Strategy: An Introduction to Game Theory
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Lecture - 41

Hello, welcome to another module in this online course Strategy, An Introduction to Game Theory. In the previous module we had looked at the sealed bid first price auction. Now, we have found the Nash equilibrium for the sealed bid first price auction, where the valuations v 1 and v 2 are distributed uniformly in 0 to 1. Let us now find another key aspect of this auction which is termed as the expected revenue of the auction.

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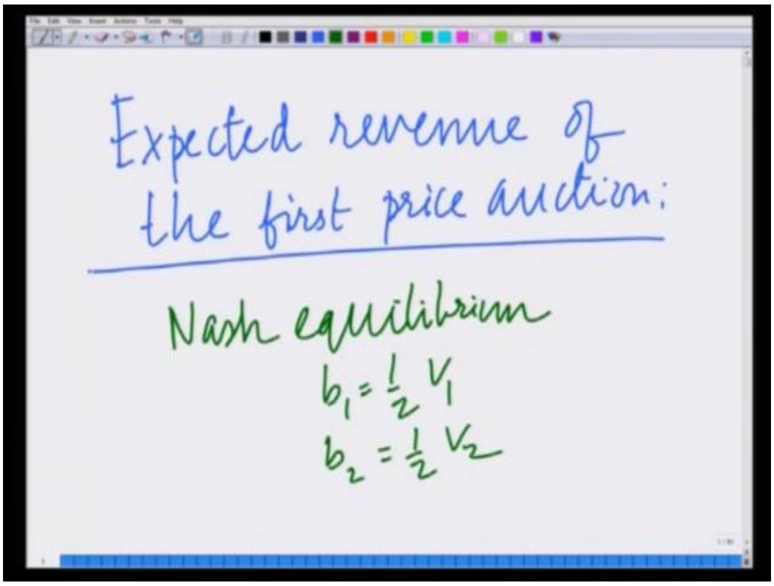

So, let us, so in this module we are going to focus on the expected revenue or the expected revenue to the auctioneer, expected revenue of the first price auction that is what is the revenue, this first price auction is expected to bring the auctioneer that is what is the average price this object which is being auction fetches. And we had already seen that the Nash equilibrium of the sealed bid first price auction, the Nash equilibrium you seen in the previous module Nash equilibrium is bidding b 1 equals half v 1, b 2 equals half v 2.

So, the Nash equilibrium bids of both the players are b 1 equals half v 1 and b 2 equals half v 2. And also remember that in the sealed bid first price auction, the player with the highest bid wins the auction and pays an amount equal to his bid therefore, the revenue to the auctioneer is the maximum of the bids b 1 and comma b 2. Therefore, since the player with the highest bid wins the auction or since the player with the maximum bid wins the auction pays an amount equal to it is bid.

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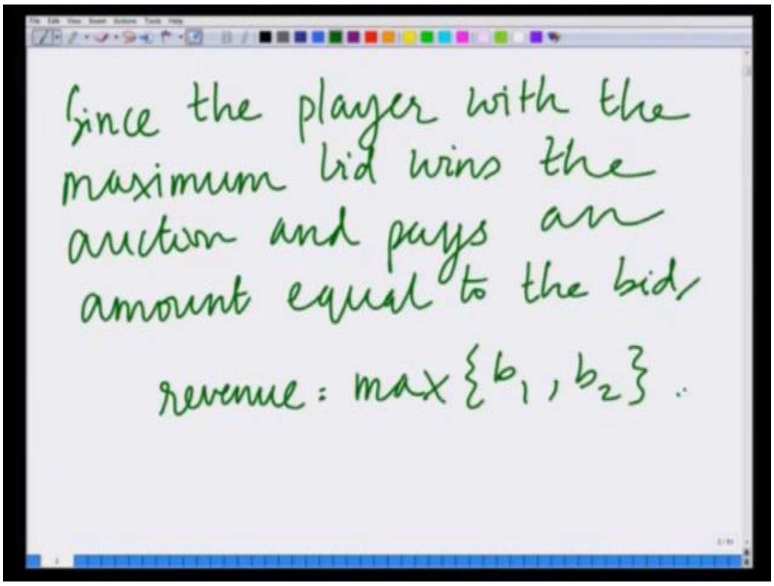

Since, the player with the maximum bid wins the auction and pays an amount equal to the bid, the revenue of the auction here revenue equals the maximum of the bids b 1 comma b 2. Since, the player of the bidder with the highest bid wins the auction and pays an amount equal to the bid, the revenue to the auctioneer is the maximum of b 1 comma b 2. However, we also know that at Nash equilibrium b 1 equals half v 1 and b 2 equals half v 2.

