Strategy: An Introduction to Game Theory
Prof. Aditya K Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 43

Hello, welcome to another module in this online course Strategy, An Introduction to Game Theory. So, we are looking at the Bayesian second price auction and we have derived the Nash equilibrium of the Bayesian second price auction.

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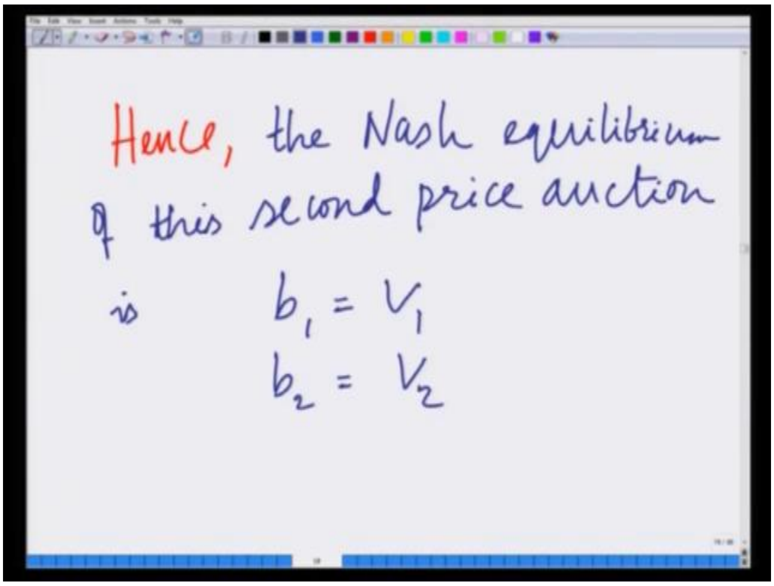

And we have said that each player bidding his true valuation b 1 equals V 1, b 2 equals V 2 is the Nash equilibrium for this second price auction. Let us now derive the expected revenue of this second price auction.

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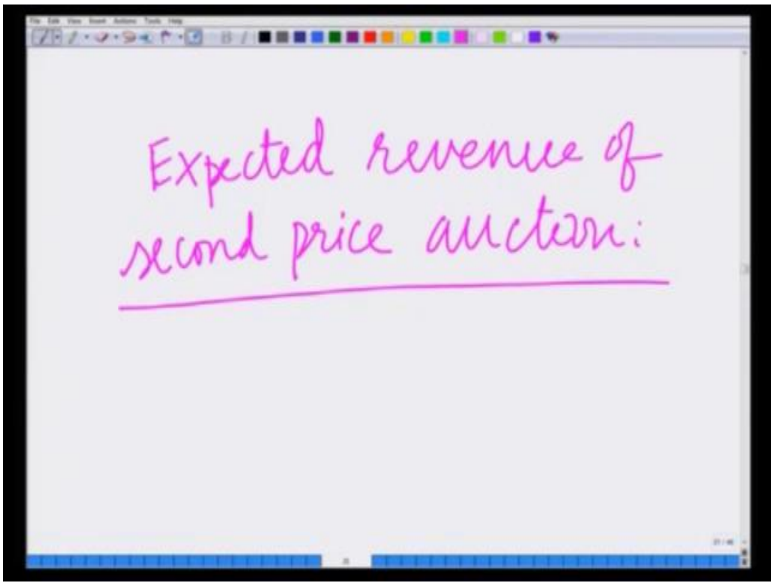

So, let us now derive the expected revenue of this Bayesian second price auction.

