Strategy: An Introduction to Game Theory
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Lecture – 08

Hello welcome to another module, this online course Strategy, An Introduction to Game Theory, what we are going to look at today is we are going to look at different game. So, if you look we seen a large number of games or you seen several examples of games so far, which are slightly simplest. So, what we are going to do right now is, we are going to look at start looking at slightly more defined and slightly more sophisticated games. In particular the kind of games, the kind of game that we are going to look at today is also known as the tragedy of problems.

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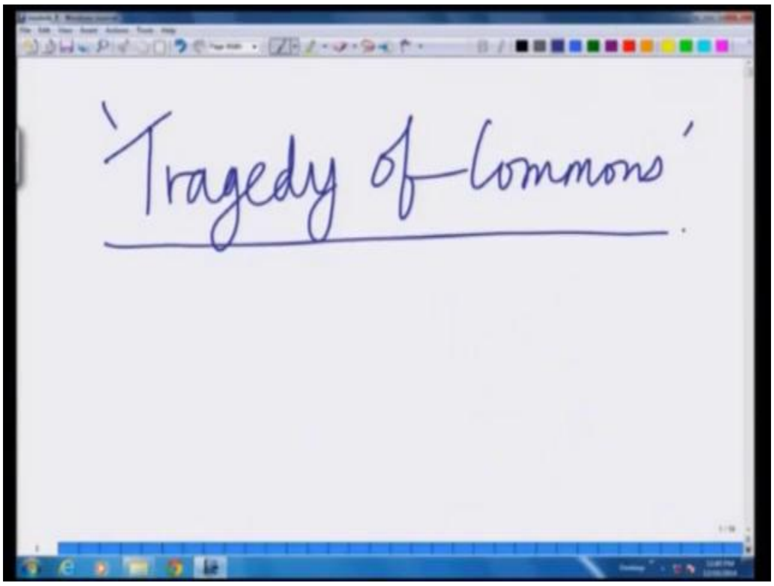

This game is write title as, the Tragedy of the Commons it is a very catchy title. What it means is it relates to the usage or rather it relates to a game, a sort of a game interaction between completing agents or between completing people related to the utilization of a common resource, such as for instant a mine minerals or a fisheries or let say a forest or so on, so the environment for that is a matter.

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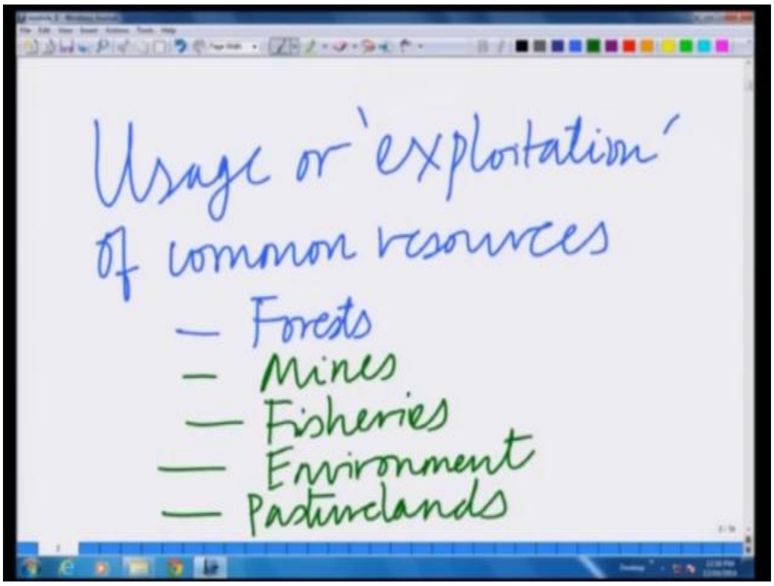

So, what this game is or this tragedy this game the tragedy of commons is about, the usage of common resources or rather exploitation. Resources such as forest, mines, fisheries or environment or pass an environment with respect to the number of the amount of green-house gases that released by the environment causing in environment pollution or pastureland's relating to the amount of grazing and also often over grazing, it leads to the depletion of these pastureland.

So, what we are looking at, we want to model the sort of a competition or studies interaction between different agents, who are using a common resource, that is common to all the people for instance for a certain state or a certain country, such as the forest or a mine, a fisheries. We should often subject to over exploitation and reading to the reverential depletion and we are trying to model this as a game and try to understand the behavior of these agents in such a game or in such an environment. Well, how do we start with it? Let us start by looking at a simple example, in which we are taking a look at the usage of a forest, let say there are two timber agencies.

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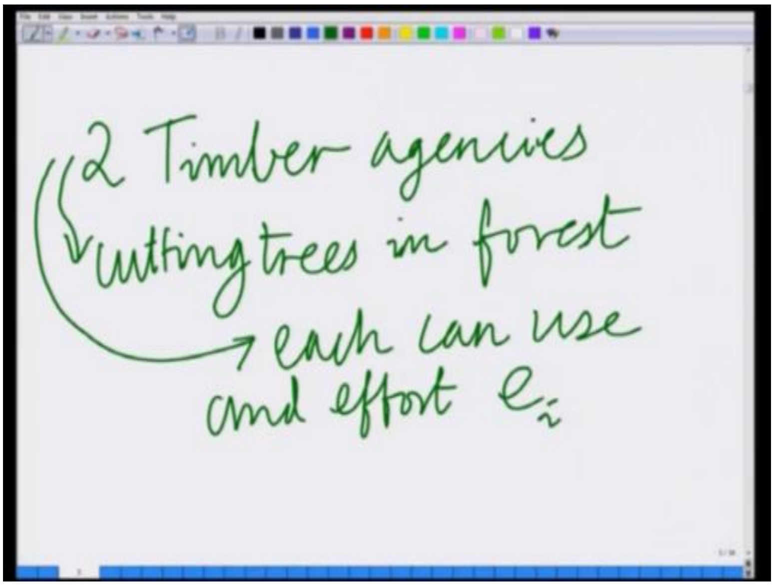

Let us consider a scenario, where there are 2 timber agencies, each is involved in cutting logs and cutting trees or lumbering in a particular forest, that is or he has a license for cutting trees in a forest towards providing lumber and each one can use an effort e i. So, each can use an effort e i, it can also cut trees, that are cut for instance the number of possible trees that are log by these two different timber agencies. And, of course, the number of trees that are cut is proportional to the effort e i.