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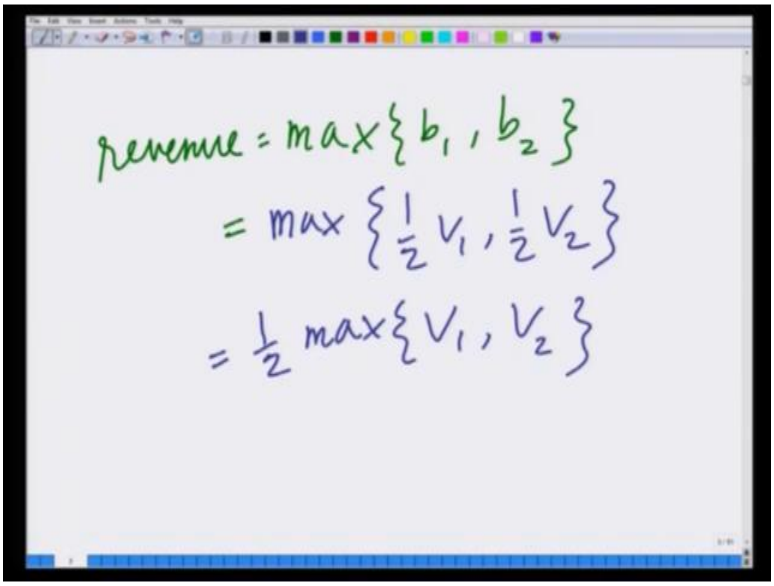

Therefore, we have revenue equals maximum of half b 1 comma b 2, but b 1 equals half v 1 and v 2 equals half v 2. Therefore, we can also say the revenue is the maximum of half v 1 comma half v 2, which is half of the maximum of v 1 comma v 2, where v 1 and v 2 are the valuations of players 1 and 2 respectively. So, basically the revenue two the auctioneer, we are sure is half of the maximum of the valuations v 1 comma v 2.

Now, since these valuations v 1 and v 2 are random variables, more precisely these are random variables which are distributed uniformly in the interval 0 to 1, we have to find the average value of the revenue, which means we have to find the average value, we have to find half of the average value of maximum of v 1 comma v 2. And we know, v 1 and v 2 are valuations, let us assume these are independent valuations which are uniformly distributed in 0 comma 1.

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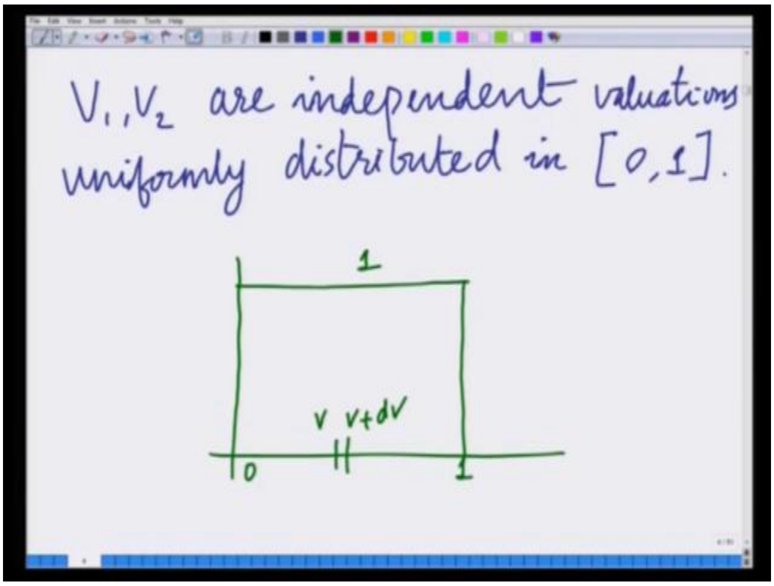

So, v 1 comma v 2 are independent valuations uniformly distributed in, these are independent valuations which are uniformly distributed in 0 comma 1. Therefore, let us now consider, let us now look at our uniform random variable which is distributed uniformly in 0 comma 1 and let us now look at a small interval around v and v plus d v. So, let us look at this infinitesimal interval between v and v plus d v and now let us ask the question what is the probability that the maximum of v 1 comma v 2 lies in this infinitesimal in small interval v to v plus d v.

