Strategy: An Introduction to Game Theory
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 11

Hello everyone, welcome to another module in this online course Strategy, An Introduction to Game Theory. So, in the last module we looked at the Cournot Duopoly.

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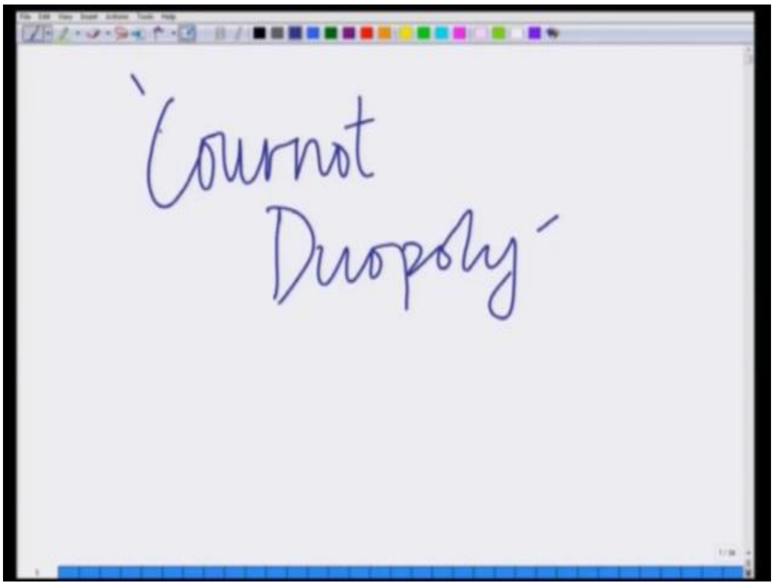

So, the example of the game we looked at as the Cournot duopoly, which is due to the economist Cournot and it models a game between two companies that is a duopoly between two companies, which are producing quantities, which are substitutes are also was known as strategic substitutes. And this game models the interaction between these two companies or the game between these two companies.

And we looked at this game and we looked at the payoff functions, we model the payoff functions of both these forms involved in the duopoly, we found out the best response of both these firms and then, we derived the Nash equilibrium of the game. So, just to do a brief recap, we said that the payoffs of both these firms…

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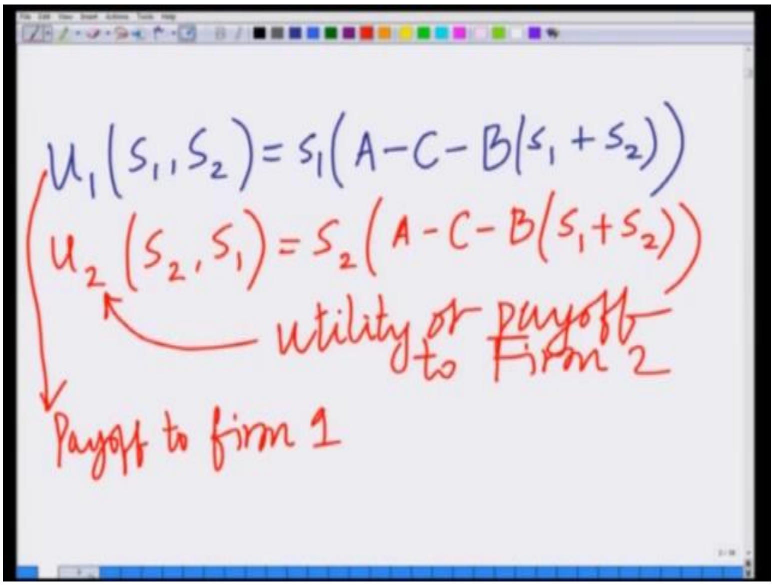

That the utility functions u 1 of s 1 comma s 2 that is utility of firm 1 as a function of, it is quantity s 1 and quantity s 2 produced by firm 2 is given by the function s 1 times A minus C minus B into s 1 plus s 2. And similarly, the payoff function u 2 of s 2 comma s 1 is given as s 2 times A minus C minus B s 1 plus s 2. So, these are the payoff functions of the two firms that is firm 1 and firm 2. So, this is the payoff to firm 1 and this is the utility function, payoff function for firm 2 and we have also derive the best responses from these payoff functions.

