Strategy: An Introduction to Game Theory

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Lecture – 40

Hello, welcome to another module in this online course Strategy, An Introduction to Game Theory. So, let us continue our discussion on auctions, let us start by looking at a sealed bid first price auction.

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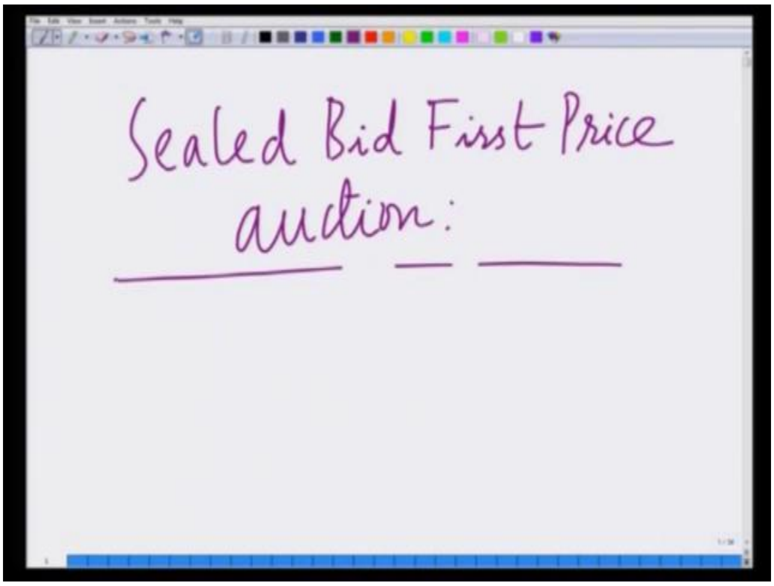

So, let us begin our discussion on auctions by looking at a sealed bid first price auction. So, we consider I have two player auctions.

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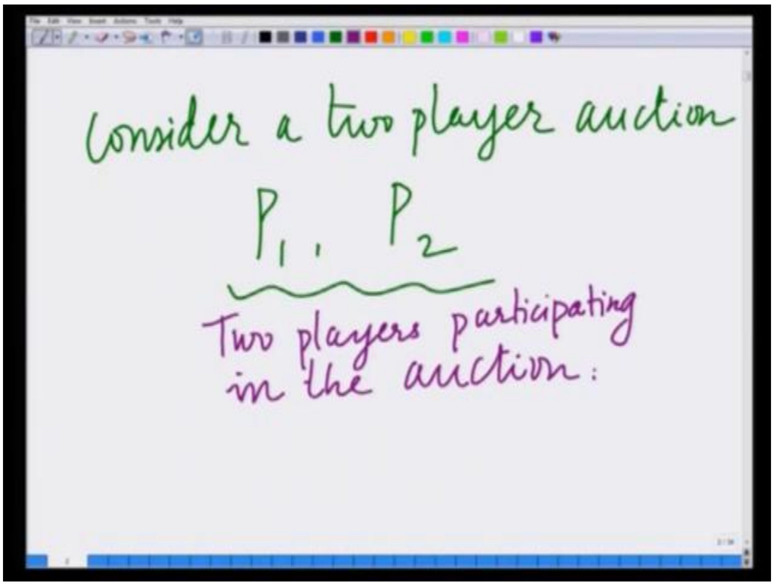

So, let us say we consider a two player auction, let us call these player 1 and player 2, these are the two players participating in the auction and both these players submit their bids for the object being auction, let say their bids are b 1 and b 2. So, these players submit their bids.

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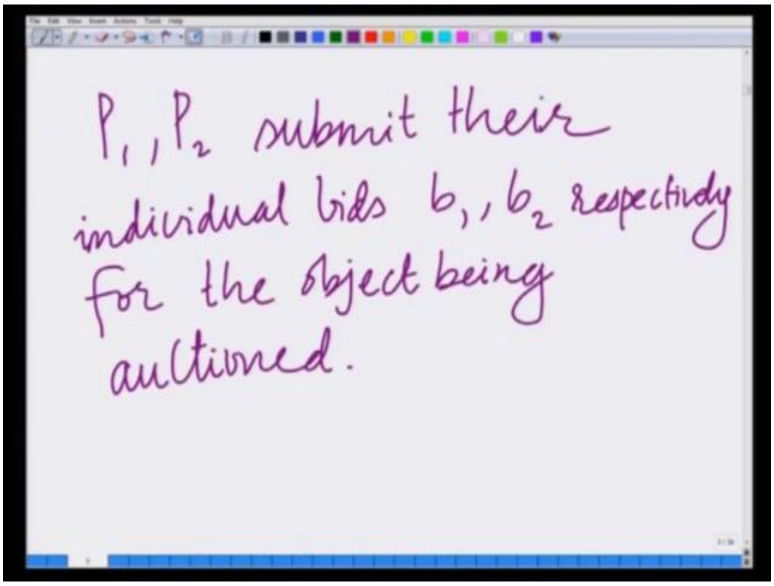

So, p 1 comma p 2 they submit their individual bids, b 1 comma b 2 respectively for the object being auction and these bids are submitted in a sealed envelope, these bids are sealed or these bids submitted in sealed envelope.

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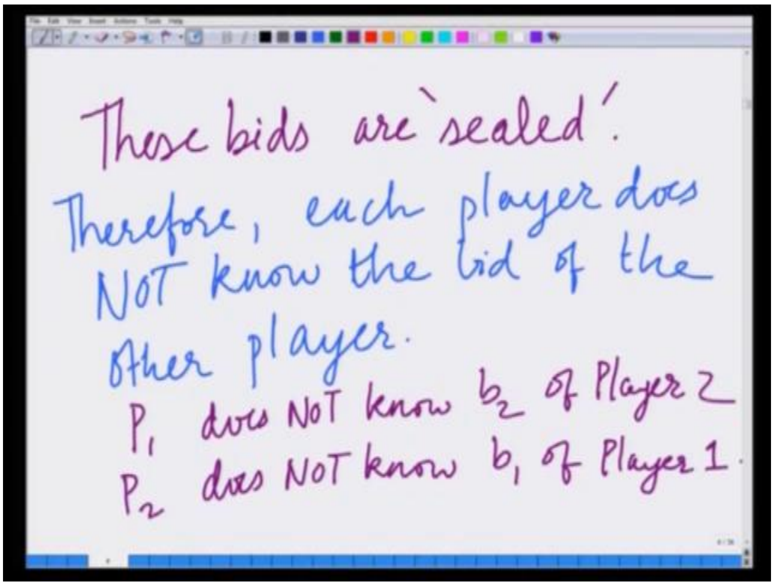

So, these bids are sealed therefore, the bid of each player is not known to the other player. So, the bid of each player is not known to the other player implies p 1 does not knows the bid b 2 of player p 2. And p 2 similarly does not know the bid b 1 of player p 1 therefore, each player does not know the bid of the other player that is p 1 does not know b 2 of player 2 and similarly p 2 player 2 does not know b 1 of player 1 are basically p 2.

So, p 2 p 1 and p 2 both the players they submit their bid b 1 and b 2 in a, in sealed envelope. So, which means they are not disclosing their bids to the other players. Therefore, p 1 does not know the bid b 2 of player 2 and p 2 that is the second player also does not know the bid b 1 of player 1 and the auction mechanism is as follows. The player with the highest bid wins the auction, this is known as the first price auction.