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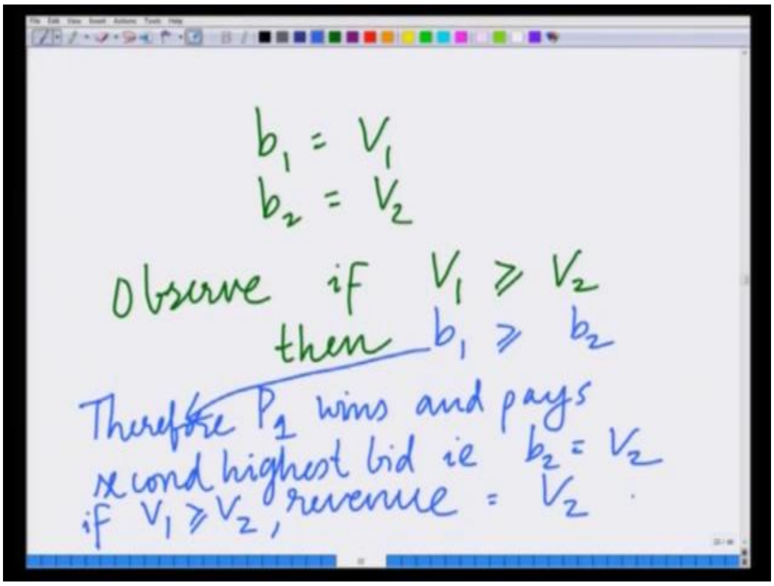

Now, the Nash equilibrium is b 1 equals V 1 and b 2 equals V 2, now observe if V 1 is greater than or equal to V 2, then we have then it follows that since b 1 equals V 1 and b 2 equals V 2, we have b 1 is greater than or equal to b 2. Therefore, player 1 wins and pays the second highest bid therefore, P 1 wins and pays second highest bid; that is b 2 equals V 2. So, if V 1 is greater than equal to V 2, then the payment to the auctioneer, revenue to the auctioneer is b 2. If V 1 is greater than or equal to V 2, revenue equals V 2, revenue to the auctioneer equals V 2.

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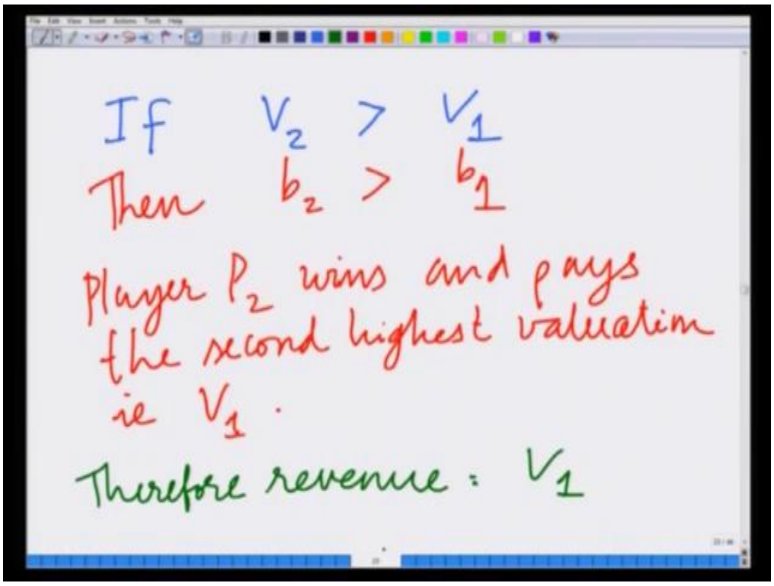

On the other hand, if V 2 is greater than V 1 then b 2 is greater than b 1, player 2 wins; player P 2 wins and pays the second highest valuation that is V 1 therefore, the revenue equals V 1. So, what we are seeing is something very simple, if V 1 is greater than or equal to V 2 ((Refer Time: 03:14)), then the revenue to the auctioneer is V 2. On the other hand, if V 2 is greater than V 1 that is valuation 2 is greater than valuation 1, then the revenue to the auctioneer is V 1.

Based on both these observations, we can conclude that the revenue for the auctioneer in these two players, Bayesian second price auction is the minimum of the valuations V 1 comma V 2.