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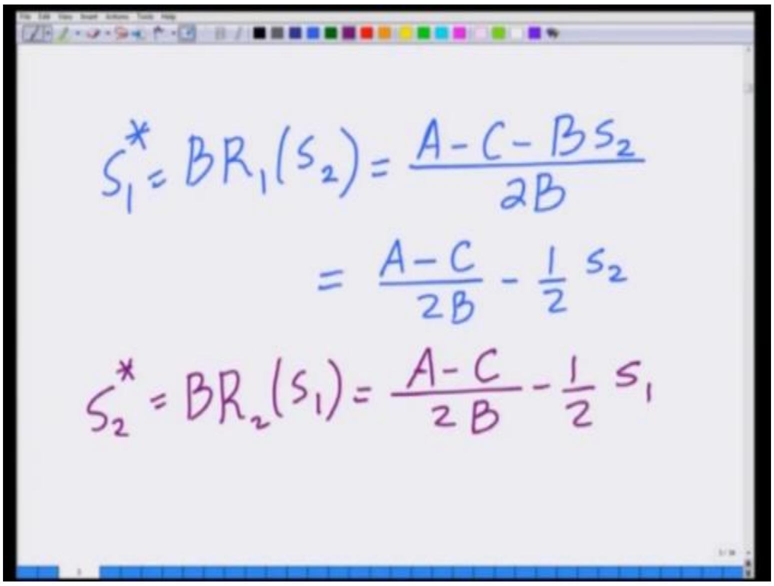

We said the best response s 1 star, which is also B R 1 of s 2 can be written as A minus C minus B times s 2 divided by 2 B, which is equal to A minus C by 2 B minus half s 2, this is the best response s 1 star. Similarly, we derived the best response s 2 star and we said the best response s 2 star, which is also B R 2 as the function of the quantity s 1 that is the best response of firm 2 is given as A minus C by 2 B minus half s 1. These are the best response quantities and let us now take an example and we again consider this example.

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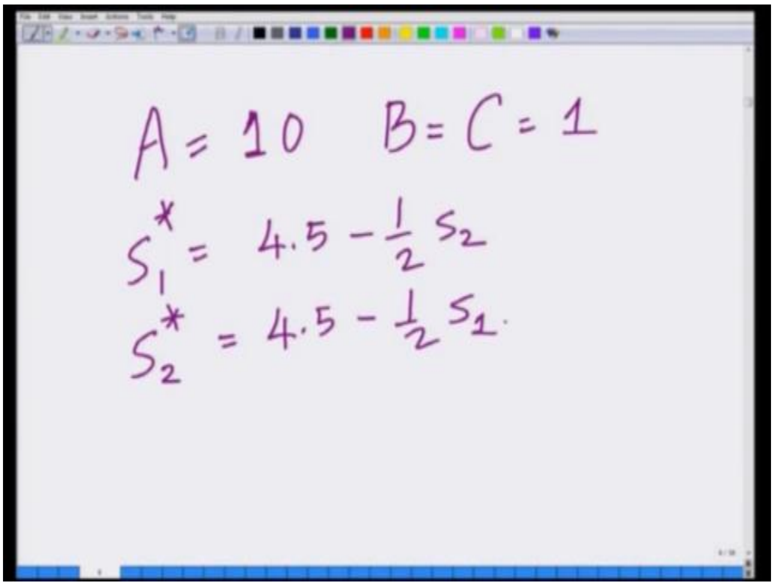

Let us consider an example, where A equals 10 and B equals C equals 1 then we have s 1 star equals 4.5 minus half s 2 and s 2 star equals 4.5 minus half s 1. So, for this specific example, where A is equal to 10 and B is equal to 1 and C is equal to 1, we have s 1 star the best response quantity of firm 1 is equal to 4.5 minus half s 2 and s 2 star. The best response quantity of firm 2 is 4.5 minus half s 1 and now let us graph these two things, although we already did it, let us graph these things once again.