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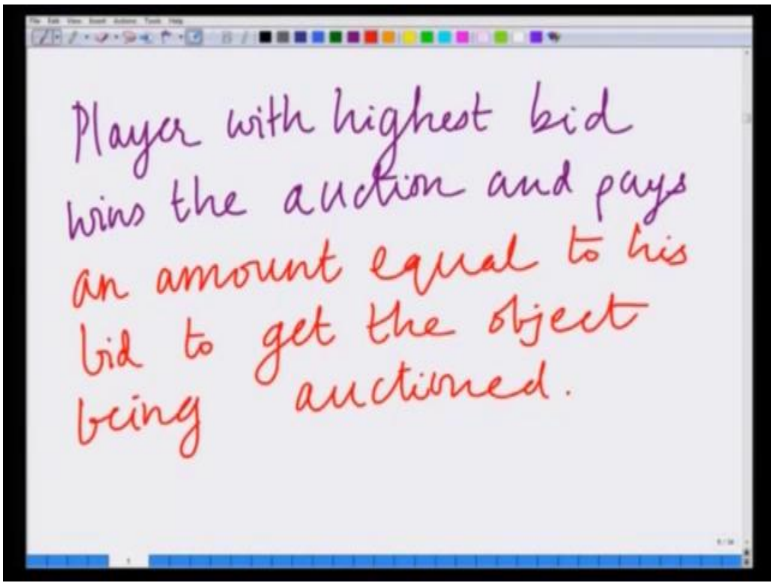

So, the player with the highest bid wins the auction and he pays an amount equal to his bid to get the object, this is known as the first price auction. So, player with highest bid wins the auction and pays an amount equal to his bid to get the object being auction.

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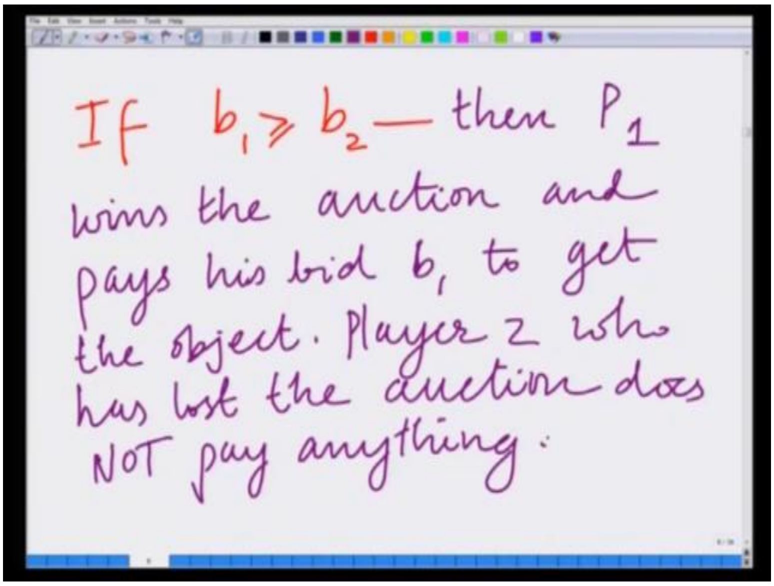

For instance, let us take an example, if b 1 is greater than or equal to b 2, if b 1 is greater than that is bid of player 1 is greater than or equal to bid b 2 of player 2, then this implies then p 1 or player 1 wins the auction and pays his bid which is b 1 to get the object. However, player 2 with his bid b 2 who has lost the auction does not pay anything. The other player who have lost auction that is player 2 who has lost the auction does not pay anything.

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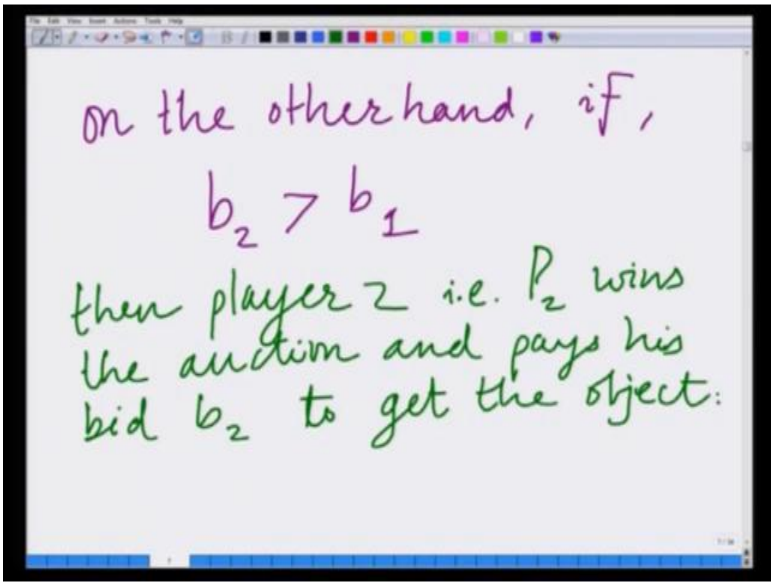

Similarly, if b 2 is greater than p 1 on the other hand, if b 2 is greater than b 1 then player 2 that is p 2 wins the auction that is he wins the auction and pays his bid amount that is b 2 to get the object. So, if b 2 is greater than b 1 that is player 2 has bid higher than player 1, then player 2 gets the object and he pays his bid amount that is b 2 and player 1 who has last the auction does not pay anything. So, this is known as the first price auction that is basically we are saying that is first price auction is an auction in which the player with the highest bid wins the auction and pays an amount equal to his bid value.

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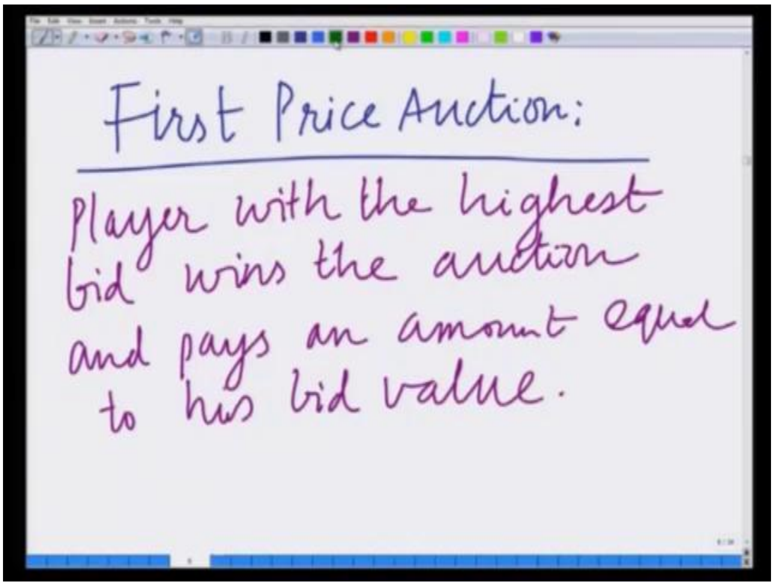

So, this is known as the first price auction, which is basically player with the highest bid wins the auction and pays an amount equal to his bid value to get auction, he pays an amount equal to his bid value that is if b 1 is greater than or equal to b 2, then player 1 wins the auction and he pays b 1 to get the auction, well player 2 does not pay anything. On the other hand, if b 2 is greater than b 1 then player 2 wins the auction and pays his bid amount b 2 to get the object and player 1 does not get anything, this is known as this auction format or this auction mechanism is also known as a first price auction. Now, in addition to this bids b 1 and b 2, in an auction each player has a private valuation for the objects.