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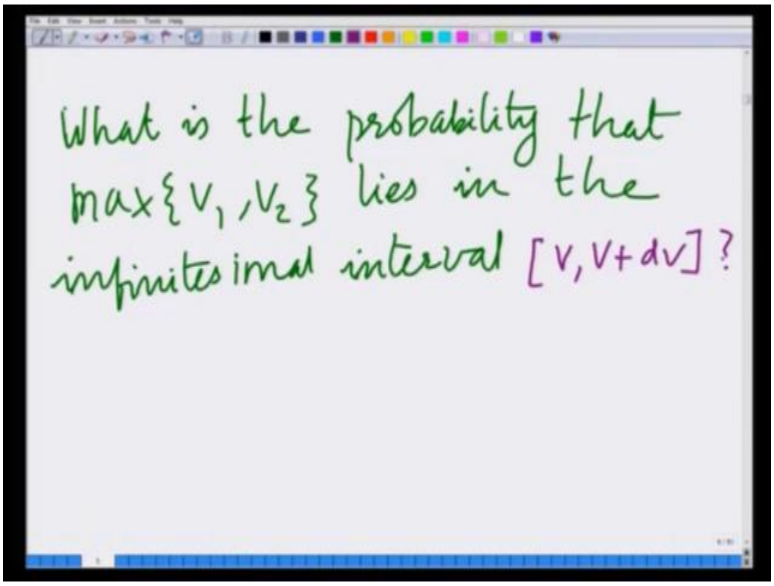

So, let us ask the question what is the probability that maximum of v 1 comma v 2 lies in the infinitesimal interval v comma v plus d v. What is the probability then the maximum of v 1 comma v 2 lies in this infinitesimal interval v to v plus d v that is lies in this small interval of with d v. Now, you can see that can occur in two possible scenarios. What is the first scenario? The first scenario is when v 1 lies in v to v plus d v and v 2 lies in 0 to 1 that is if v 1 is the maximum, then v 1 should lie in this interval v to v plus d v and the other valuation v 2 should lie in the interval 0 to v.

And similarly, the other scenario is where v 2 is the maximum valuation and v 2 lies in this interval v to v plus d v and v 1 lies in the interval 0 to v. So, there can be two possible scenarios.

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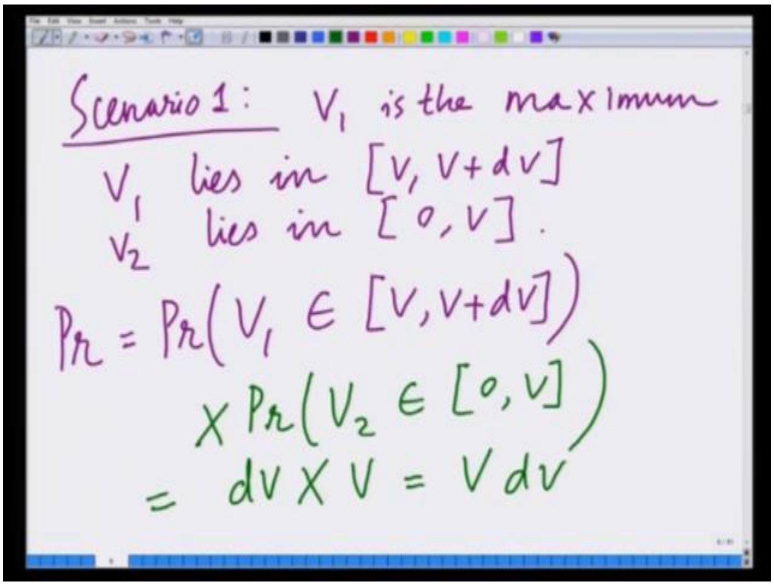

Let us look at each of them. What is the first scenario? Scenario 1, v 1 is the maximum v 1 lies in v 2 v plus d v and v 2 lies in 0 to v that is what we are saying is v 1 lies in this infinitesimal is small interval v to v plus d v and also v 1 is the maximum therefore, v 2 has to be less than v 1. So, v 2 can only lie between 0 to v and what is the probability of this event, probability of this event equals when since valuations we are assuming valuations v 1 and v 2 are independent, the probability of this is basically the probability v 1 lies in the interval v comma v plus d v multiplied by the probability v 2 belongs to the interval 0 to v.