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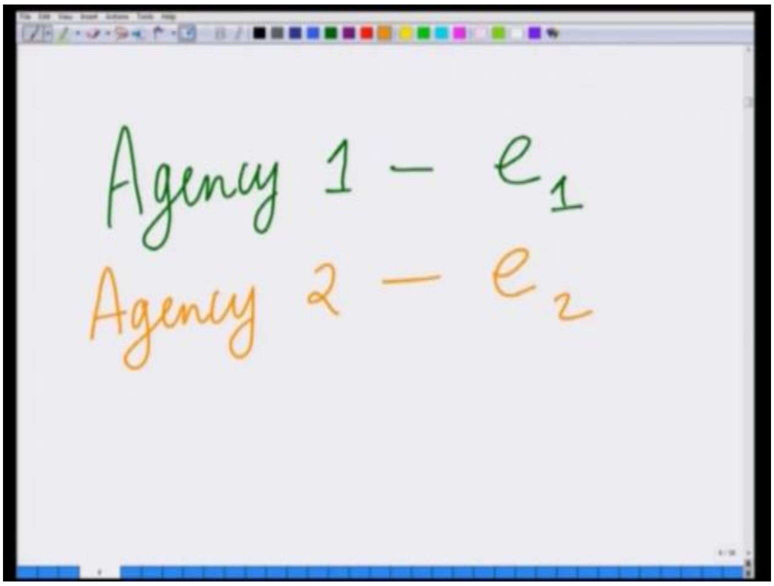

So, we can look at 2 timber agency the effort of each is denoted, so the effort of timber agency 1, agency 1 has an effort of e 1 and agency 2 has an effort of e 2. So, we are looking at this strategic interaction between these 2 timber agencies which are cutting or logging trees towards providing timber and their actions or their efforts, effort e 1 of agency 1, e 2 of agency e 2 of agency 2 and we trying to understand the strategic interaction between these two a timber agencies.

Of course, now we have to specify payoff function corresponding with these two efforts, we already said that the number of trees that are logged or cut is proportional to the effort putting by each number agency. So, the payoff can be model as follows.

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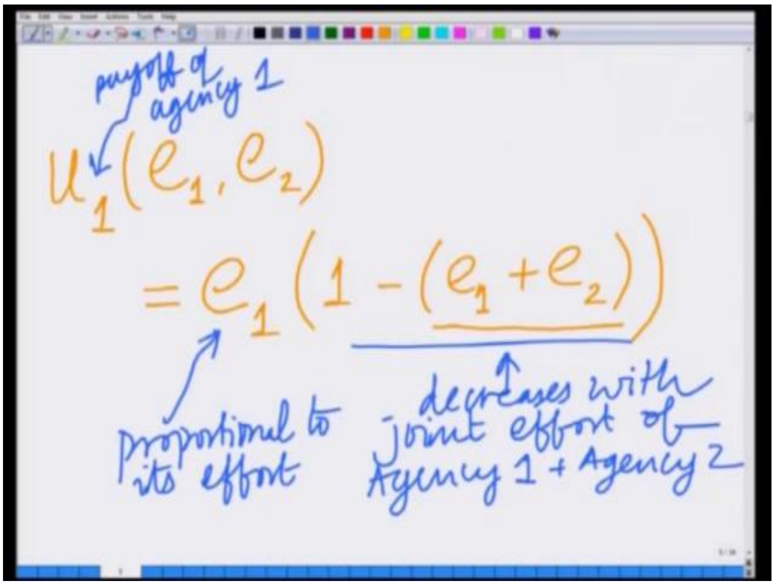

The payoff u 1 of agency 1 has a function of each effort e 1 and the effort e 2 of the other timber agencies can be modeled as e 1 times 1 minus e 1 plus e 2. Now, of course, this requires some explaining, the first factor e 1, we are saying the utility or the payoff is proportional to it is own effort or with the number of trees cut or the number of trees log is proportional to each other. But, also the payoff decreases with the total effort put in by both the timber agencies that is e 1 and plus e 2, because the payoff is also proportional as you can take e to 1 minus e 1 plus e 2.

Because, the number of trees that are logged by this timber agencies together, the less is left to be used in the future, so the less is left for exploration, which means the future payoff are going to be lower as compared to the both logging less. So, this is an interesting payoff function, which takes into account not only the current payoff, it also takes in to account the possible future payoffs, so related arising from over exploitation of a certain resource, such as the forest.

So, it is proportional to e 1 that is it decreases with e 1, but the payoff also decreases with the total effort that is 1 minus e 1 plus e 2 is in being that the more effort, but both of them put into together. The more trees are cut, the more trees are log therefore, less is left for nourishment and less is left for future. So, this let me just highlight the important aspects, this shows the payoff of agency 1 has a function of it is effort e 1 and effort e 2 of the other agency proportional to it is effort, but decreases with joint effort of agency 1 plus agency 2.

So, we saying it is proportional it is on effort, but it also decreases with the sum or the net effort put by agency 1 and agency 2. The reason being, the more the joint effort put by them is exploitation of the resource giving less for possible future use.