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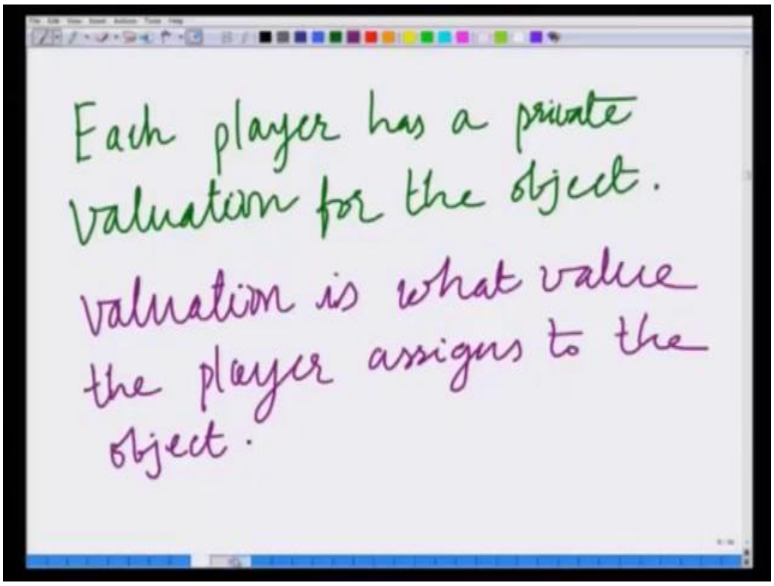

So, in addition to their bids each player has a private valuation for the object. What is the valuation? The valuation basically is how much the bidder value is the object, the valuation is I simply put a valuation is what value the player or the bidder player assigns to the objects. So, each player has a private valuation and this valuation is basically nothing but, the value that the player assigns to the object and this can be different from the bid, this need not be equal to the bid that is the player might bid differently and he might have a different value that he attaches to the object.

And ultimately his aim is to get secure the object or secure the object at a bid that is much lower than the value he places on it, so that he can make a profit out of it, so let us call these different private valuations, valuations of player 1 and player 2.

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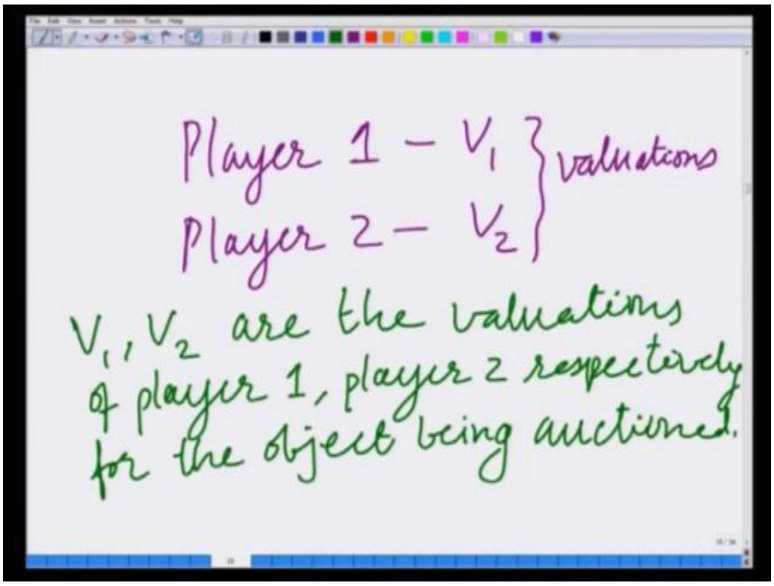

Let say player 1 has a valuation v 1 and player 2 has a valuation v 2. What are v 1 and v 2? V 1 and v 2 these are the valuations, so v 1 and v 2 are the valuations of player 1 and player 2 for the object being auctioned. So, v 1 comma v 2 are the valuations of the player 1 comma player 2 respectively for the object, these are the valuations of player 1 and player 2 respectively for the object being auctioned. And these valuations are private, which means player 1 does not know the valuation v 2 of player 2 and player 2 does not know the valuation v 1 of player 1.

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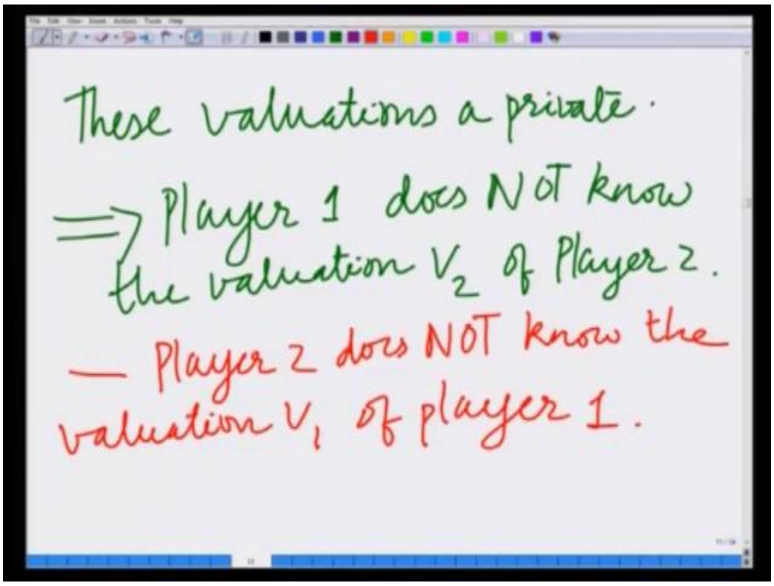

So, therefore, these valuations are private, implies player 1 does not know the valuation v 2 of player 2, further player 2 does not know the valuation of player 1. So, these each player does not know the valuation of the other. So, player 1 does not know v 2 which is the valuation of player 2 and player 2 does not know v 1 which is the valuation of player 1 therefore, these are known as private valuations. And; however, what is known is let say some information related to these valuations, some statistical information. For instance, let say with these valuations are distributed uniformly in the interval 0 to 1.

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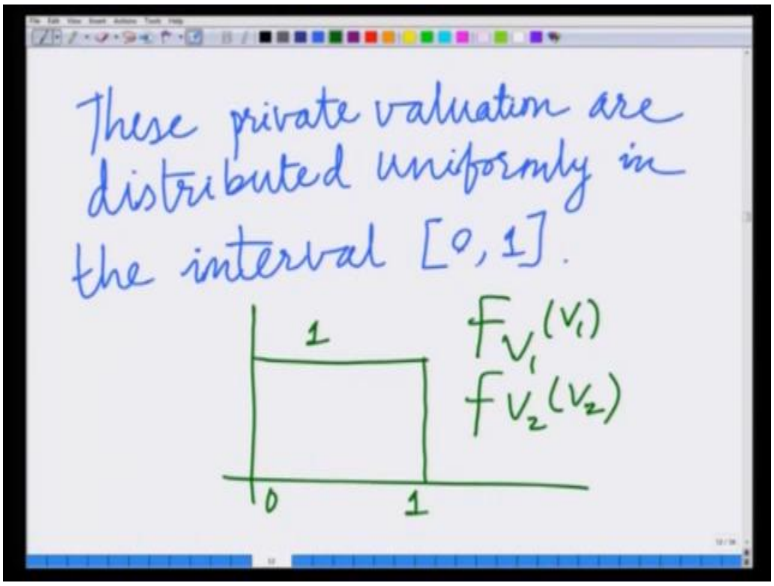

So, what we are going to assume as that these private valuations are what is known is that these private valuations are distributed uniformly in the interval 0 to 1. That is probability density of both v 1 and v 2 is uniform in the interval 0 to 1, this is what we are already seen, we are already seen an example of a uniform random variable in the interval 0 to 1 that is a random variable which is distributed uniformly in the interval 0 to 1. And what we are saying now is, we are using this uniform random variable in the interval 0 to 1 to characterized the valuations v 1 and v 2.