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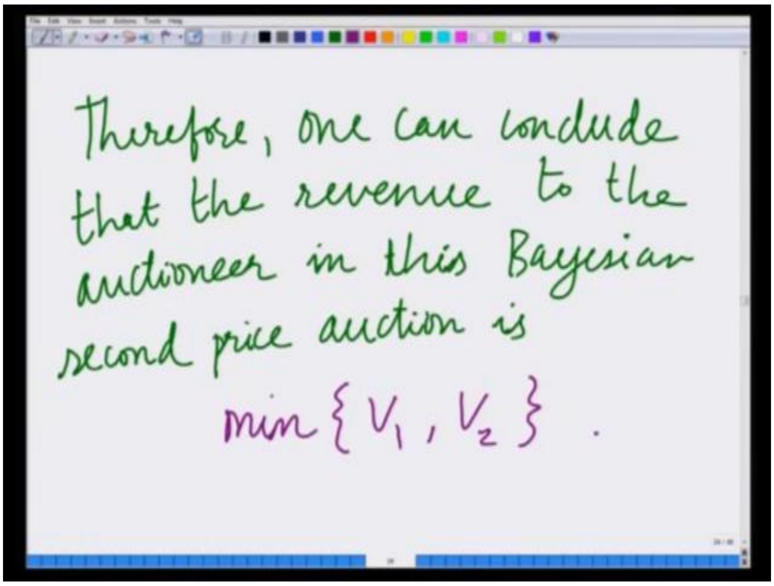

Therefore, one can conclude that the revenue to the auctioneer in this Bayesian second price auction is the minimum of V 1 comma V 2. Therefore, what we can conclude is that the revenue to the auctioneer is the minimum of V 1 comma V 2. Remember, we also assumed that these valuations V 1 comma V 2 are distributed as independent random variables uniform with uniform probability density function in the interval 0 to 1.

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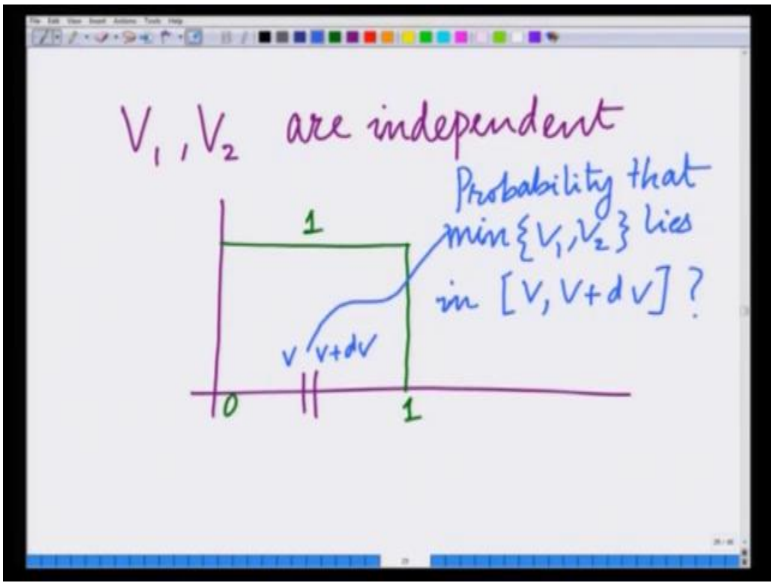

So, each, so V 1 comma V 2 are independent and they are distributed as uniform random variables in the interval 0 to 1. Now, let us ask the question what is the probability that the minimum of these two random variables lies in the interval v plus d v. So, what is the probability that minimum of V 1 comma V 2 lies in v comma v plus d v? What is the probability that the minimum of these two random variables V 1 comma V 2 lies in the interval v comma v plus d v?

We can analyze this again similar to the first price auctions scenario, remember there we have considered the maximum of V 1 comma V 2. Now, we are considering the minimum of V 1 comma V 2, well this occurs in two scenarios.

