and increasing relative risk aversion property.

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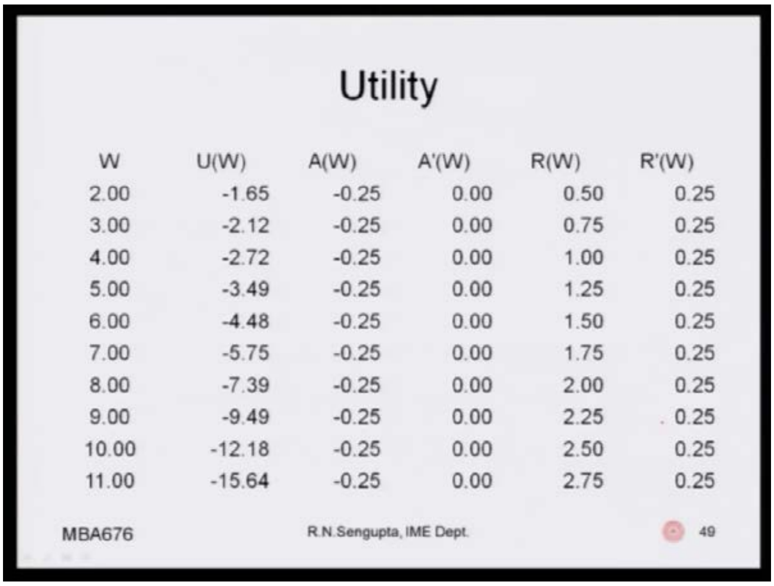

Utility

W	U(W)	A(W)	A′(W)	R(W)	R′(W)
2.00	-1.65	-0.25	0.00	0.50	0.25
3.00	-2.12	-0.25	0.00	0.75	0.25
4.00	-2.72	-0.25	0.00	1.00	0.25
5.00	-3.49	-0.25	0.00	1.25	0.25
6.00	-4.48	-0.25	0.00	1.50	0.25
7.00	-5.75	-0.25	0.00	1.75	0.25
8.00	-7.39	-0.25	0.00	2.00	0.25
9.00	-9.49	-0.25	0.00	2.25	0.25
10.00	-12.18	-0.25	0.00	2.50	0.25
11.00	-15.64	-0.25	0.00	2.75	0.25

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We take the values given as in this excel sheet, which has been transmitted in this PPT slides. The first column again are W values. The second column are basically corresponding values of U W considering that you have basically the exponential utility function and then the corresponding values of A bar. And, the R bar given if you provide.

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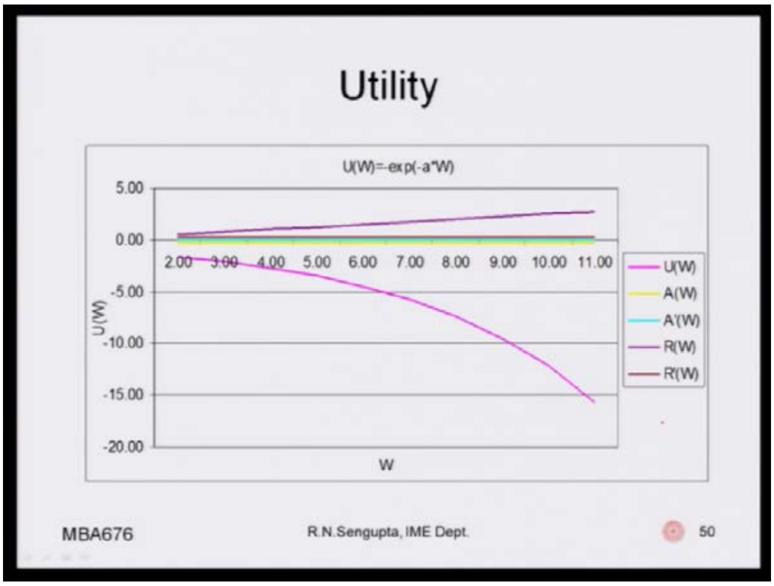

Again you will get the characteristics and how the utility functions look like.