We are saying that the densities f of v 1 of v 1 and f of v 2 of v 2 at both follow I uniform random variable which is distributed uniformly in the interval 0 to 1. So, we are saying these valuations v 1 and v 2 are distributed uniformly in the interval 0 to 1 and what we wish to find is now the Nash equilibrium of this auction game or this the Bayesian auction game. Now, you can clearly see why this auction game is Bayesian nature, because there is uncertainty in this game that is uncertainly regarding the valuations.

So, each bidder or each player does not know the valuation of the other player, but here certain probability, a certain statistically information about the valuation all that type of the other. So, this context of an auction if you want to think of it as a Bayesian game, you can think of the valuation of the other player as the type of the other player. So, each type of the other way player has the different valuation and you can see there is a infinite number of types, because the valuation can take any real number between 0 to 1. So, this is the Bayesian games, since there is uncertainty regarding the valuations of the other place.

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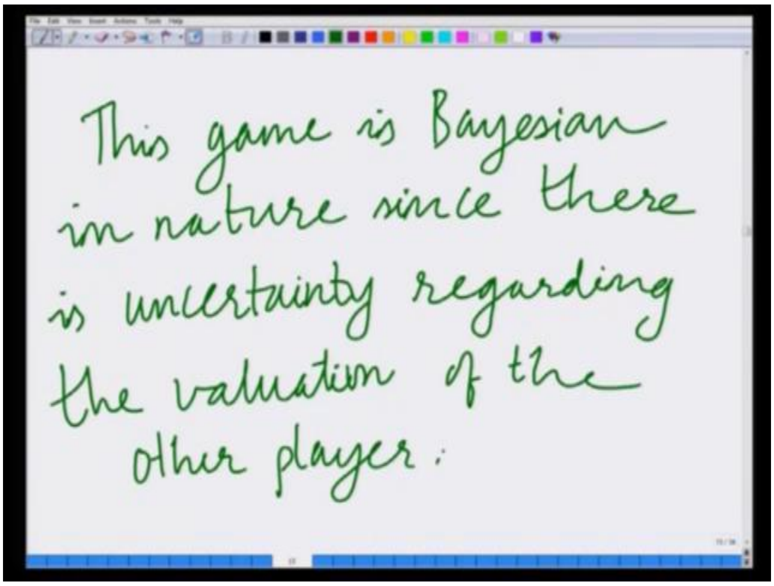

So, this game is this game of this auction game is Bayesian in nature, since there is uncertainty regarding the valuation of the other player. And therefore, we now have to analyze this game similar to the other games that we have consider before we have to find what is the bidding strategy of each player of each time. Therefore, we have to come up with the Nash equilibrium bidding strategy of each player participating in the auction. So, you would like to analysis this game to find the Nash equilibrium bidding strategy of each player.

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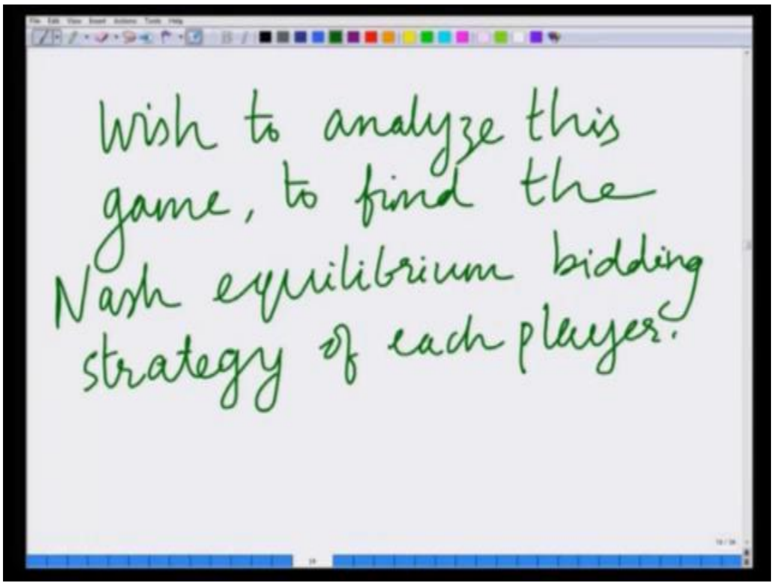

So, we wish to analyze this game to find the Nash equilibrium bidding strategy of each player, at Nash equilibrium what is the strategy employed by each bidder or each of this players in this first prize auction. And towards this what we are going to demonstrate is that we are going to demonstrate that the bidding strategy.

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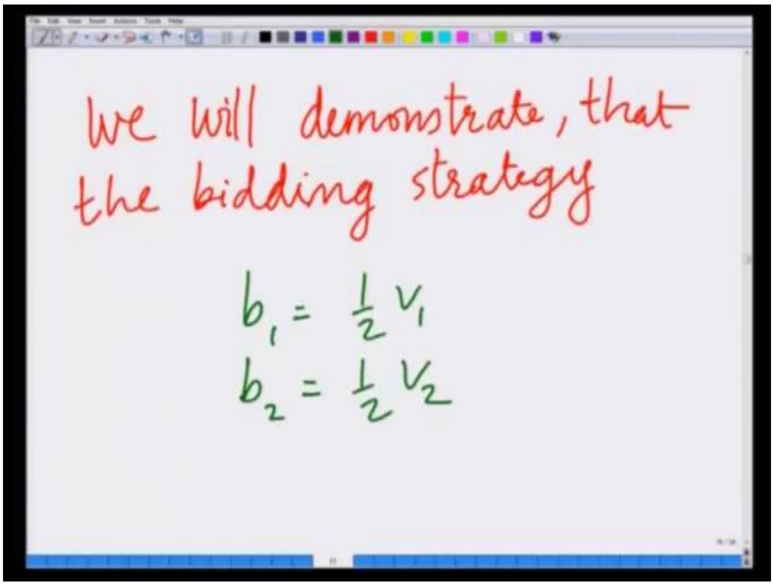

So, we will demonstrate that the bidding strategy b 1 equals half v 1 and b 2 equals half v 2 is the Nash equilibrium bidding strategy for this game that is b 1 equals half b 1, b 2 equals half v 2 of the Nash equilibrium bids to each player in this first price auction that is with each player is bidding half his for her valuation is the Nash equilibrium bidding strategy.

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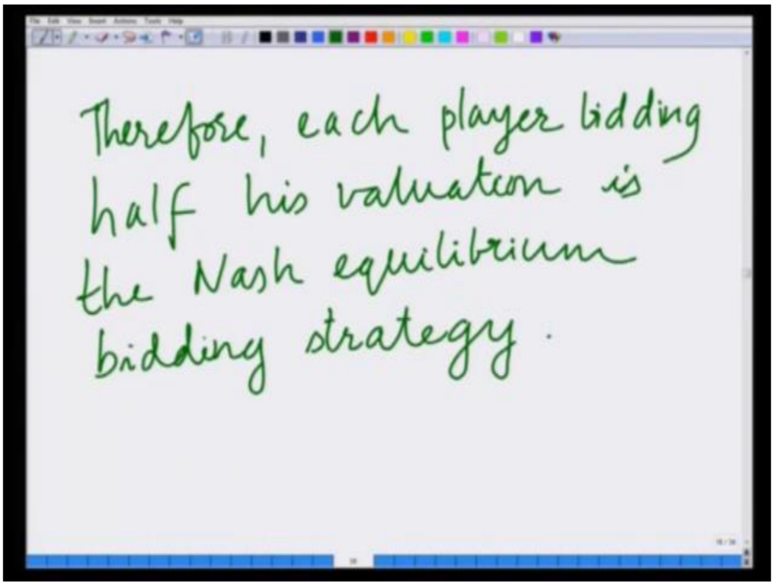

Therefore, half his valuation is the Nash equilibrium therefore, each player bidding half his valuation is the Nash equilibrium bidding strategy. So; however, we are going to demonstratives, again similar to what we have done in the context of previous game that is by showing that each player is playing is or her best response. That is we want to demonstrate that is Nash equilibrium b 1 equals half v 1 is the best response to bid b 2 equals half v 2 of player 2.