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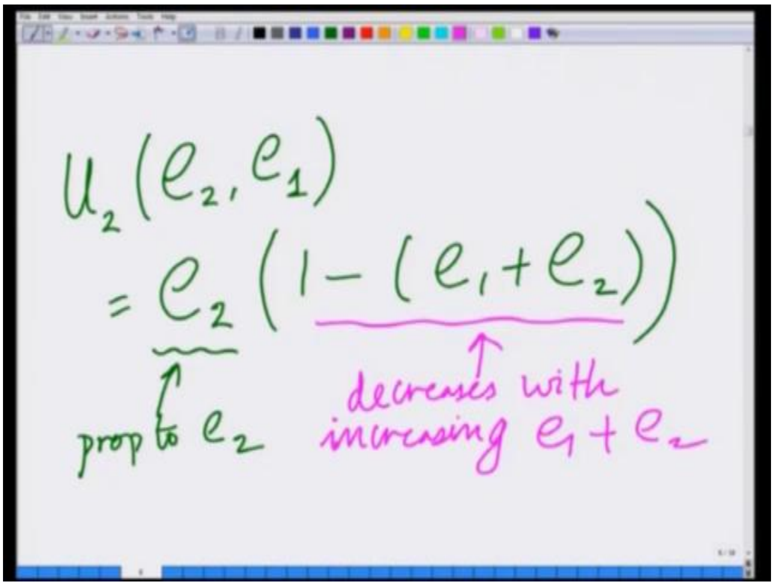

Similarly, now the payoff of the second agency u 2, e 2, remember we have to put the action of the second agency first while taking about it is payoff u 2 equals e 2 into 1 minus e 1 plus e 2. Also again proportional to e 2 and inversely I am decreases with increasing e 1 plus e 2. So, there are two components one which is proportional to e 2 another component, which is decreasing with e 1 plus e 2, which is the reason be similar to, what we have said in the case of the payoff function of agency 1.

So, as you can see this is a strategic interaction, because the payoff for each agency, the payoff of the timbering agency or the payoff of each logging operation depends not only on it is own effort, but also depends on the effort put in bites, competitor, agents. So, this is naturally an example of strategic interaction between these 2 timber agencies, what is the best, what is the, how does this game evolve or what you expect to see with respect to the efforts putting by both these agencies in practice.

And, of course, this game is now slightly more refined or slightly more advanced compared to the other game that gives in before, because now these efforts e 1 and e 2 are not confined to a finite set.

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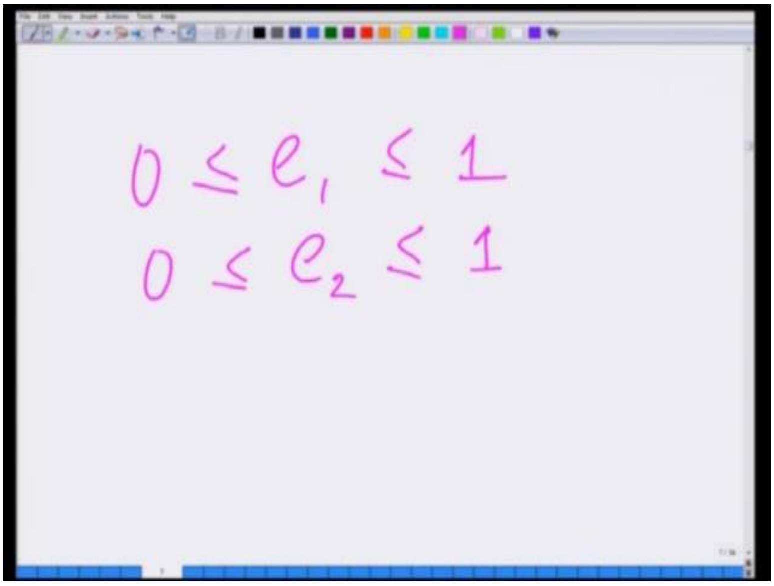

But, each e 1 e 2 can be any real number between 0 and 1, so we are saying 0 less than equal to e 1 less than equal to 1, 0 less than equal to e 2 less than equal to 1, which means e 1 and e 2 can take any real value between 0 1. So, we have the continuing set of efforts that can be put by these timber agencies, so the action sets are infinite and continue actions compared to the discrete action set.

Remember, in all the games that we have seen previously, we had two players and each player had a finite set of action. In fact, each player, in fact most of the games each of the player had possibility of two actions to choose sets. So, now, we are moving to a slightly different scenario slightly more advanced scenario, where the set of possible actions is infinite that is each timber agency you can chose an effort and this effort can be any real number between 0 and 1, so this is a more advanced game. So, we will have an infinite set of possible actions that also means, because we have an infinite set of possible actions, I can no longer draw the game table.

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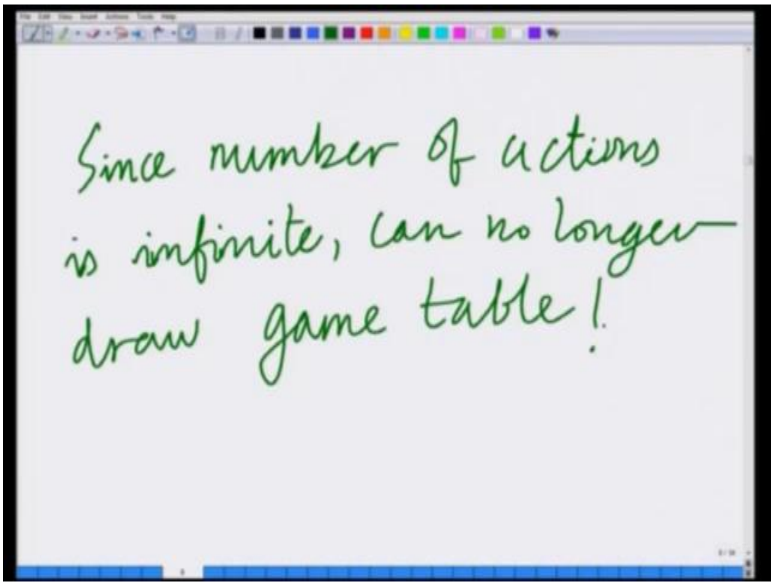

So, I can since, I can no longer draw the game table, because in game table I have only a finite number of rows for a certain finite number of actions. Because the number of actions is infinite, I can no longer draw game table, but therefore, to analyze this game or to come up with the reason into interpret the behavior of the different agents in this game, I have to come up with the different frame work.

And that is, what we are going to talk about in the next couple of a minute that is, how to characterize the behavior or how to characterize the outcome in this particular game, which involves an impossibly finite, which involves infinite set of action for both the players. So, naturally as we said before that is to analyze the behavior of any game, we have to first start by looking at the best response of each agent or the best response of each player.