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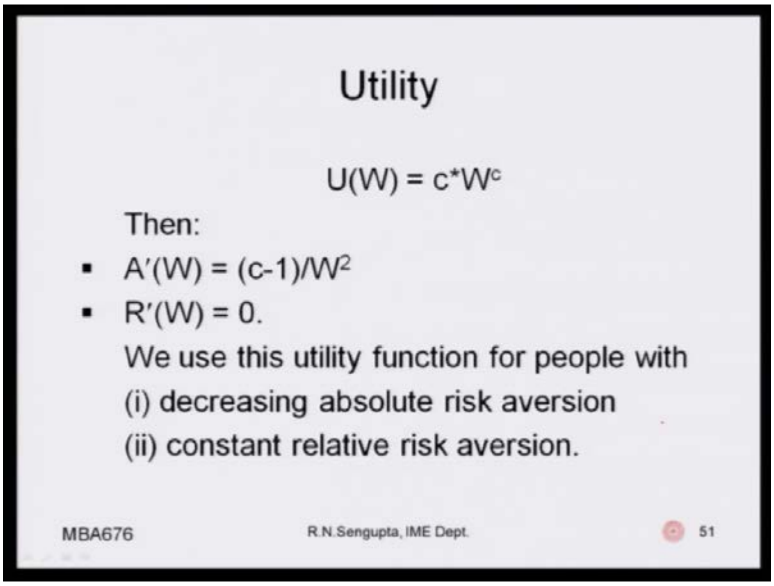

Similarly, coming to the power utility functions, you have A bar and R bar value given. Again R bar value is 0. And, depending on the C value, you will basically have A bar as positive or negative. If we see the denominator, which is W square would always be positive. So, only c minus 1 would basically dictate what is the sign of the A bar value or A dash value. And, again you can characterize the human being or a decision maker who is taking the decision as a decreasing absolute risk aversion property and a constant relative risk aversion property if that person has a power utility function as given.

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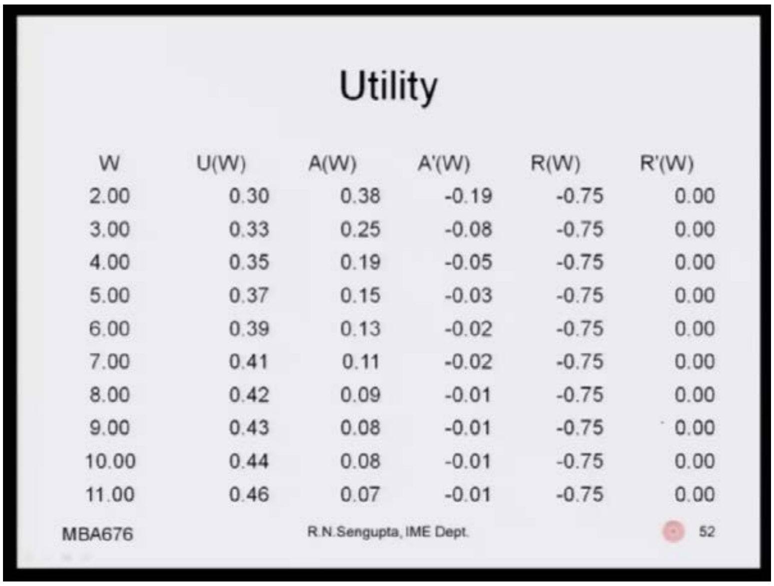
Utility

W	U(W)	A(W)	A'(W)	R(W)	R'(W)
2.00	0.30	0.38	-0.19	-0.75	0.00
3.00	0.33	0.25	-0.08	-0.75	0.00
4.00	0.35	0.19	-0.05	-0.75	0.00
5.00	0.37	0.15	-0.03	-0.75	0.00
6.00	0.39	0.13	-0.02	-0.75	0.00
7.00	0.41	0.11	-0.02	-0.75	0.00
8.00	0.42	0.09	-0.01	-0.75	0.00
9.00	0.43	0.08	-0.01	-0.75	0.00
10.00	0.44	0.08	-0.01	-0.75	0.00
11.00	0.46	0.07	-0.01	-0.75	0.00

Again concentrating in the same trend, we consider the W values, U W values and corresponding the last column. And, next to next that, which is the A bar value, we basically have the A bar and R bar for this theoretical example.