And similarly for player 2 b 2 equals half b 2 is the best response to b 1 equals half v 1 of player 1. So, we want to demonstrate that each of the players is playing his or her best response that is we wished to demonstrate that b 1 and b 2 that is b 1 equals half v 1 and v 2 equals half v 2 are the best responses to each other.

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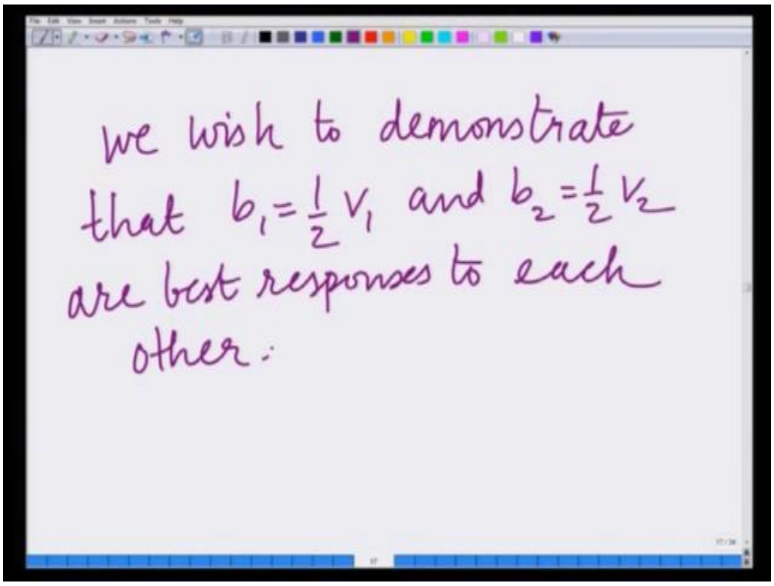

So, we wish we wish to demonstrate that b 1 equals half v 1 and b 2 equals half v 2 are best responses to each other. Now, let us start by considering that player 2 is bidding b 2 equals half b 2 and let us find, what is the best response b 1 of player 1.

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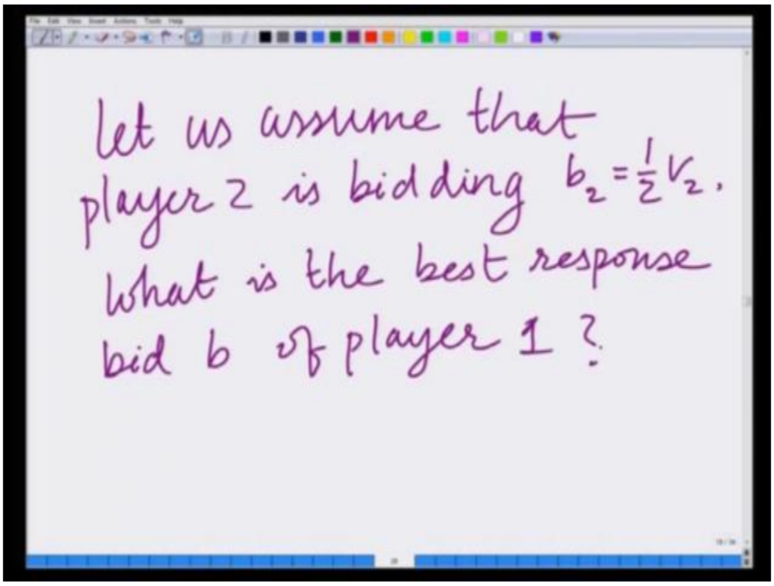

So, let us assume indeed player 2 is bidding b 2 equals half b 2. What is the best response bid b of player 1? To find the best response bid b of player 1 we have to find the valuation or we have to find the payoff to player 1 as the function of the bid b and then we have to find the value of b or the bid b at which his payoff is maximized. So, first we have to find the payoff as a function of b and we have to choose that particular value of b for which this payoff is maximized and that is his best response bid b to the bid b 2 equals half b 2 of player 2. So, let us find the best response, so now, first let us start by finding the payoff of player 1 or the payoff to player 1 as a function of his bid b.

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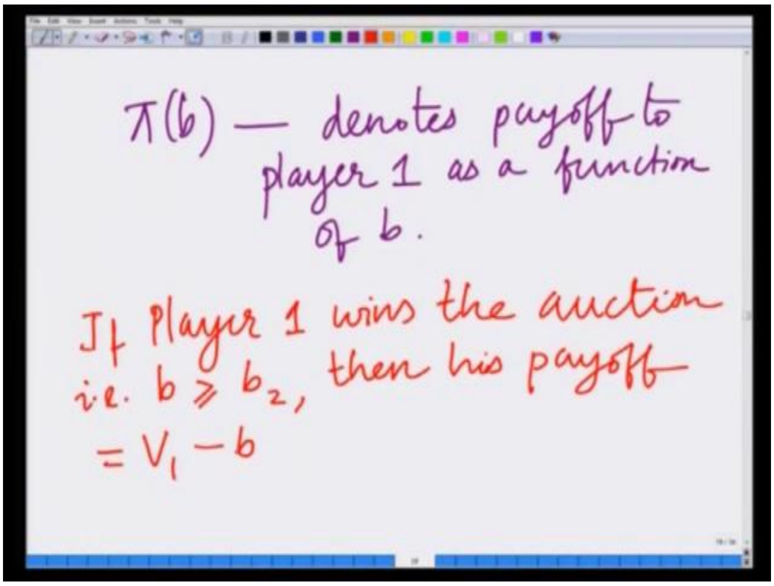

So, let us denote this quantity by pi of b that is pi of b denotes payoff to player 1 as a function of the bid b. And remember the payoff to the bid b payoff to player 1 as a function of bid b depends on one of two scenarios that is either player 1 wins the auction, if player 1 wins the auction which means is bid b is higher than the bid b 2. So, let us considered the first scenario that is player 1, if player 1 wins the auction and this corresponds to the scenario that is bid b being higher than equal to b 2 when his payoff is equal to, because if he wins the auction with bid b then he pays an amount equals to the bid b.

And therefore, he loses his bid b; however, he games in terms out of the valuation, because now he is going to get the object which has a valuation to which he attaches a value v 1. So, his net payoff is v 1 minus the amount paid which is the bid b, so is net payoff is v 1 that is his valuation minus the bid amount b.

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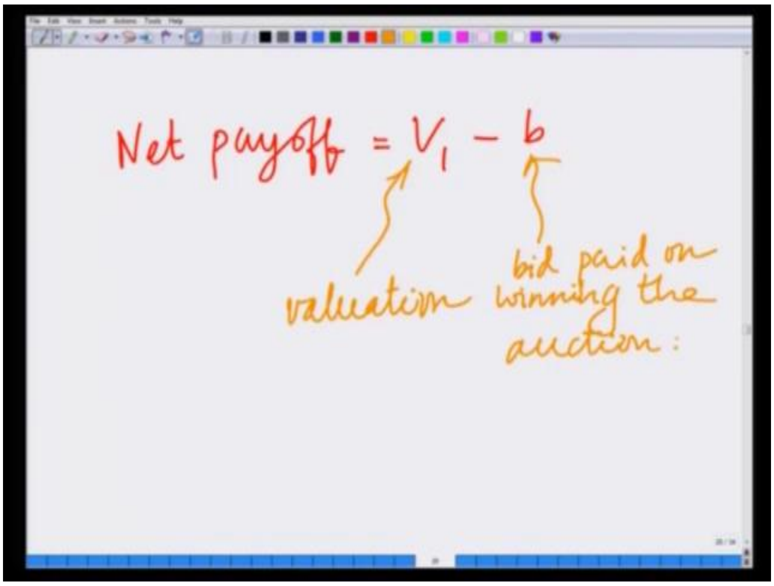

So, net payoff equals v 1 minus b, v 1 is this valuation while bid b is the bid paid on wining the auction. So, as paid an amount b to get the object and he has the value of v 1 for the object. So, his net profit or his net payoff is v 1 minus v if he wins the auction on the other hand if he loses the auction that is if b is less than or equal to bid b 2 of player 2 then his net payoff is 0, because it does not pay anything and need a does he get the object, so his net payoff is 0.