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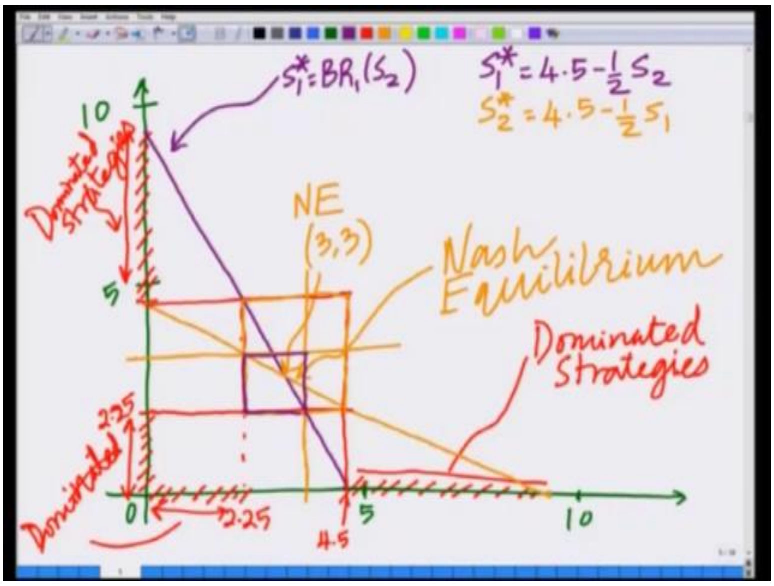

So, that I can illustrate a slightly different point compare to what I illustrated last time, let us take the scale between 0 and 10, again here I am going to take consider between 0 and 10. So, 5 is approximately here at the midpoint and what we said is, s 1 star equals 4.5 minus half s 2. So, s 2 is equal to 9, then s 1 star equal to 0 and if s 2 is equal to 0, s 1 star equals 4.5 which is approximately somewhere around this and hence the best response is going to look something like this, this is s 1 star equals B R 1 that is the B R best response as a function of s 2.

Now, similarly we have s 2 star equals 4.5 minus half s 1 that is the best response s 2 star as a function of s 1, which means if s 1 is equal to 9, s 2 star is equal to 0. Similarly, if s 1 is equal to 0, s 2 star is 4.5 which is approximately around here and once again I can join these two and I get this and we said this is the point 3 comma 3, where the best responses intersect and this is the Nash equilibrium. This is the point 3 comma 3, where the best responses intersect and this is the Nash equilibrium, we have already seen this.

Now, let us take a slightly different approach, let us try to analyze this game a little more by looking at this graph. Now, if you look at this plot, you can see that as s 2 varies that is for s 2 is equal to 9, the best response s 1 star is 0 and if s 2 is equal to 0, the best response s 1 star equals 4.5. So, for s 2 possible values of s 2 the best response s 1 star always lies between 0 and 4.5. So, these quantities which are greater than 4.5, the quantities s 1 star or the best response s 1 greater than 4.5 is never a best response.

So, these strategies that is, this quantities s 1 which are greater than 4.5, these are dominated strategies. So, these are never going to be used by firm 1, so these strategies that is, quantity greater than 4.5 are not going to be used by firm 1. Let me repeat that argument again, when s 2 is equal to 9, s 1 star is equal to 0 and as s 2 decreases from 9 to 0, s 1 star increases from 0 to 4.5 finally, reaching 4.5 when s 2 reaches 0 and s 1 star does not increase beyond 4.5.

So, s 1 star always lies between 0 and 4.5, as a result all quantities s 1 greater than 4.5 these are the dominated strategies. Similarly, if you look at all quantities again, if s 1 is equal to 9, s 2 star is equal to 0, if s 1 is equal to 0, s 2 star is equal to 4.5 therefore, s 2 star again lies between 0 and 4.5. Therefore, all values of s 2 star greater than 4.5 are dominated, these belong to the dominated, these are dominated strategies that is, these are never the best response to any quantity produced by firm 1.

So, what we are saying is the best response of firm 1 always lies between 0 and 4.5, the best responses of firm 2 also always lie between 0 and 4.5. Hence, this game can now be reduced to this box, where s 1 star and s 2 star lie always between 0 and 4.5. Since, the best response is s 1 star are always between 0 and 4.5, best response is s 2 star are always between 0 and 4.5. Therefore, this game can now be reduced to this smaller box, where s 1 is restricted to 0 and 4.5 and s 2 is restricted to 0 and 4.5.