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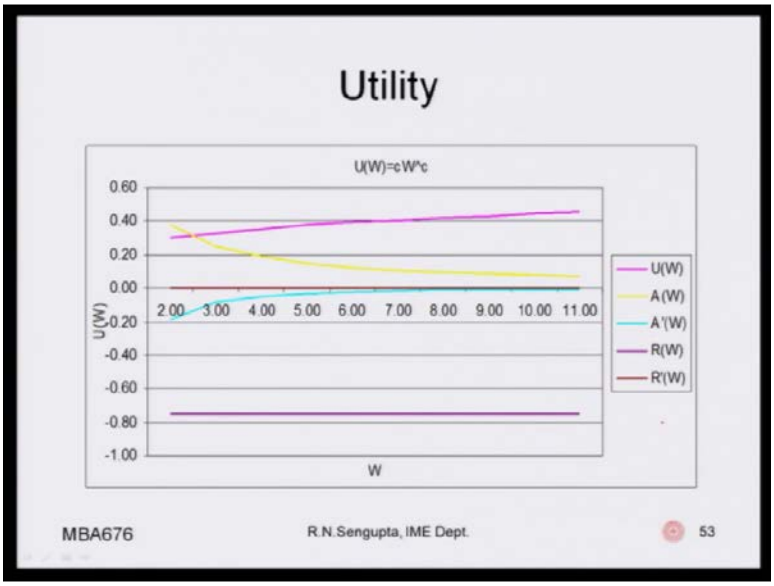

And, if we draw the utility function values, you have as it is given. So, remember that, when you are basically trying to draw them, I have used different colors schemes for them. So, if you see, the pink one are obviously, the utility function; then, the corresponding yellow, light green and the violet, so on and so forth. If are able to do that, basically, I have the same coloring scheme and you will be able to find out how the characteristics of the utility function looks like corresponding to U W, A bar W and R W.

Now, before I proceed, about what is certainty equivalent, let we pause here and give you a brief utilization of the quadratic utility function which I did mention that, why it is important. Now, we all know and we will consider the later on also that, in probability distribution, one of the very important distributions is the normal distribution. Normal distribution is basically bell-shaped curve, where the mean, median, mode are the same values.

Now, it should be remembered for any distribution and any investment purposes, given amount of Money, which you are investing, you will basically get some return. Now, if you try to plot the returned distribution and if the utility function is quadratic, then it can we proved mathematically then, the return distribution would be normal; which in general sense means if the distributions are normal, then utilizing different properties of normal distributions give you fantastic results and you are able to basically quantify this to the best possible extent. And obviously, you get the diversification to the maximum possible extent, which you will be considering later on. But, in general, returns are seen first or different type of stocks, different type of portfolio, different type of options are never non-normal. So, if it is non-normal, which means that, the corresponding utility function based on which you will be trying to utilize different situations are not quadratic. So, which means that the utility function concept which we will consider initially would be of quadratic nature and hence the returns as normal.

And, later on as we go into the concept of different type of extreme value distributions, how they would basically change your utility functions, how they will basically change your returns would also come into the pictures as they are trying to analyze the portfolios, trying to analyze different types of stocks, how to analyze the different types of risk; and, return perspective would basically change as we proceed from the case of utility function being quadratic to the case where the utility functions are not quadratic. Continue your utility discussions – utility function discussions; we did mention that, what is the implication of quadratic utility function and why it is important. We will consider few important concepts, which may not be very apparent in the initial stage. But, later on when we do the portfolio analysis from the quantitative finance point of view, it will be important.

So, consider the very simple concept of certainty equivalent. Certainty equivalent in a very simple sense means that, what is the overall certain value, which you will get given a situation. So, if you consider a utility function given a U W with some utility function depending on whatever the property distribution is for that utility function, we can find out the certainty equivalent to be that value of C, which will give you the same expected value. So, let us consider this with a very simple example, so that how… And, we will consider that why this value of C is important. The value of C is important in a very general sense.