So, this probability remember the first scenario occurs when v 1 lies in the interval v to v plus d v and v 2 lies in the interval 0 to v. Therefore, the probability of this joint occurrence is the product of the individual probabilities, since v 1 and v 2 are independent random variables. So, this is obtained by multiplying the probability that v 1 lies in the interval v to v plus d v times the probability that v 2 lies in the interval 0 to v.

And this is given as now you can see the probability that v 1 lies in the interval v to v plus d v is equal to the length of the interval. Remember, since we said that v 1 and v 2 are uniform random variables in the interval 0 to 1, the probability that v 1 lies in the interval v to v plus d v is equal to the length of the interval that is d v. And similarly the probability that v 2 lies in the interval 0 to v is the length of the interval which is equal to v.

So, therefore this is equal to the first probabilities the length of the interval which is d v times the second probability that v 2 belongs 0 to v is v. So, the net probability is v d v, so what are we derived we have derived the fact that the probability that the maximum that if you consider the valuations v 1 and v 2 which are uniformly distributed in 0 to 1, the probability that v 1 is maximum and it lies in the interval v to v plus d v and v 2 lies in the interval 0 to v is v times d v.

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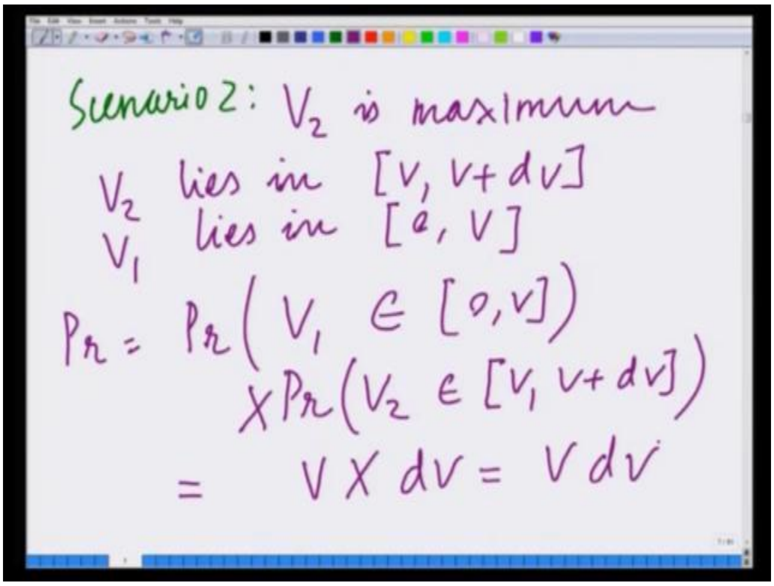

Now, let us consider the other scenario that is scenario 2, scenario 2 is where v 2 is maximum and therefore, v 2 lies in the interval v 2 v plus d v and v 1 since v 2 is maximum v 1 has to be less than v 2. So, v 1 can only lie in the interval 0 to v and the probability of this event equals again the probability that now v 1 belongs to the interval 0 to v times the probability that v 2 belongs to the interval v comma v plus d v. The probability that v 2 lies in the interval v to v plus d v and v 1 lies in the interval 0 to v is equal to the multiplication of these individual probabilities.

And now again the probability that v 2 lies in the interval v to v plus d v is the length of the interval d v times the probability that v 2 that the probability that v 1 lies in the interval 0 to v is the length of the interval v times the probability that v 2 lies in the interval v to v plus d v is the length of the interval d v. So, this is also similarly again v times d v.