Therefore, we have eliminated the dominated strategies or eliminated the strategies which are not best responses to the strategies of the other form, as the result we have been able to reduce this game. Now, if you look at this game, now s 2 lies only between 0 and 4.5, when s 2 is equal to 0, s 1 is s 1 star equals 4.5. When s 2 is equal to 4.5, s 1 star you can see you can verify is equal to 2.25, so we are further saying since s 2 is now after elimination of the dominated strategies only lies between 0 and 4.5.

The best response s 1 star in turn lies only between 2.25 and 4.5. So, this strategy is s 1 star between 0 and 2.25 it can now be eliminated, because these are in the reduced game, these are dominated strategies, since these are no longer the best response. Similarly, now if you look at this point, the strategies s 2 star if s 1 is 0, s 2 star equals 4.5 if s 1 is equal to 4.5, s 2 star is equal to 2.25 therefore, s 2 star varies only between 0 and 2.25. Therefore, s 2 star varies only between 2.25 and 5, s 2 star values between 0 and 2.25 are in turn dominated strategies.

So, these are also now in the reduced game dominated, these are again dominated strategies. So, these in the reduced game these are the dominated strategies, so eliminating these dominated strategies further reduces the game to this small box, let me draw this box in this orange color, this reduces this game to further this small box. And now, again in this reduce game you can see that one can further eliminate the dominated strategies, one can further eliminate the dominated strategies in this reduced game, in which case the box is going to keep shrinking even further and further.

So, since now you can further eliminate the dominated strategies, this box is going to keep shrinking further and further and this box is going to converge to this Nash equilibrium. This is what are we doing here, what we have doing is we are eliminating the dominated strategies. In the first instance, we said that the strategies s 1 which are greater than 4.5 and s 2, which are greater than 4.5 are dominated strategies. So, these can be eliminated therefore, the game reduces to the strategies s 1 between 0 and 4.5 and s 2 between 0 and 4.5.

Now, in this reduced game we are saying that further this strategies s 1 between 0 and 2.25 and s 2 between 0 and 2.25 are dominated strategies. So, these can be eliminated and therefore, the game further reduces to s 1 between 2.25 and 4.5 and s 2 between that is the quantity s 2 between 2.25 and 5. And therefore, now we have a reduced game then and therefore, that is repeating this process that is if we keep on repeating this process, that is eliminating these dominated strategies further and further in each successful iteration, this box is going to progressively keep shrinking and proven, if this box is progressively shrinking it converges to the Nash equilibrium.

For instance, now we can see if s 2 is between 2.25 and 5, s 1 lies only in this range and if s 1 is between 2.25 and 5, then s 2 lies only in this range. So, the game further shrinks to this box and you can see this box progressively converges to the Nash equilibrium, that is elimination of dominated strategies in this Cournot game or this game of Cournot duopoly, iteratively or ultimately leads to converges to the Nash equilibrium. So, this is an interesting perspective of the Cournot duopoly.

Let us look at another aspect in this game, let us also now try to look again similar to the tragedy of commons, let us try to look if there is another outcome which yields a higher payoff for both the players.

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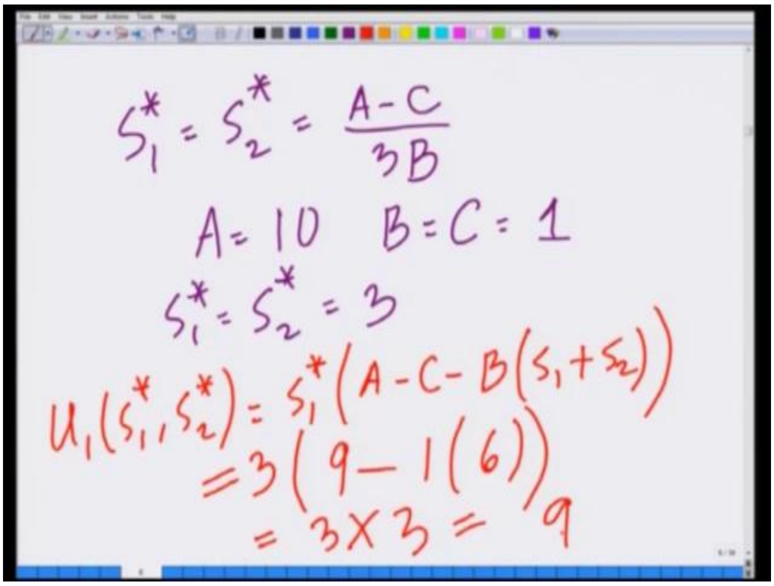