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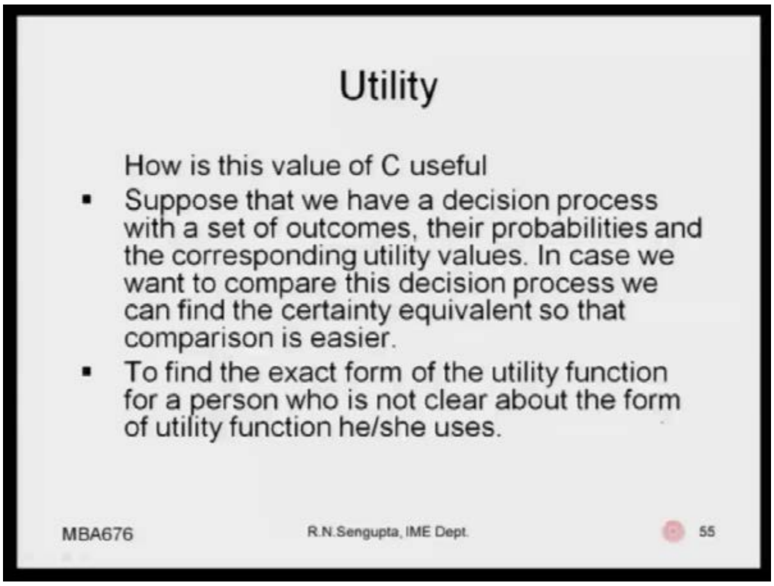

Considering that you have different type of portfolios, where the outcomes are non-deterministic. So, when you are trying to analyze different type of portfolios, our main concern is to find out the average value of the portfolios as the ranking becomes very important for you and you are able to rank the portfolios in such a way that you are able to take the decision, which is best to your knowledge considering whether you want to be basically minimize your risk, whether you want to basically maximize the risk you return or whether you want to basically take a decision, which is in between depending on it is basically compromised, where you increase the expected value of the return to the maximum possible extent and try to basically decrease the risk to the minimum possible extent as that the compromise can be made.

Now, how is this value of C is useful is that, suppose that we have decision process with a set of outcomes; and, given the set of outcomes, you want to rank them. That is point number 1. And, you want to also find out that what is the exact form the utility function based on which you can basically analyze any decision. So, our human being is basically able to give you the outcomes; but, he or she is basically not able to quantify the utility function based on which he or she is taking a decision. So, your main task would be –

given the certainty equivalents, you basically can find and use different types of mathematical techniques as that you are able to find out what type of person he or she is and then what type of utility function you will be able to analyze. So, consider this very simple situation.

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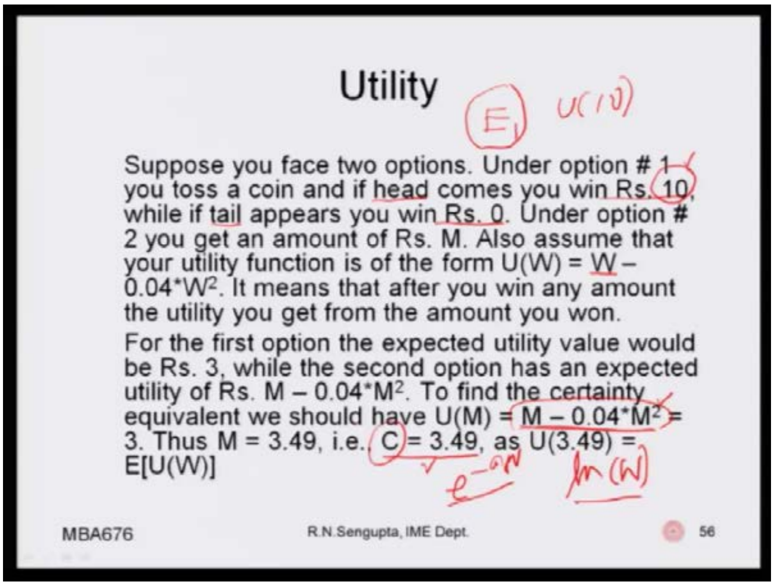