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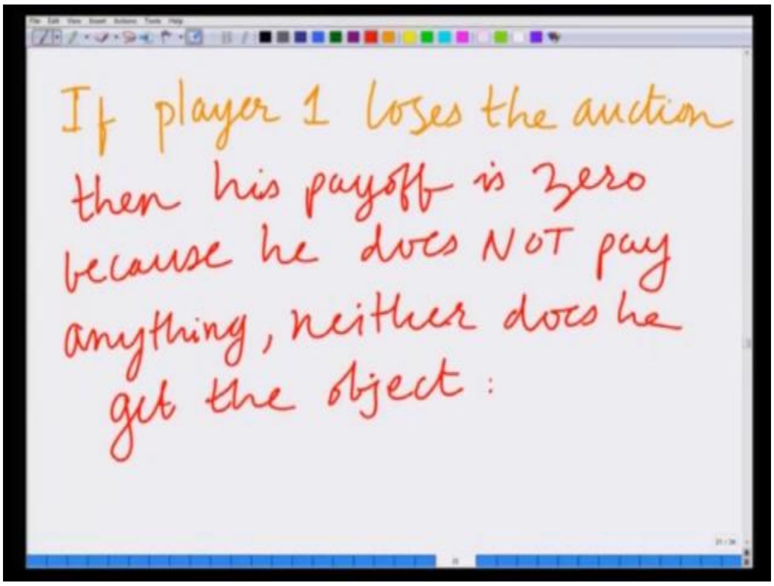

So, if he loses the auction if player 1 loses then his payoff is 0, because he does not pay anything and neither does he get the object. So, if player 1 loses the auction which occurs when his bid b is less than the bid b 2 of player 2, then he loses the auction he does not pay anything. So, the amount paid by him 0 neither does he get the object, so therefore, his net payoff is 0.

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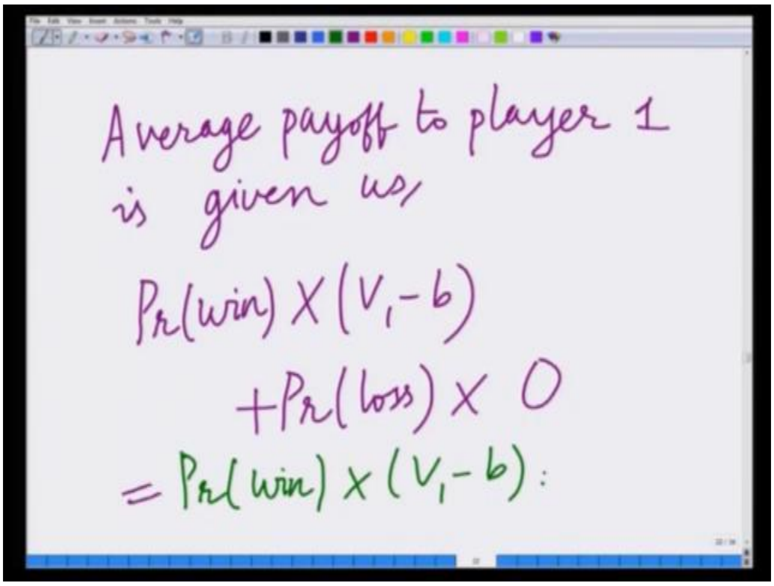

So, therefore, the average payoff of player 1 is given as probability winning the probability he wins the auction times v 1 minus b. Because, his payoff if he wins the auction is v 1 minus b plus the probability that he loses the auction the probability of lose times 0, because if he loses the auction his net payoff is 0. So, what we are saying, we are saying that if he wins the auction then his payoff is v 1 minus b that is valuation b 1 minus bid amount b paid.

And therefore, we are multiplying that by the probability of winning probability of winning times v 1 minus b plus the probability of loss times 0. Because, we loses the auction then he does not get the object and either does he pay anything. So, the net profit and the loss of auction is 0, which means the average profit or the average payoff is probability of winning times v 1 minus p plus probability of lose times 0 which can be simplified as probability of win times v 1 minus b.

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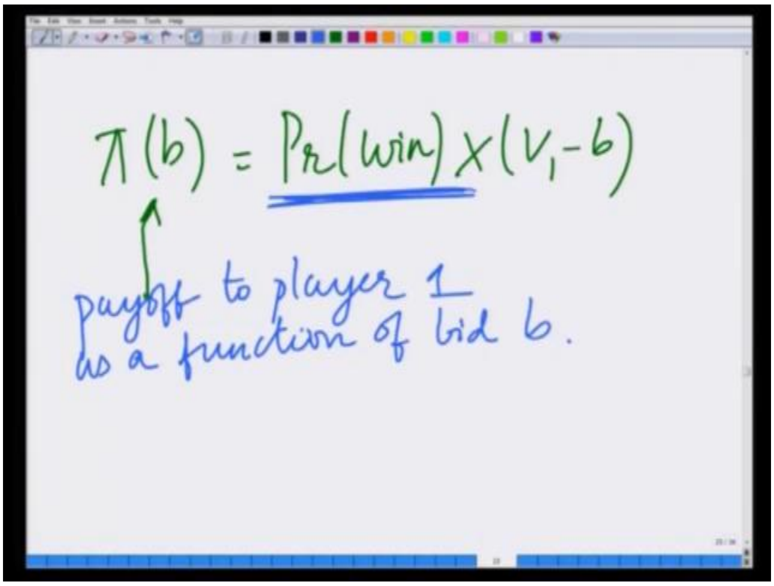

So, net payoff pi b to player b equals the probability of winning the auction times v 1 minus b. So, what is pi of b payoff to player 1 as a function of bid b, so pi of b which is the payoff player 1 as a function of the bid b is equal to the probability of winning times v 1 minus v. It now remains to find what this quantity probability of winning this that is as a function of the bid b, what is the probability that he wins the auction.