We have seen the Nash equilibrium, the Nash equilibrium is s 1 star equals s 2 star equals A minus C by 3 B and if A equals 10 and B equals C equals 1, we have s 1 star equals s 2 star equals 10 minus 1 by 3 equals 3. Let us look at the Nash payoff, the Nash payoff that is u 1 of s 1 star comma s 2 star equals s 1 star into A minus C minus B times s 1 plus s 2, which is equal to 3 times A minus C is 10 minus 1 9 minus B, which is 1 times s 1 plus s 2 is 6 that is 3 times 9 minus 1 that is 3 times 3 equals 9.

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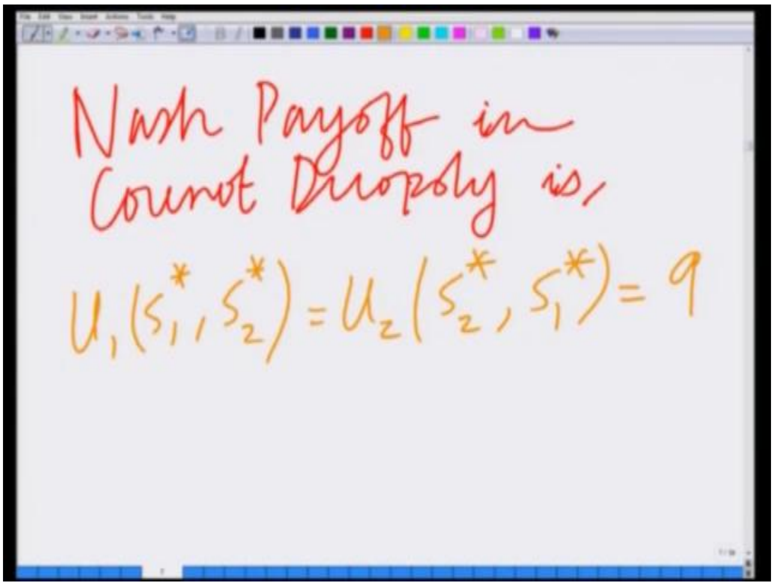

So, the Nash payoff in the Cournot duopoly game is u 1 of s 1 star comma s 2 star equals u 2 of s 2 star comma s 1 star equals 9. Now, let us try to see is there any outcome which yields a higher payoff for both.

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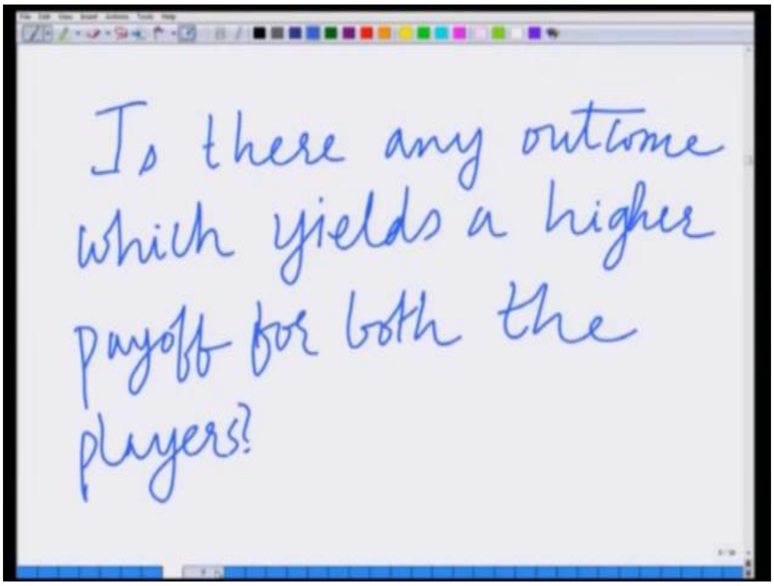

So, the question that we are asking is, is there any outcome which yields a higher payoff for both the players, this is the same question that we have asked in the tragedy of commons. To look at this, let us look at a scenario where both these firms are collaborating.