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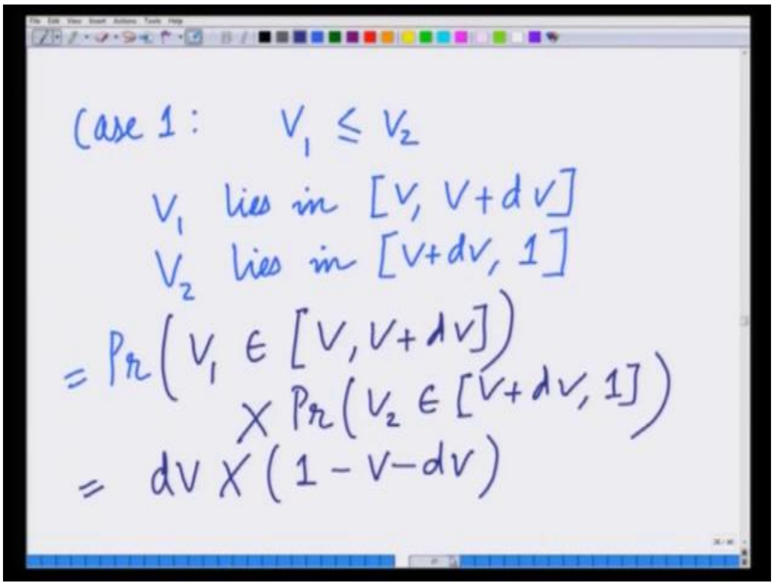

In case 1, when V 1 is the minimum that is V 1 is less than or equal to V 2 therefore, V 1 lies in the interval v comma v plus d v; that is it implies in an infinite decimal interval v to v plus d v and V 2 lies to the right of V 1, V 2 lies in v plus d v comma 1. So, the first case is where V 1 is the minimum and V 1 lies in the interval v to v plus d v and V 2 lies to the right of V 1; that is in the interval v plus d v to 1.

The probability of this, so therefore, probability of this event equals probability V 1 lies in or belongs to the interval v plus d v times the probability V 2 belongs to the interval v plus d v comma 1. We said since these are uniform random variables distributed uniformly in the interval 0 to 1, if the probability that it lies in any particular interval is equal to the length of the interval. Therefore, the probability that V 1 lies in v to v plus d v is equal to the length of the interval d v times the probability V 2 lies in v plus d v to 1 is the length of the interval 1 minus v minus d v.

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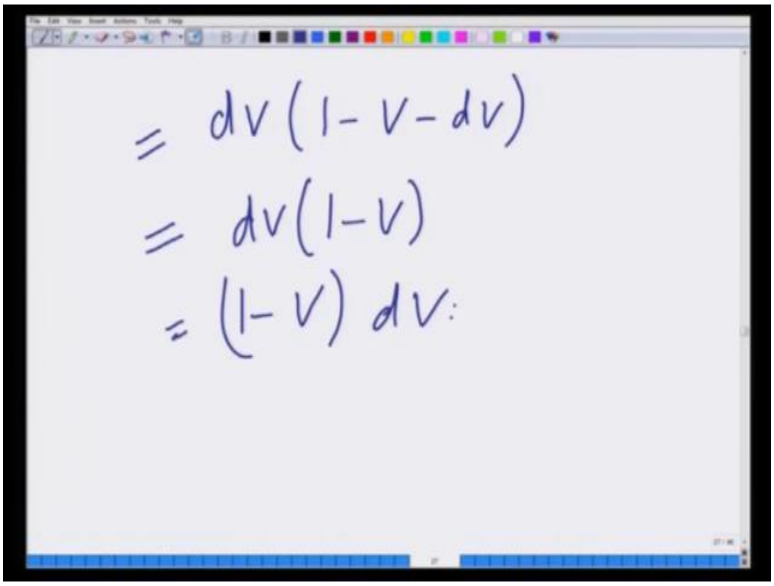

So, therefore, the probability is equal to d v times 1 minus v minus d v, now this term d v here is small in comparison to v, so I can neglect this. Since, these are infinite decimal quantities that we are talking about and therefore, that probability is equal to 1 minus v d v, equals 1 minus v d v. So, the probability that the minimum lies in v to v plus d v, we are analyzing it by splitting into two cases; one is V 1 is the minimum and V 1 lies in the interval v to v plus d v and V 2 lies to the right of V 1 that is in the interval v plus d v to 1.