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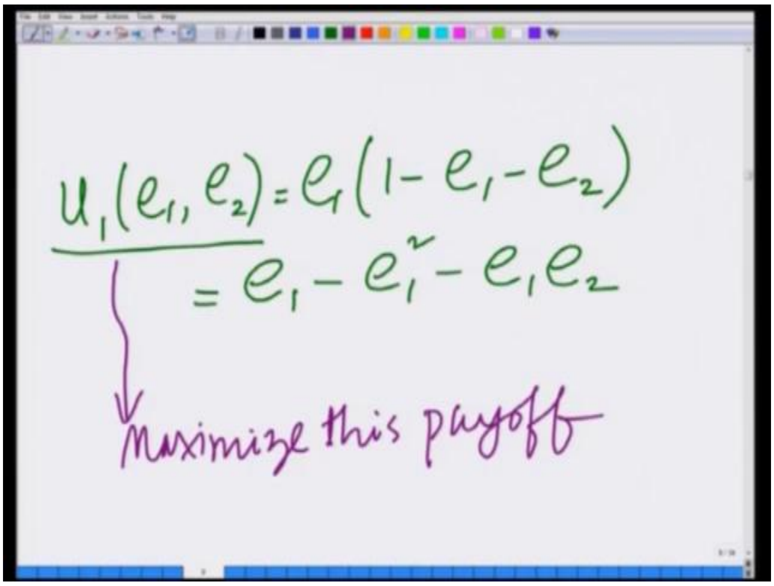

Well, let start by looking at the utility function of player 1 that is u 1 to characterize it is best response. We have u 1 of e 1 comma e 2 equals e 1 into 1 minus e 1 minus e 2, which is equal to I can expand this to write it as e 1 minus e 1 square minus e 1 e 2. Now, I have a payoff of u 1, which is a function of both e 1 and e 2, I have to find the best response e 1 for a given 1 e 2 or for a given strategy e 2 of agency 2, which means I have to maximize this utility u 1, I have to maximize. Find the best response, the best response is, where the payoff is maximum for given effort e 2 by player 2.

And therefore, to maximize this continues function most of you, as most of you must be familiar from an introductory knowledge of differential calculus. I can differentiate this utility function and set it equal to 0 to find the maximum, to find the point at, which this payoff is maximum, therefore I am going to differentiate this with respect to u 1 and set it equal to 0.

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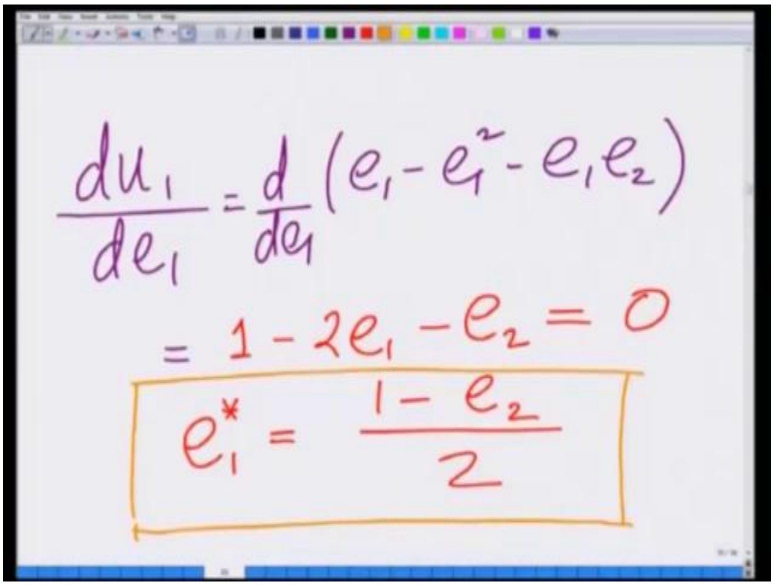

So, d u 1 by d e 1, when I differentiate this with respect to e 1, I am going to have let me write to down here, that is d by d e 1 of e 1 minus e 1 square, minus e 1 e 2, which is equal to derivative of e 1, is 1 derivative of e 1 square is 2 e 1 derivative of e 1 e 2 is e 2. And therefore, to find the maximum that is the effort e 1 at which, it is maximum I have to equate to 0 and, now solving this equation I have e 1 star, which is the best response equals 1 minus e 2 divided by 2.

So, the optimal, so the best response even star equal 1 minus e 2 by 2 I hope every 1 was able to follow that argument that is, basically we have payoff function, which is the function of the effort e 1 to find the best response I have to maximize this payoff function, this is now a continues function with respect to e 1 this is, in fact differentiable function. So, I am going to differentiate this with respect to e 1 and set it equal to 0 to find the best response even star and the best response e 1 star is 1 minus e 2 by 2.

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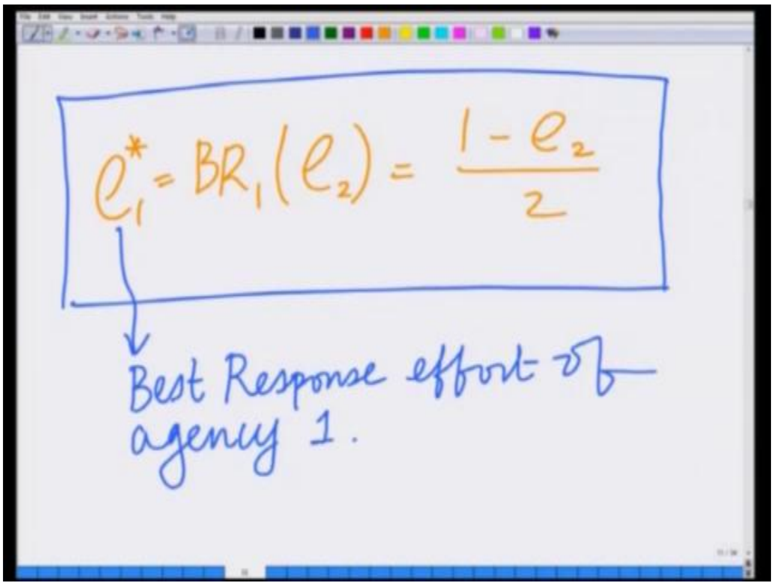

In fact, this can be return as e 1 star equals best response 1 with respect to e 2, which is equal to 1 minus e 2 divided by 2. So, we have found the best response e 1 star, as a function of e 2, this is the best effort of agency e 1, as a function for a given effort e 2 by the timber agency 2. So, this is the best response effort of agency 1, what do I do next, next I have to find the best response effort of agency 2, which I can obtain from the utility function u 2 u 2 of e 2 comma e 1 equals remember that is equal to e 2 into 1 minus e 1 minus e 2 that x payoff the utility function is given as e 2 into 1 minus e 1 e 2.