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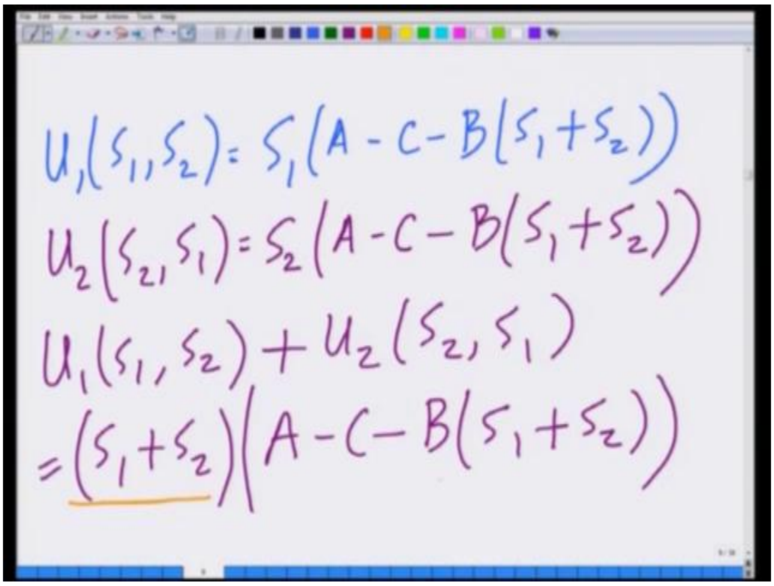

So, we have u 1 of s 1 comma s 2 equals s 1 into A minus C minus B times s 1 plus s 2 and u 2 of s 2 comma s 1 equals s 2 into A minus C minus B into s 1 plus s 2 therefore, if we look at u 1 of s 1 comma s 2 plus u 2 of s 2 comma s 1 this is equal to s 1 plus s 2 into A minus C minus B s 1 plus s 2 and you can see this quantity only depends on the some quantity s 1 plus s 2.

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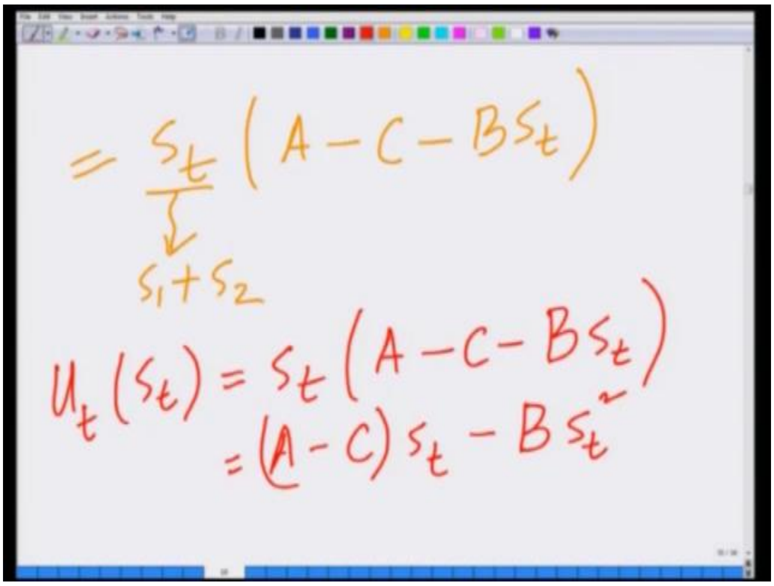

So, I can write this as equal to s t total times A minus C minus B times s t, where s t is equal to s 1 plus s 2. So, the total utility that is the sum of the utility that is u 1 plus u 2 depends only on the total quantity that is s 1 plus s 2. So, I can write this as u t that is the total utility of as a function of s t equals s t into a minus c minus b times s t which is equal to A minus C s t minus B s t square.