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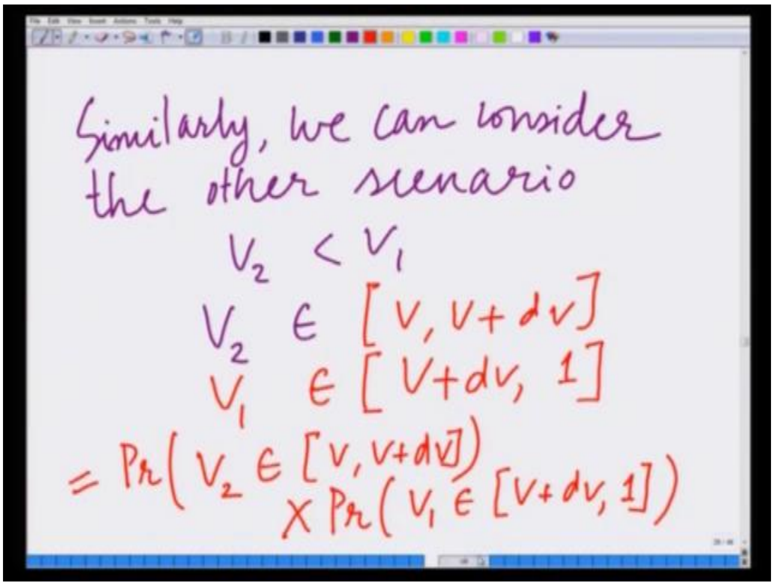

Similarly, we can consider the other scenario. What is the other scenario? The other scenario is V 2 less than V 1, V 2 belongs to the interval or rise in the interval v to v plus d v and V 1 lies to the right of V 2 that is v plus d v to 1. So, we are considering the other scenario, where V 2 is the minimum of V 1 comma V 2 and V 2 therefore, lies in the interval v to v plus d v. Since, we are saying the minimum must lie in the interval v to v plus d v and V 1 lies to the right of V 2; that is in the interval v plus d v to 1.

And therefore, the probability of this is equal to the probability V 2 lies in v to v plus d v times the probability V 1 lies in v plus d v to 1. Again, we are saying that each probability, since these are uniform in the interval 0 to 1.

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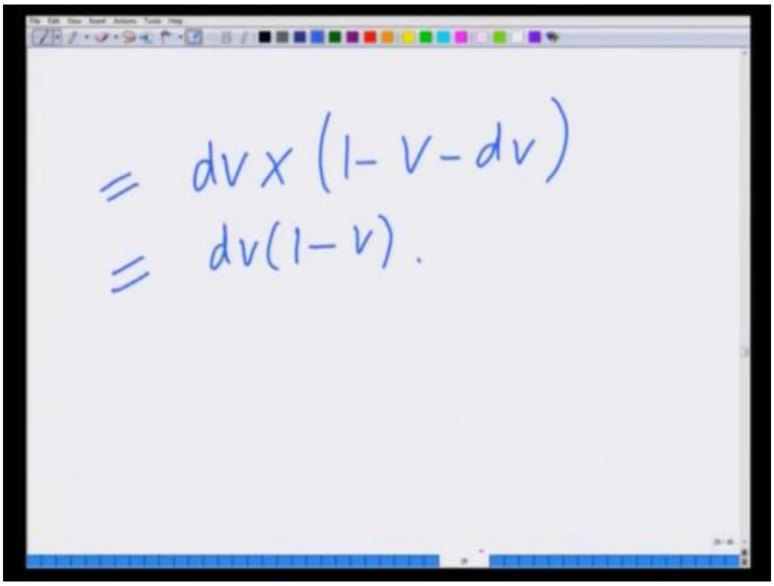

The probability, each probability is simply the length of the interval, so this is equal to d v times 1 minus v minus d v. Again, since d v is an infinite decimal, I can ignore it in comparison to the v in the second terms, so this is again equal to d v 1 minus v. Therefore, now considering the two cases where V 1 is less than or equal to V 2 and V 2 is less than V 1.