And remember, now I have to find the best response e 2 that is that e 2 for, which this payoff is maximize, for a given effort e 1 by it competitor, who is agency 1 or who is player 1.

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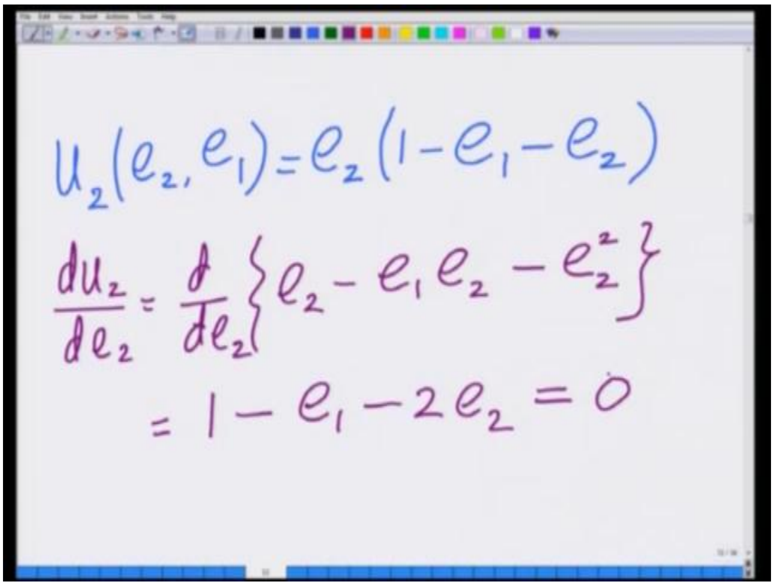

And therefore, now I have to differentiate this with respect to e 2 and set it equal to 0, find that e 2, where this is maximize or find the best response e 2 therefore, differentiating this with respect to e 2, I have d by d e 2 of e 2 minus e 1 e 2 minus e 2 square, which is equal to 1 minus e 1 minus 2 e 2, which I am now going to equate to 0 this is equal to 0.

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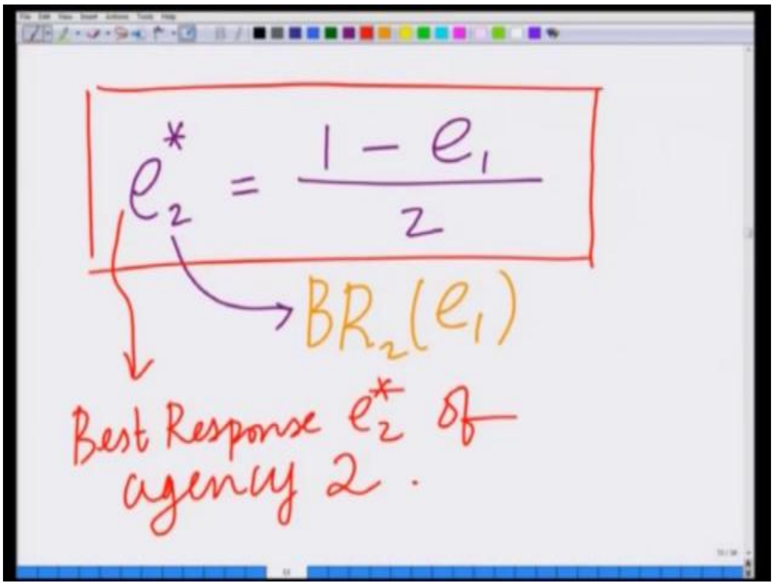

And, solving this I am going to obtain e 2 star equals 1 minus e 1 by 2, that is the best effort e 2 star equals 1 minus e 1 by 2. So, this is. In fact, e 2 star, which is the best response of 2 given action 1 or effort e 1 of agency 1, so we have the best response or e 2 star this is the best response, best response e 2 star of agency 2 or player 2.

So, what are we founds, so for we have e 1 star, which is the best response of agency 1, we have also for e 2 star, which is the best response of agency 2. So, we have now found the best responses of both the players.

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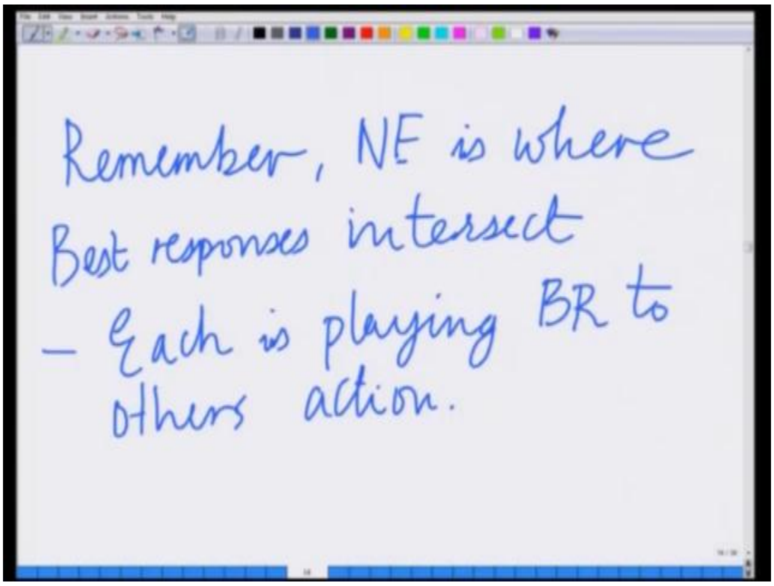

Now, to find the Nash equilibrium we of course, find the point, where the best responses intersect remember Nash equilibrium remember, is where best responses intersect, which means each is playing best response to others action or strategy, which means, what we are saying is the Nash equilibrium is, where the best responses intersect therefore, each is playing is best response action to the action of the other player.