And now, if I differentiate this with respect to the total quantity s t to find the optimal quantity s t for which the total utility that is the total payoff to firm 1 and firm 2 is maximized, remember this is the function of s t at differential function of s t. So, I can differentiate this with respect to s t to find the total quantity s t for which the some utility u t is maximized.

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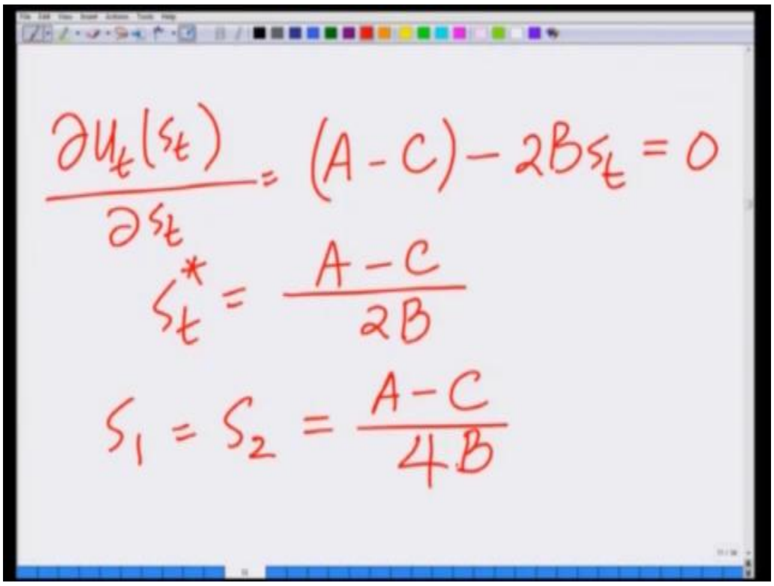

And if I do that I get dou u t s t by dou s t equals A minus C minus 2 B s t which I equal to 0 to maximize which means s t star that is the quantity for which the total utility is maximized is A minus C by 2 B. And now similar to again the strategy of commons we can assume that both s 1 that is if firm 1 and firm 2 produce half of this quantity. So, we can set s 1 equals s 2 equals A minus C by 4 B.

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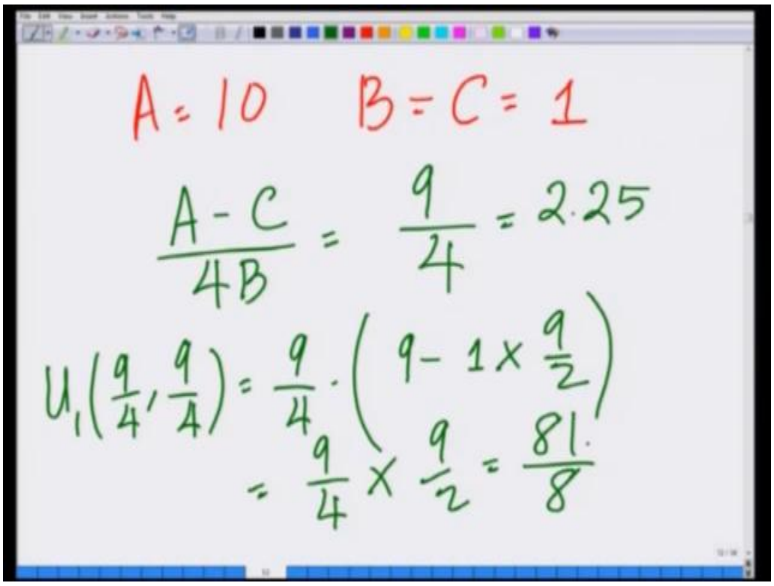

And again let go back to our previous example, where A equals 10 B equals C equals 1 and we have A minus C by 4 B equals 9 divided by 4 equals 2.25. So, both of them can produce a quantity s 1 equals s 2 equals 2.25 to get a higher some utility. Now, let us look at the individual utility, what is u 1 of 9 by 4 comma 9 by 4 that is equal to 9 by 4 times A minus C that is 9 minus B 1 times s 1 plus 2 that is 9 by 2 which is equal to 9 by 4 times 9 by 2 which is equal to 81 by 8.