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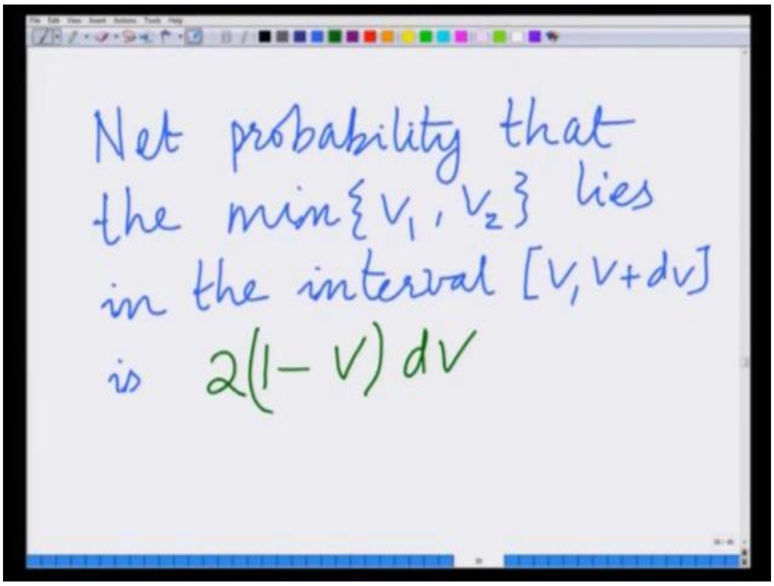

The net probability that the minimum of V 1 comma V 2 lies in the interval v to v plus d v is given as 1 minus v times d v twice 1 minus v. So, the net probability that the minimum of V 1 comma V 2 lies in the interval v to v plus d v is given as twice of 1 minus v times d v. Therefore, now we have the probability that the minimum lies in the interval v to v plus d v. If the minimum lies in the interval v to v plus d v, then the revenue to the auctioneer is equal to v.

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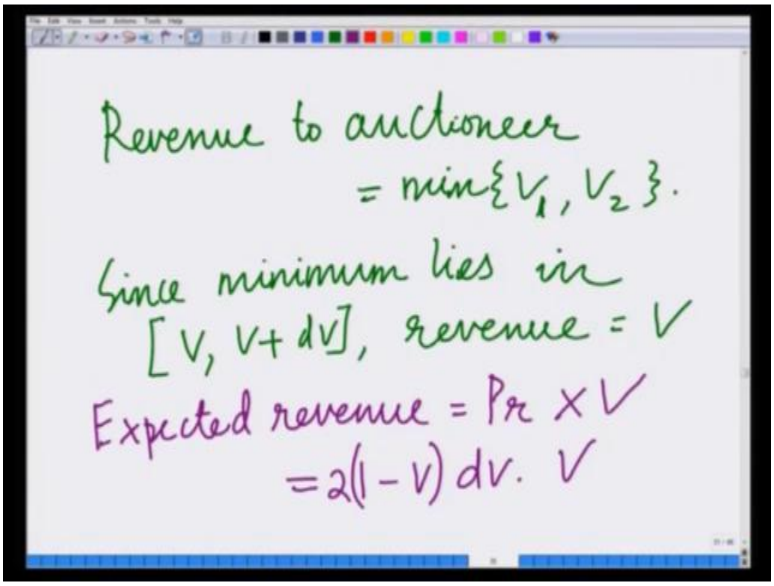

Remember, revenue to the auctioneer equals minimum of V 1 comma V 2, since minimum lies in the interval v to v plus d v, revenue to the auctioneer equals v. Therefore, the expected revenue equals the probability times v, which is basically equal to twice 1 minus v d v times v. So, with probability twice 1 minus v into d v corresponding to the minimum of V 1 comma V 2 lying in this infinite decimal is for interval v to v plus d v, the auctioneer gets a revenue of v.