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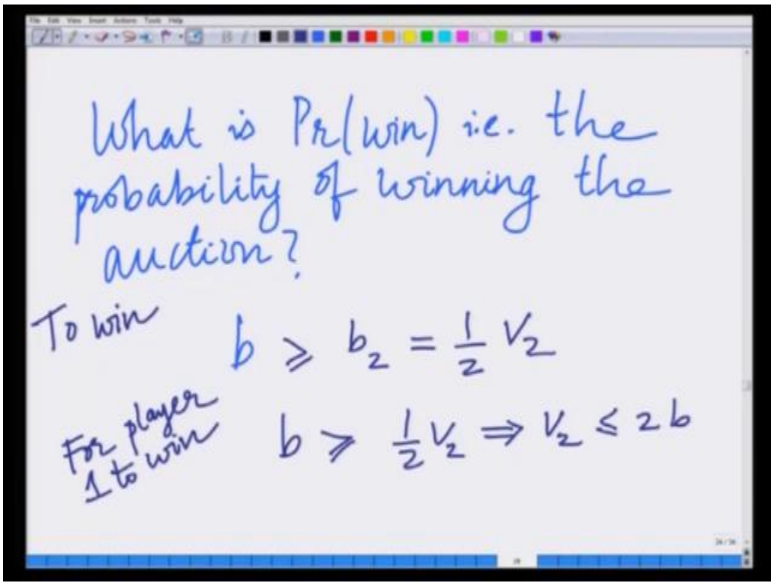

What is p r win that is the probability of wining the auction, remember to win the auction the bid b must be greater than equal to bid b 2 of player 2, but b 2 of player 2 remember we are assumed b 2 is equal to half v 2. So, to win the auction bid b that is to win we must have to win or for player 1 to win we must have b greater than or equal to b 2 equals half of v 2. So, we must have for player 1 to win b must be greater than or equal to half v 2 which implies that v 2 less than or equal to 2 b.

So, to win the auction bid b must be greater than equal to half v 2 which means v 2 must be less than or equal to twice B. Therefore, it means that v 2 which is remembered the valuations are uniformly distributed in the interval 0 to 1.

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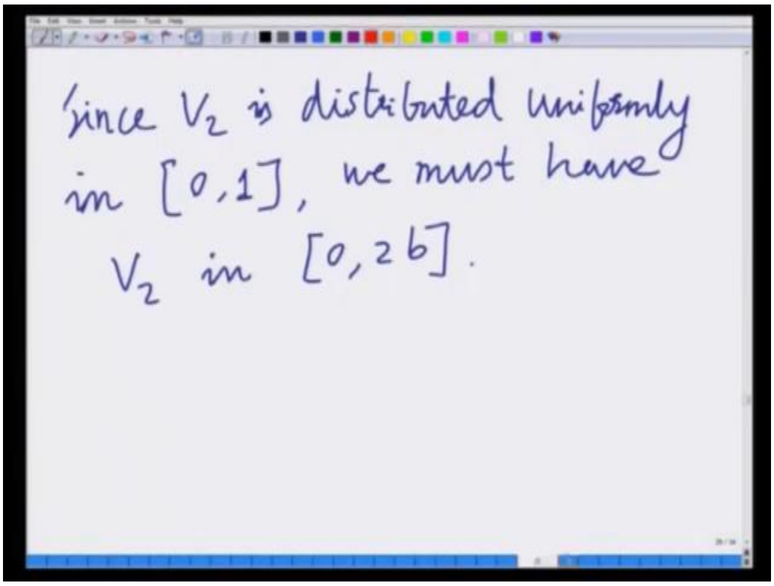

Therefore, it means that v 2 must lie in the interval 0 to 2 v, this is the valuation v 2 is distributed uniformly in 0 to 1 it we must have v 2 in or v 2 in 0 to 2 b. Therefore, if v 2 is less than or equal to 2 b that is for player 1 to win the auction, it must be the case that v 1 lies in 0 to 2 b.

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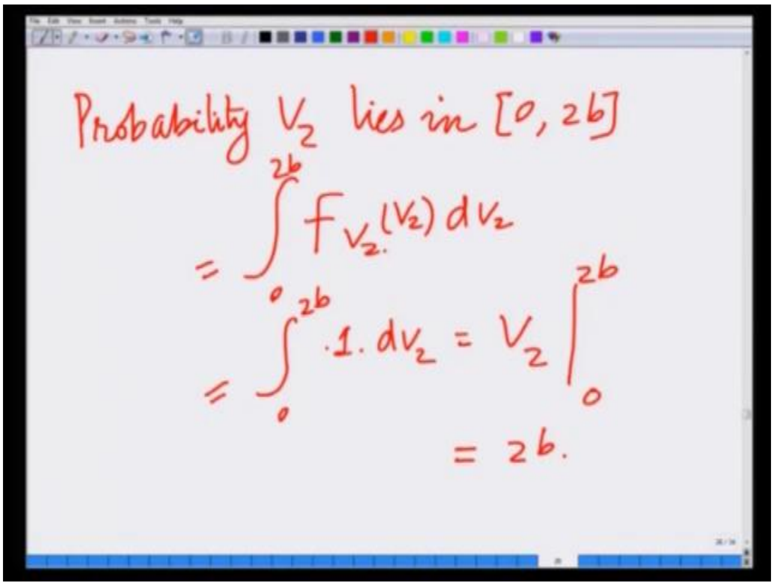

And what is the probability that v 2 lies in 0 to 2 b the probability, remember probability v 2 lies in 0 to 2 b is equal to the integral 0 to 2 b f of v 2 that is the probability density of f of v 2 integrated between 0 to 2 b. But, between 0 to 2 b since this integral is contain in the interval 0 to 1 the probability density is 1 therefore, we contrives this as 0 to 2 b 1 times d v 2 which is v 2 integrated between the limits 0 to 2 b which is equal to 2 b.

And therefore, what we have shown is that the probability that the valuation v 2 of player 2 lies in the interval 0 to 2 b is equal to 2 b and therefore, this is the probability that player 1 wins the auction.

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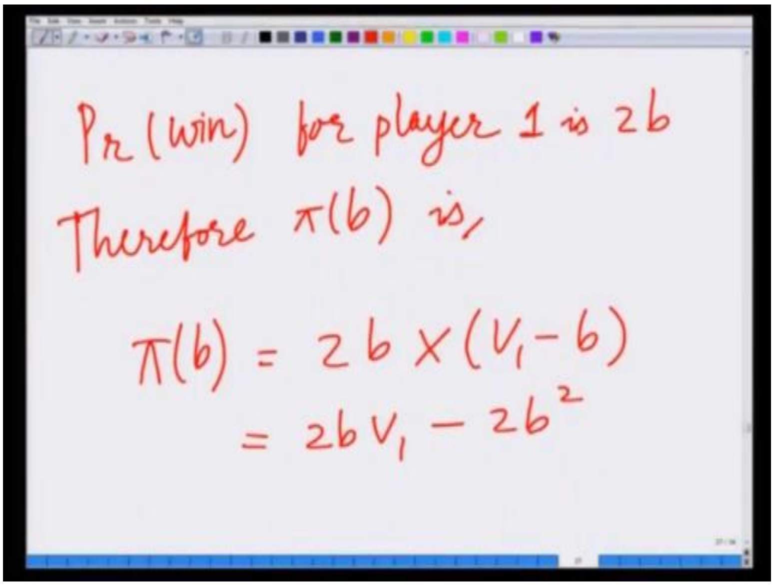

Before the probability of win for player 1 as a function of his bid b 2 b and therefore, now if we go back to this expression over here, where we are characterizing the payoff we now the probability of win, the probability of win is 2 b as a function of his bid b. Therefore, the net payoff pi of b to player 1 as a function of his bid b is pi of b equals the probability of winning 2 b times v 1 minus b which is the payoff or bidding, which is equal to 2 b v 1 minus 2 b square. So, pi of b which is the payoff to player 1 as a function of his bid b is 2 b v 1 minus 2 b square.

Now, we have the payoff as a function of b we have to find the b for which the payoff is maximum for this purpose we have to differentiate this function of b and set it equal to 0 and there by solve it to find the value the best response b.