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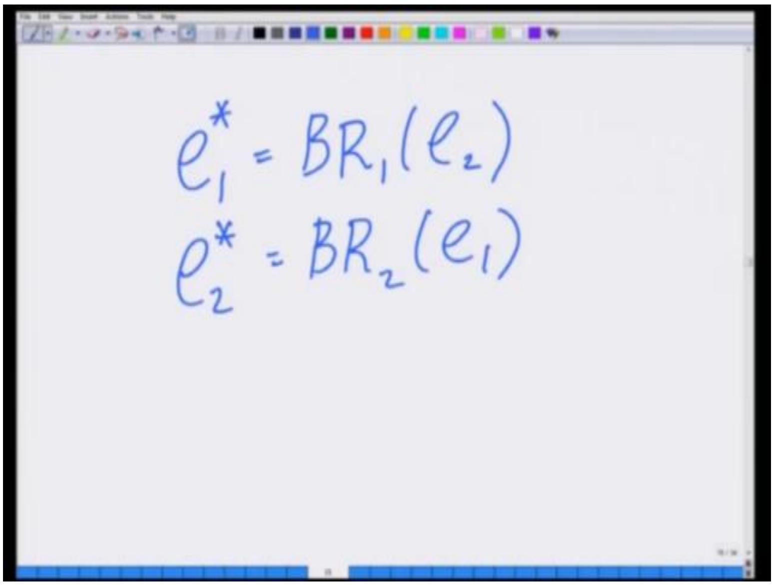

So, naturally we have found remember e 1 star equals B R 1, e 2 star equals B R 2 e 1, now at the Nash equilibrium everyone is playing his best response.

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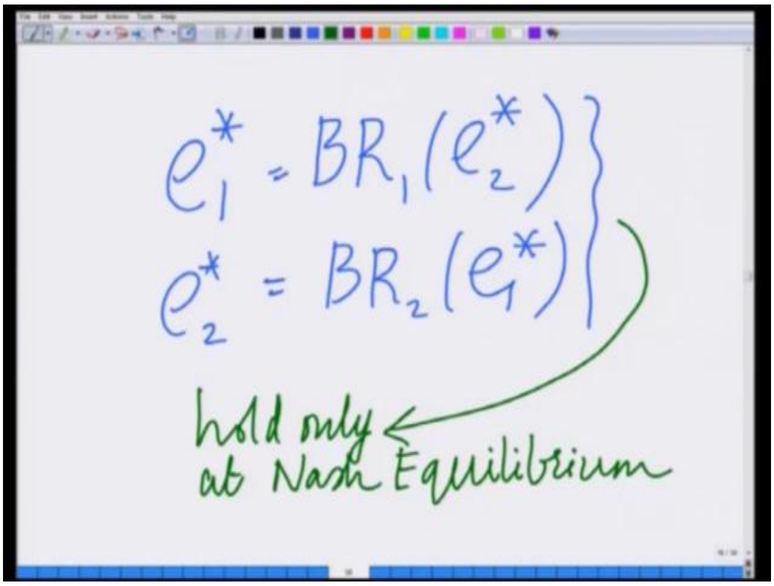

So, what we have is. in fact, we have e 1 star equals B R 1, e 2 star, because a Nash equilibrium player 2 are also playing is best response. And, similarly e 2 star equals B R 2, e 1 star, this equations holed only at Nash equilibrium, hold only at these equations only Nash equilibrium because, both the players are they are playing their best response.

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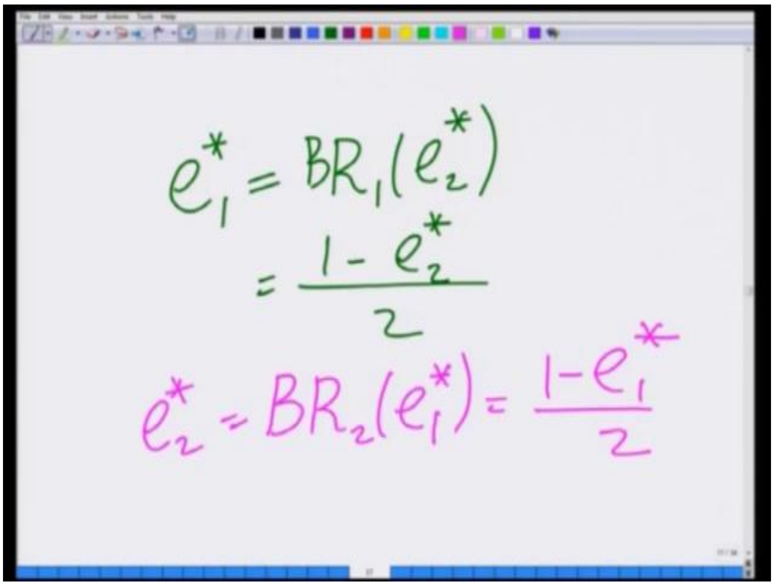

As a result, what we have is we have e 1 star, equals best response 1 e 2 star, which is 1 minus e 2 star divided by 2. Similarly, we have e 2 star equals best response 2 best responses 2 of e 1 star equals 1 minus e 1 star by 2. So, you have a system of linear equations in terms of e 1 star and e 2 star, let me write them down again, clearly.