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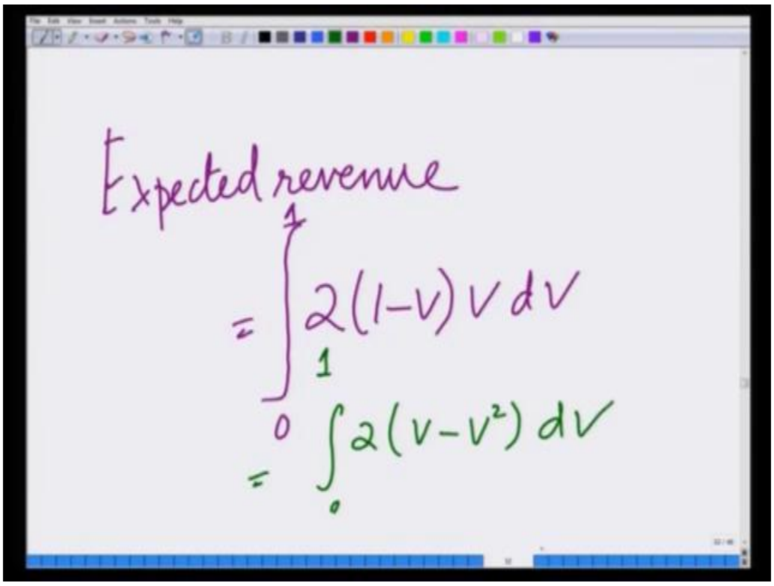

So, expected revenue equals twice 1 minus v into v d v and now, this has to be integrated corresponding to each infinite decimal interval of width d v between 0 and 1. So, therefore, I have the integral 0 to 1 twice into 1 minus v into v d v. So, I have this integral twice into v into 1 minus v d1v between the limits 0 to 1, which is also the same as integral evaluated between the limit 0 to 1 twice v minus v square d v.

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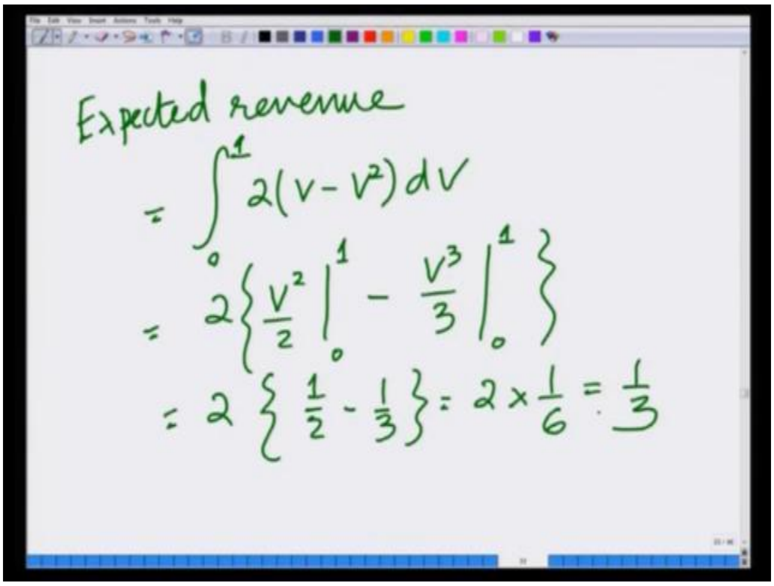

So, this is equal to the expected revenue equals integral 0 to v twice v minus v square d v, which is also equal to twice. Then, the integral of v is v square divided by 2, evaluated between the limits 0 to 1 minus v cube divided by 3 evaluated between the limits 0 to 1, which is twice into half minus one-third which is twice into 1 6 which is equal to 1 by 3.