Suppose you face two options, that means there are two outcomes. Under option 1, you toss a coin; and, if head comes, you win 10 rupees; while if a tail appears, you win 0. Under option 2, you get an amount of M; M is a value which is unknown to you. Now, your main concern is basically to find out the value of M, which is the certainty equivalent. So, if you solve this problem considering that there is a utility function, which is quadratic in nature, which is given with the value of W minus 0.04 into W square. So, it can be a logarithmic utility function, it can be an exponential utility function, it can be bar utility function. But, for our case, we are taking the simple case of the quadratic utility function. So, if you find the quadratic utility function, what your main concern is that, given the value of M; so, M is the input, which you are giving, which is the value. And, if you basically convert that M value into its utility that, utility function is given by M minus 0.04 into M square.

So, consider that is the overall expected value for that case of M. And, also consider that when you have option 1 – when you are considering option 1, you have basically the outcome is given as 10; and, if a head comes and if you have value of 0 for tail comes. So, if you find out the value of this utility, what you will do is basically, you multiply 10 into half plus 0 into half and that will give you an expected value. And, based on that expected value, you will basically find out that, at what value of M would you get the same value of the expected value, which is for the non-deterministic case. So, if we basically are able to solve, it will be this 10 value, which when a head comes considering this basic ((Refer Slide Time: 19:26)). If you multiply 10 into half, and then you will basically multiply 0 into half if a tail comes. So, from these two situations, you will get the expected value of the utility for the case, which we will consider as E 1.

And, for the next case, which is a certainty case, you already have a value of W. So, when we are considering the value of W, this will be replaced by M and the corresponding utility would be given by this. Now, remember one thing which I did not mention, but considering that you have already understood it that, the value of 10 which you have is not the value, which you will be utilizing in your calculation. This is basically the 10 value, which is going as an input. And, the corresponding output, which is coming would be U 10. So, corresponding to 10, you will have the utility, where this M value will be replaced by 10 minus 0.04 into 10 square. So, that will give you the utility corresponding to 10. That will be multiplied by 0 as the first term – plus when you have a value of 0 which is the actual wealth that, when converted in the utility function, obviously, 0, because 0 minus 0.04 into 0 square is also 0. That will be multiplied by a half. So, based on that, you will be finding out the value of E 1. And, once you find out the value of E 1, you have to basically equate it to on the right-hand side; on the left-hand side, you already have a utility based on the value of M.

Now, remember that, certainty equivalent element basically means a coin for which the probabilities, the outcomes are deterministic. In this sense, the probability is 1. So, once you multiply the utility, which is here – multiplied by the probabilities is 1, then the value of the certainty equivalent, which you find out is basically 3.49; which means that, given two situations in one case, where you input 3.49 and you utilize the utility function as given here, probability is 1. And, on the other case, when we have basically two arms, where 10 is the input; corresponding to that, you find another utility function; 0 is the input; based on that you find out this utility function; multiply each of them by half and half corresponding to the case if it is an unbiased coin. When you equate that, you find out the certainty equivalent. Before we go to the next slide, we should be remembered that, the value of C, which is the certainty equivalent will change depending on what your utility function is. So, if I change the utility function, obviously, you will get different values of C. So, one should remember that, if a human being has a quadratic equality function or if human being has here for example the logarithmic utility function or if the human being has basically this exponential utility function. So, depending on what utility you are using, the value of C would change depending on the situation.