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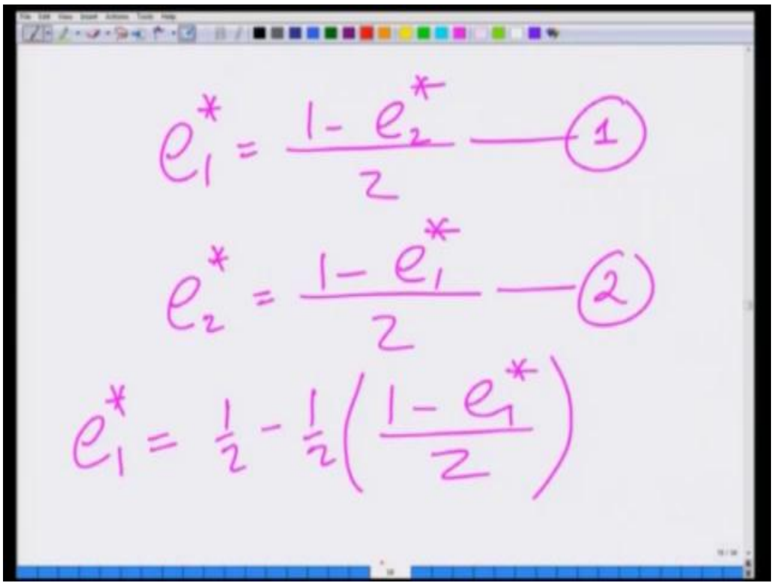

I have e 1 star equals 1 minus e 2 star divided by 2, I have e 2 star equals 1 minus e 1 star divided by 2 this is the set of equations, I have to find the Nash equilibrium actions even star into star I can substitute e 2 star from the second equation in the first and then I will have e 1 star equals half minus half e 2 star, but e 2 star is 1 minus e 1 star divided by 2, so we have substituted. So, let us call this the second equation let us call this the first equation have substituted for e 2 star and the second equation in the first equation.

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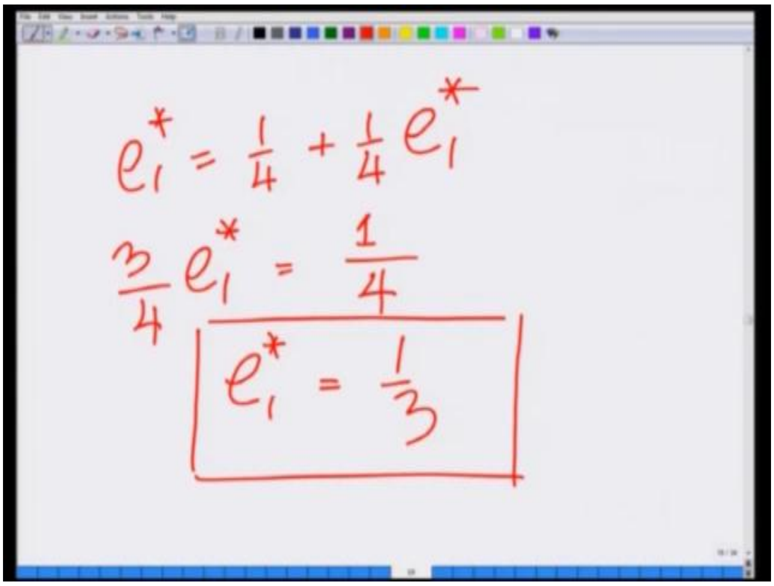

And therefore, I have e 1 star equals half minus half 1 minus e 1 stars divided by 2 which after simplification gives me e 1 star equals as you can clearly see 1 4th plus 1 4th e 1 star, which means 3 4 e 1 star equals, 1 4th and therefore, e 1 star equals, e 1 star equals only this set of equations, I get e 1 star equals 1 by 3. So, that is the Nash equilibrium effort e 1 star of agency 1. What about e 2 star?

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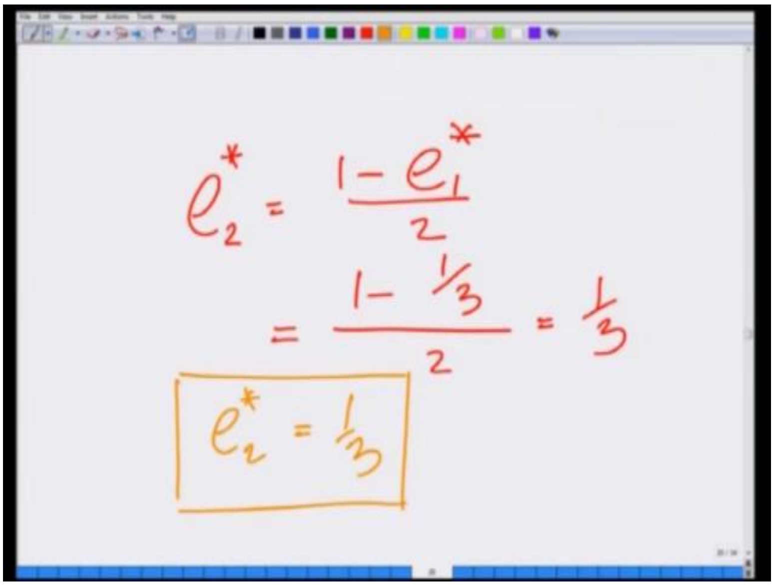

Well, e 2 star equals 1 minus e 1 star divided by 2, which is 1 minus 1 by 3 divided by 2. we can also see this is 1 by 3, anyway we could also guess that from the symmetric of the game, therefore, we get that the effort e 2 star equals 1 by 3. The Nash equilibrium effort and therefore, what we have is we have an expression, we have interestingly found, what the Nash equilibrium effort for both the timber agencies is…

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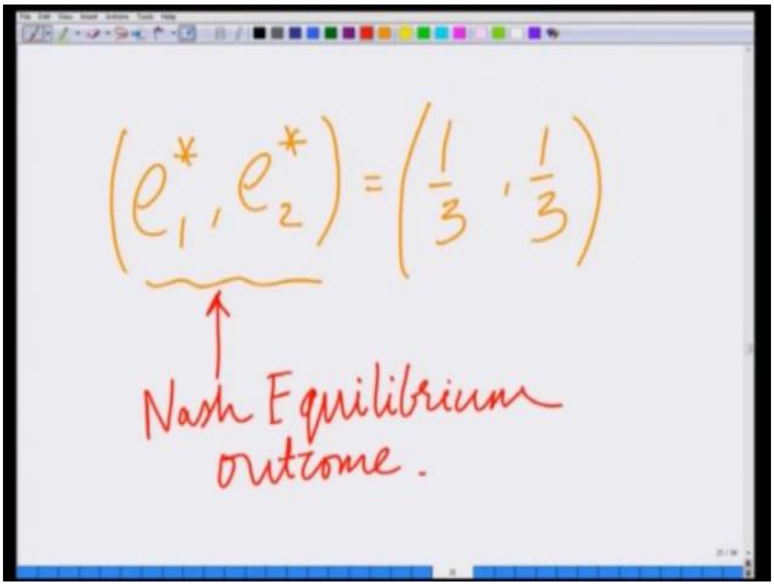

The Nash equilibrium effort or the Nash equilibrium outcome e 1 star comma e 2 star equals 1 by 3 comma 1 by 3 therefore, this is the Nash equilibrium this is the Nash equilibrium outcome each of them is using in an effort e 1 star equals 1 by 3 e 2 star equals 1 by 3 that is to say that e 1 star is the best e 1 star equal to 1 by 3 is the best effort of agency 1 to the effort e 2 equals star equals 1 by 3 by agency 2.