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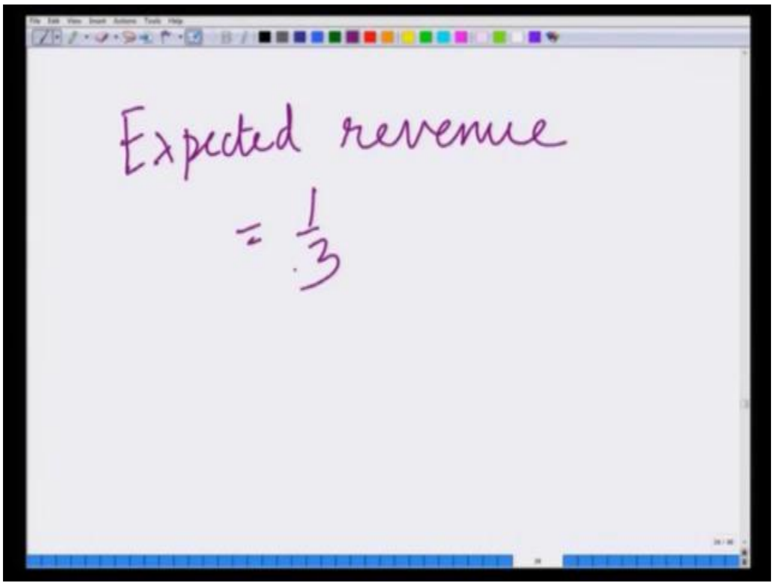

Therefore, what we are seeing is something that is very interesting. We are seeing that the expected revenue is equal to one-third and this is the same as the expected revenue, remember of the first price auction. Remember, we looked at the Bayesian first price auction earlier in one of the previous module and there also, we had seen that the expected revenue when the valuations for distributed uniformly in the interval 0 to 1 was one-third.

Therefore, what we have seen is the revenue of these two auction formats is actually equal. Although, these are two very different auction formats, we are seeing that the revenue of these two different auction formats is equal. This is termed as the revenue equivalence principle.

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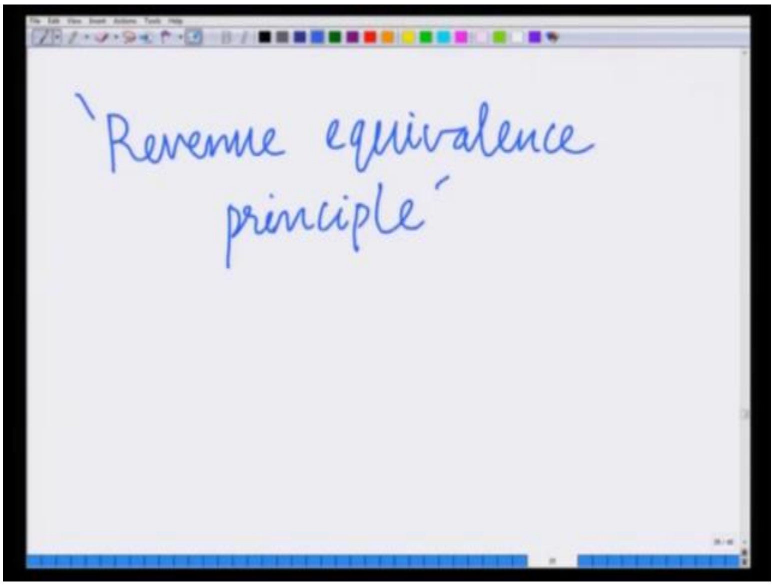

The system has the revenue equivalence principle. So, what we have seeing these, that across these different irrespective of the bays, irrespective of the format of the auction. We are seeing that the revenue to the auctioneer, that is the net revenue to the auctioneer is equal across these different auction formats and this is termed as the revenue equivalence principle and this is a very important principle in auction theory.

We say that irrespective of the auction format, the revenue auctioneer is always equal and this is termed as the revenue equivalence principle. So, what you have done so far is we have looked at the Bayesian second price auction, we have derived the Nash equilibrium of the Bayesian second price auction, that in each player bidding his true valuation that is b 1 equals V 1, b 2 equals V 2 is the Nash equilibrium, the Bayesian Nash equilibrium of the second price auction. And now, we have also derived the expected revenue of the second price auction. When we have said that the expected revenue is 1 by 3, which is same as that of the first price auction and this is, this illustrates the revenue equivalence principle.

Thank you very much.