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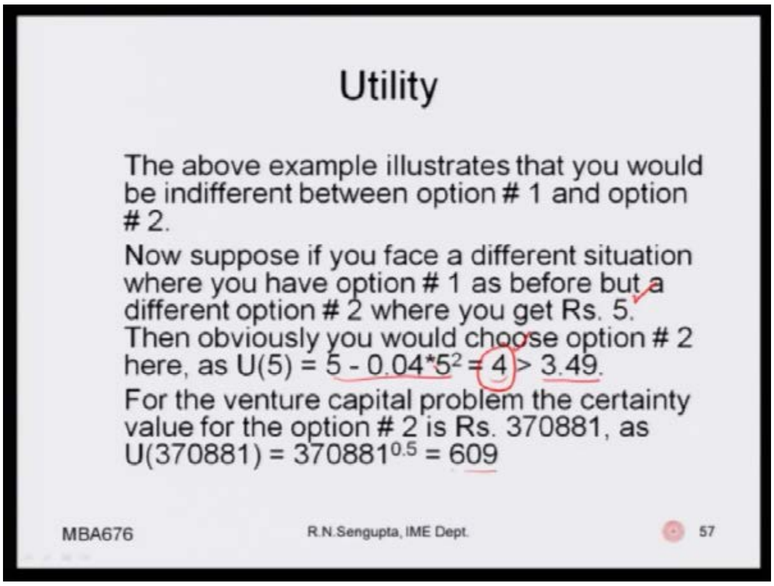

So, the above example illustrates that, when you will basically would be indifferent between option 1 and option 2. And, based on that, you have to find the value of C, which you have already found out in certainty equivalent. Now, if you face a different situation as I mentioned; and, if you change the value of and you get different option 2, where you get a value of 5; so, based on that, if you use the same utility function, which is the quadratic one, you will get a value of 4; which means that, for a value of input of 5, you will get a expected value of that certainty equivalent coming out be 4. Here certain time using for the case that, the amount of input value is 5.

Now, this when you consider with or try to basically compare with the value of 3.49, obviously it means that your expected value for this decision where you are inputting 5 has got higher returns or expected value. Hence, you basically take the one which is given by 4. Now, for the venture capital problem, which you have already considered; if you considered, the certain equivalent value comes out to be 609. Only remember that, in this case, the venture capital example – the utility function, which you are using is not the quadratic one; would be corresponding to the case, which you have already considered in the venture capital example one.

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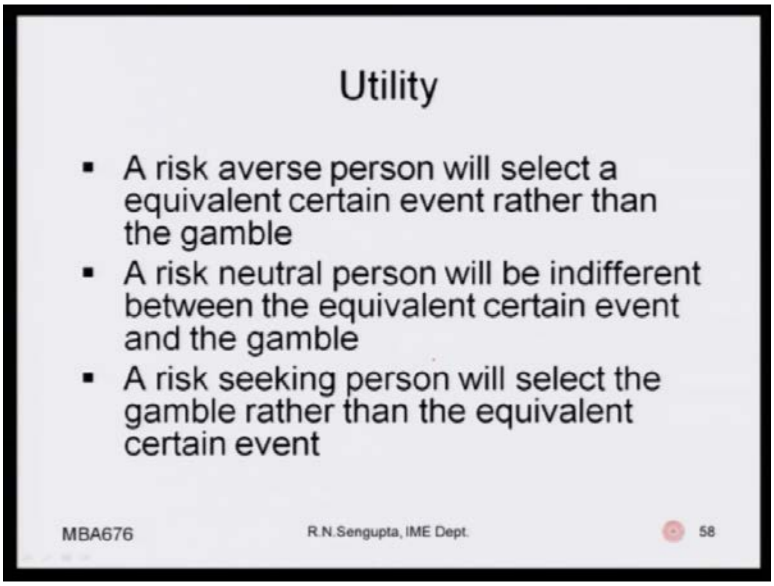

So, a risk averse person will select an equivalent certainty rather than the gamble, because he or she obviously, does not want to take any risk. A risk neutral person is indifferent between them. And, a risk seeker person would basically select the one where the certainty equivalent is not that important, because he or she basically analyze a situation, where the outcome is more based on the probabilistic one and he or she basically thinking of that outcome when the value actually is must higher.