Similarly, e 2 star equal to 1 by 3 is the best response of agency 2 to the effort e 1 star equals to 1 by 3 by agency 1. And since, both of them are playing their best responses therefore, this is Nash equilibrium that is each one is playing the best response to the action of the other player which intern is a best response to the action of all. So, this is the Nash equilibrium outcome of this game.

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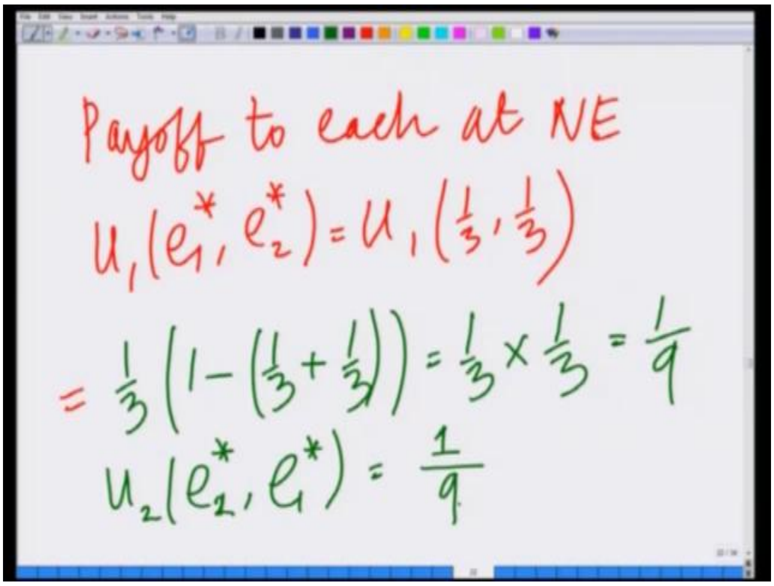

And finally, let us calculate, what the Nash payoff s, what is the payoff to each at Nash equilibrium at Nash equilibrium that is, what is u 1 of e 1 star e 2 star equals u 1 of 1 by 3 comma 1 by 3 this, we can say is equal to 1 by 3 1 minus e 1 plus e 2 1 by 3 plus1 by 3 which is equal to 1 by 3 into 1 by 3 equals 1 by 9. Similarly, you can show that because of course, both of payoff of symmetric u 2 of e 2 star e 1 star equals 1 by 9.

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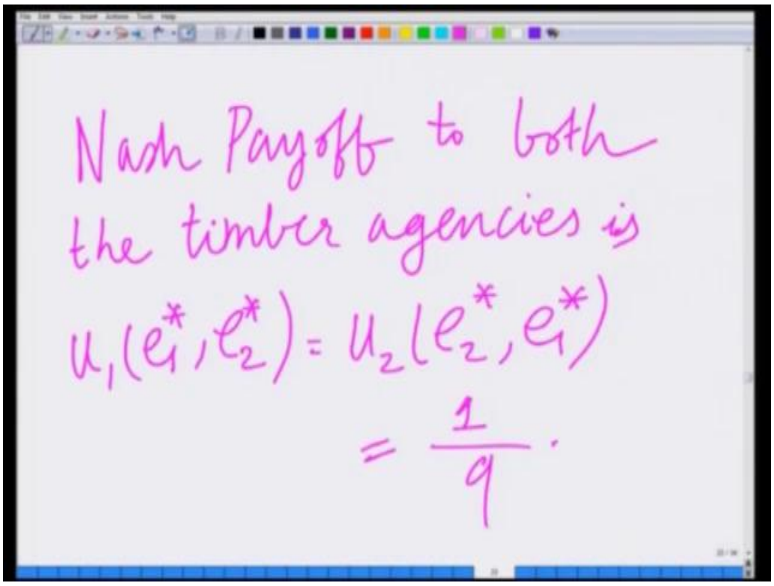

Therefore, what we can say is that the Nash payoff the Nash payoff are the payoff at Nash equilibrium to both the timber agencies is u 1 or u 1 of e 1 star comma e 2 star equals e 2 of e 2 star comma e 1 star equals 1 by 9, which is the Nash payoff of both this completing agencies are the payoff both this completing agencies at the Nash equilibrium.

So, let me summarize what learn in this game, so for we have looked at an interesting game, which is based on the tragedy of commons paradigm, which is basically. We are looking at interaction between two companies or two agents individual common resources for instants in this example. We consider two timber agencies logging forest for timber and there payoff is proportional to not only there effort, but inversely proportional to the joint effort put by everyone together.

Because that these two depletion of the resource this is a novel game compute what we have looked at previously because the efforts the possible set of efforts by each agent is infinite. So, the infinite set of possible actions of each player and we use differential calculus to compute the best response at the Nash equilibrium each of them is using each of the players playing is his or her or their best response.

Therefore, we have use this principle to solve a simultaneous set of linear equation to compute the Nash equilibrium. Nash equilibrium the equilibrium efforts to be even star equal to 1 by 3 into star 1 by 3 and also we found out the payoff to each of this from the Nash equilibrium as a last node even though we have consider only two for the agents in this example this can easily, we extended set of more such agency that is more such agents or more such an individual involved in the depletion this resource and that we need to interesting are conclusions I lead this I leave this as an effort or as a an exercise ah to the people are going this lecture. So, let us stop at this point and the next module we are going to analyze, what is a outcome of this game that is by, of the equilibrium, what is the equilibrium outcome what is effect on resource on the other aspects of this game the tragedy of commons.

Thank you.