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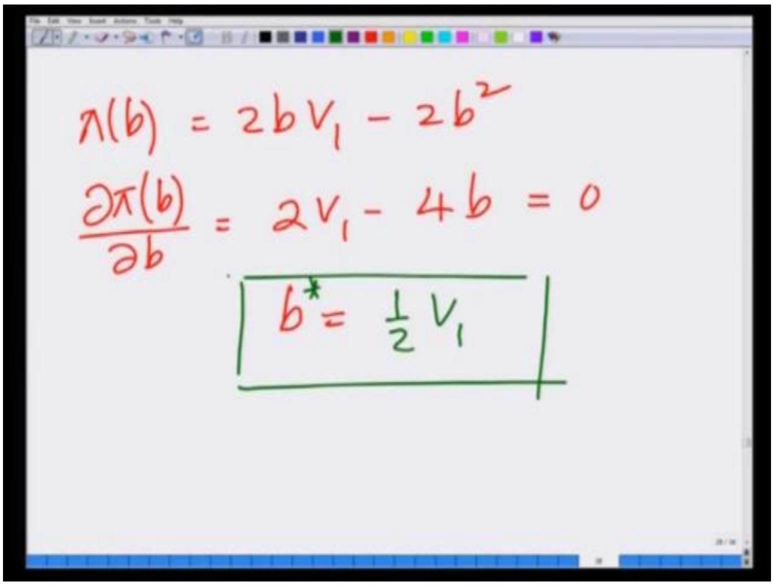

So, duo by b we have pi of b equals 2 b v 1 minus 2 b square therefore, dou pi of b by dou b equals twice v 1 minus 4 times b which we equate to 0 which implies gives us the result that b equals half v 1 that is the best response bid b star is equal to half v 1. So, what are we shown, we have shown that if b 2 is bidding player 2 is bidding b 2 equals half v 2 then the bid b star equals half v 1 is the best response bid of player 1.

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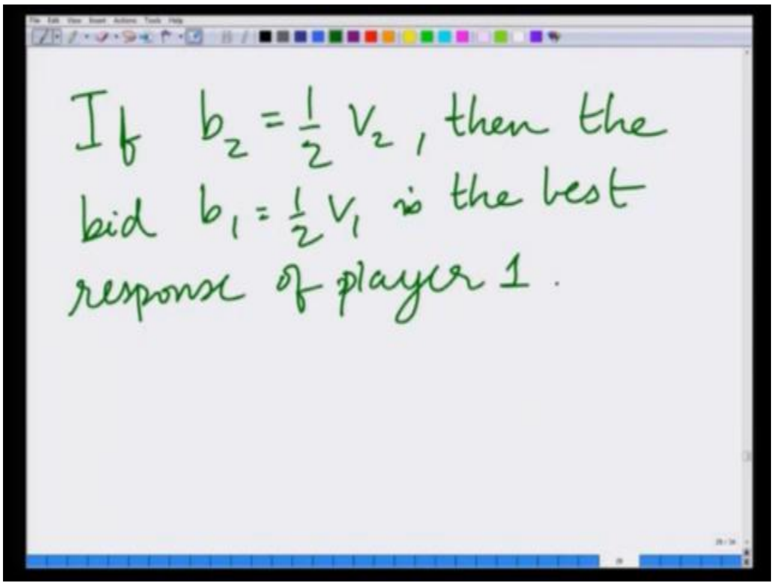

So, if b 2 equals half v 2 then the bid b 1 equals half v 1 is the best response of player 1. Similarly, by symmetry it can be shown that if player 1 is bidding b 1 equals half v 1 b 2 equals half v 2 is the best response for player 2 using a similar procedure, if I repeating the same procedure.

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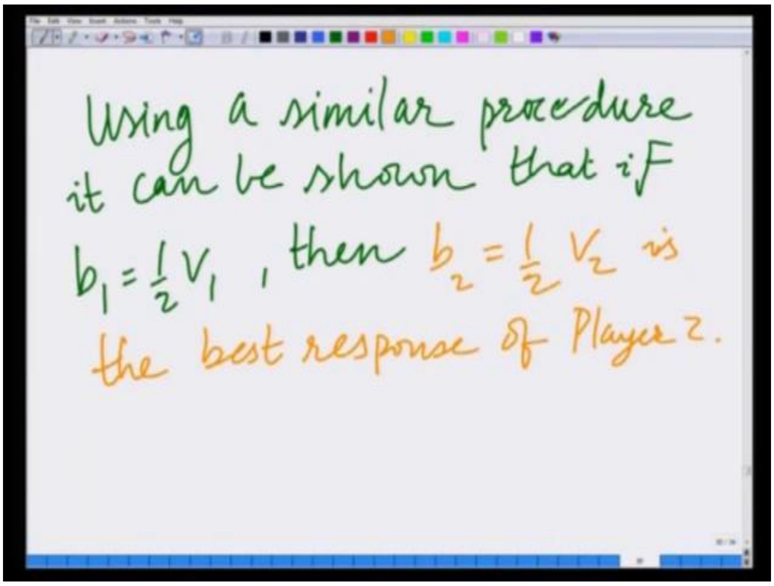

Using a similar procedure it can be shown that if b 1 equals half v 1 that is player 1 is bidding b 1 equals half v 1 then b 2 equals half v 2 is the best response of player 2. So, what we have shown is that if player 2 is bidding b 2 equals half v 2 then b 1 equals half v 1 is the best response of player 1. Similarly, we can show that if player 1 is bidding b 1 equals half v 1 then b 2 equals half v 2 is the best response bid of player 2.

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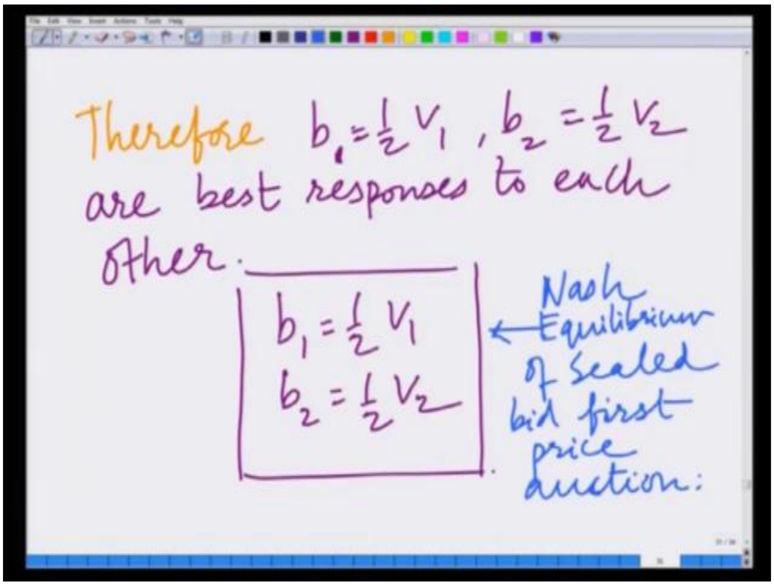

And therefore, since these bids b 1 equals half v 1 and b 2 equals half v 2 are best responses to each other, this is the Nash equilibrium of this first price auction here. Therefore, we have b 1 equals half v 1 comma b 2 equals half v 2 are best responses to each other, therefore, b 1 equals half v 1 and b 2 equals half v 2 is the Nash equilibrium of this first price auctions. So, this is the Nash equilibrium of our sealed bid.

Therefore, this is the Nash equilibrium of our sealed bid first price auction that is what we are saying is, in the sealed bid first price auction in which the two players that is the two players v 1 and v 2 are also in this bidders submit their bids in sealed envelope that is the bid of each player is unknown to the other with private valuations v 1 and v 2 distributed uniformly in the interval 0 to 1 the Nash equilibrium for this Bayesian sealed bid first price auction is b 1 equals half v 1 and b 2 equals half b 2. So, let us stop with this derivation of the Nash equilibrium here and we will continue our discussion on this auction format in the next module.

Thank you.