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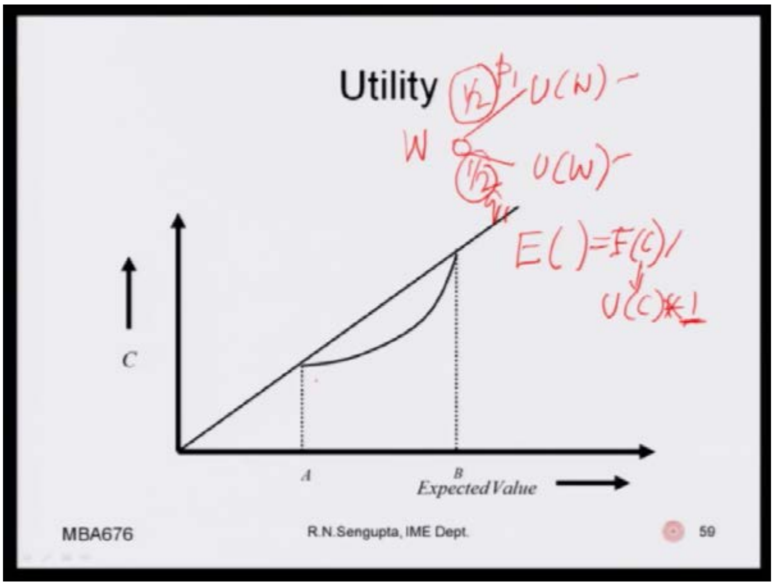

Now, we will consider very simply without going to the theoretical details that, how the expected value and the certainty equivalent can be matched in order to basically make a decision, where you are basically indifferent between them and try to find out those values. So, consider this very simple example – a theoretical one, where you have 2 values given as A and B. Now, what you want to find out is basically certainty equivalent. So, if you consider a very simple quadratic utility function or whatever it is; so, what you will do is that, as you keep changing the values on two arms. So, consider you have this case example – both the probabilities for our case consider they are half and half. And, what we are considering is that, you are basically inputting a value of W; and, based on that, this expected value, you will basically find out a utility function of U W for some value, which you are inputting; and, U W for the value which is input. So, obviously, these values are different.

Now, consider for the time being, these probabilities are p 1 and q 1. So, as p 1 and q 1 changes, obviously, the value of the utility also changes. So, what you are trying… And, your actual task is to find out the expected value of this equated to a value of expected value of C, which is the certainty equivalent multi… And, for this case, remember this U C is being multiplied by 1. This one is basically the probability is 1. So, as you keep changing the values of p 1 and p 2 and as you keep changing the values of utility functions, you will find out different expected values. So, these A's and B's are expected values. So, as you keep changing them, you will try to find out a convex combination of A and B such that the overall value or the expected value of A and B in the long run should be exactly equal to C such that, the certainty equivalent for the human being, for the decision maker can be found out. You will considerably be seeing simple examples later on as we do the problems in portfolio analysis.

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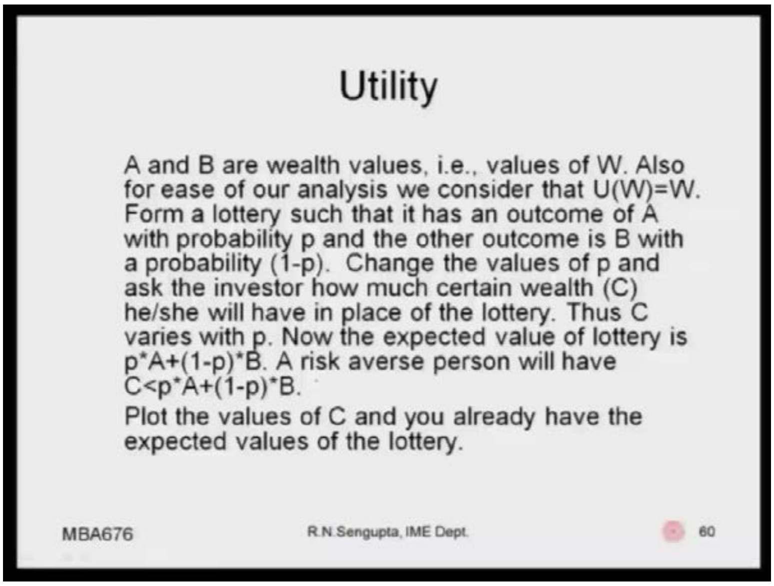

Now, A and B are wealth values considered for our time being. The values… And, also for the case of our analysis, we consider the U W is a linear function which is W.