Therefore, what are we done? We have now computed the probability that the maximum of v 1 comma v 2 lies in this small interval v to v plus d v. What is the probability? Well, we have analyzed it by splitting it into two scenarios, in the first scenario v 1 lies in the interval v to v plus d v and v 2 lies to the left of v 1 in the second scenario v 2 is the maximum and it lies in the interval v to v plus d v and v 1 lies to the left of v 2 and therefore, the net probability that the maximum of v 1 comma v 2 belongs to this interval v to v plus d v is equal to 2 v d v that is the sum of the probabilities corresponding to these two scenarios.

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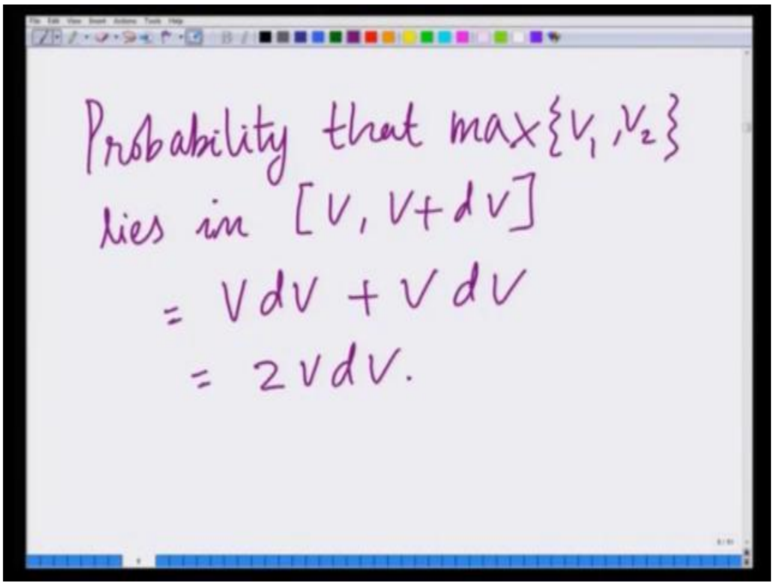

So, therefore, probability that max v 1 comma v 2 lies in v comma v plus d v is equal to v d v plus v d v equals 2 v d v. So, what are we computed we have basically computed the probability that the max of v 1 comma v 2 lies in the interval v 2 v plus d v. Now, what is the average revenue to the auctioneer? The average revenue to the auctioneer remember is, if the maximum lies between v to v plus d v average is if the revenue is half of maximum of v 1 comma v 2 that is half v times the probability.

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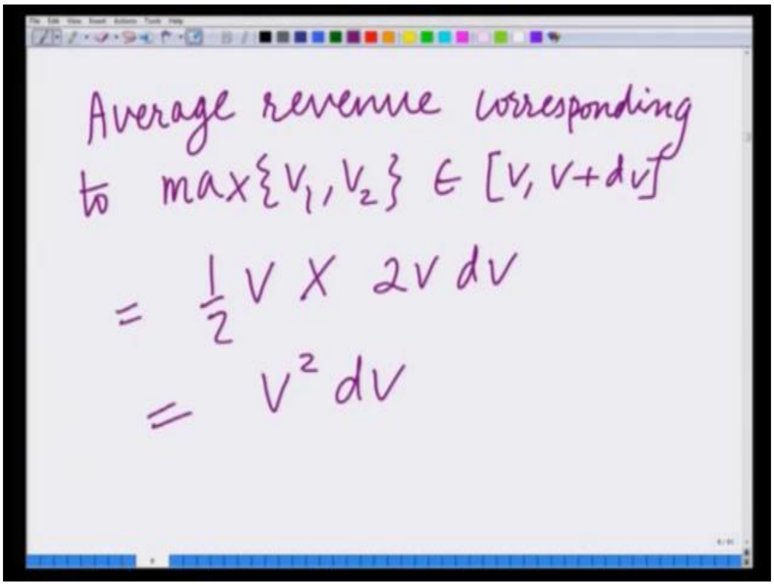

So, now, the average revenue a new corresponding to max v 1 comma v 2 belonging to the interval v comma v plus d v equals half times maximum of v 1 comma v 2, but max of v 1 comma v 2 belongs to the interval v to v plus d v since therefore, the revenue is half of v half of v times the probability, the probability is 2 v times d v. So, what are we saying the average revenue corresponds to the probability 2 v d v multiplied by half of v since the maximum of v 1 comma v 2 lies in the interval v to v plus d v.

So, the average revenue is the probability multiplied by half of the maximum which is half of v and therefore, this can be simplified as basically v square d v. So, the average revenue corresponding to the maximum of v 1 comma v 2 lying in this small interval v to v plus d v is v square d v. And now all that is remaining to be done is to integrate this quantity between the limits 0 to 1 to obtained the net average revenue to the auctioneer.

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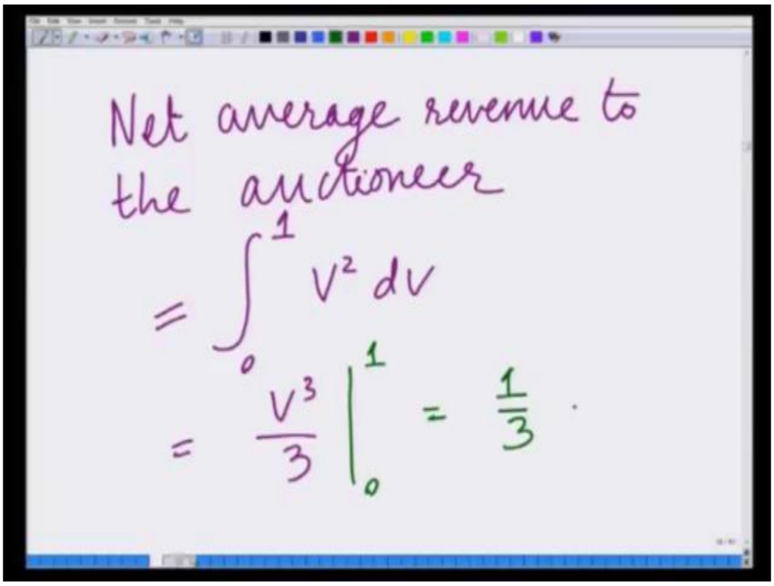

So, the net average revenue is equal to the integral of 0 integral between 0 to 1 v square d v which is also equal to v cube by 3 evaluated between the limit 0 to 1 equals 1 by 3. So, the average revenue to the auctioneer is 1 by 3 this is the expected revenue of the auctioneer.

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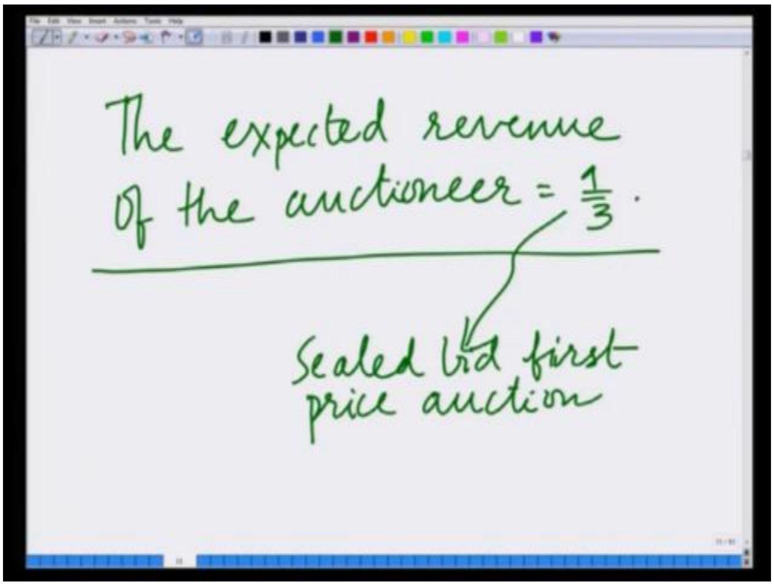

Auctioneer equal to 1 by 3 and this is for the sealed bid that is what are we shown, in this module we are shown that for the sealed bid first price auction between two players with valuations v 1 and v 2 which are distributed uniformly in the intervals 0 to 1. The average revenue to the auctioneer at the Nash equilibrium is equal to 1 by 3 or 1 by 3 is the average revenue to the auctioneer, this has the lot of significance that is this expected revenue to the auctioneer in the context of this Bayesian auction has a lot of significance it is in fact, key property of this auction and we are going to explore this more in the subsequence modules on auctions.

Thank you very much.