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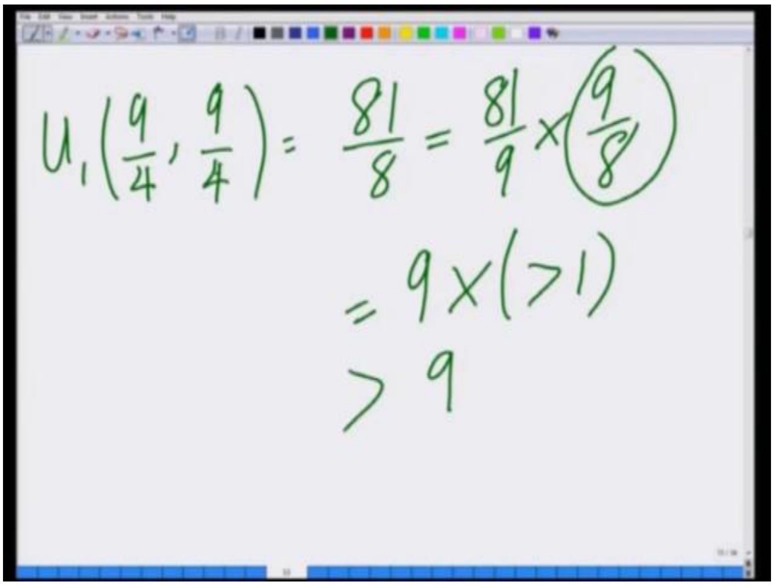

And if you look at this quantity 81 by 8 that is u 1 of 9 by 4 comma 9 by 4 that is equal to 81 by 8 which is equal to 9 times I can write it 81 by 8 as 81 by 9 into 9 by 8 which is 9 times 9 by 8 a quantity which is greater than 1. So, this is strictly greater than 9 which is the Nash payoff, so when we look at the payoff it 9 by 4 comma 9 by 4 each of them is getting a payoff of 81 by 8 which is greater than the Nash payoff of 9.

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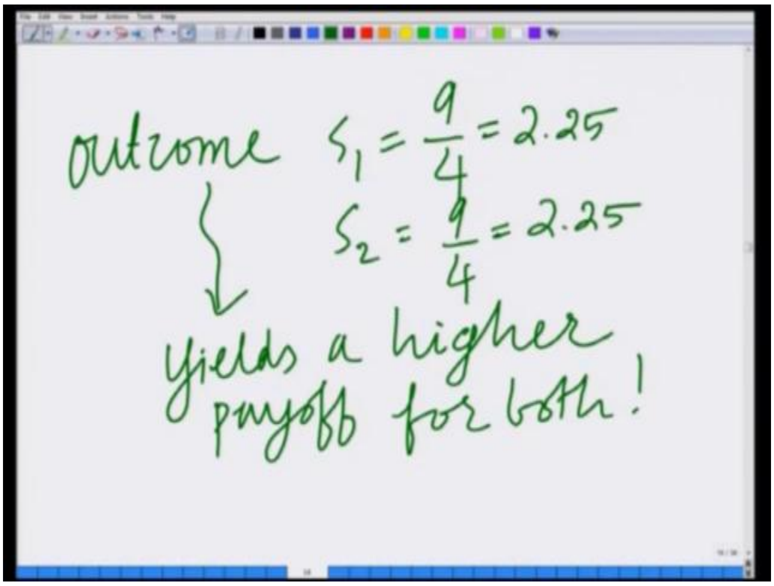

So, the outcome s 1 equals 9 by 4, s 2 equals 9 by 4 that is basically 2.25 yields a higher payoff for both. And therefore, the Nash equilibrium is not talked about, remember the Nash equilibrium of this game is at 3 comma 3, s 1 is equal to 3, s 2 is equal to 3, where the Nash payoff is 9 for both the firms. We are saying that when s 1 is equal to 2.25 and s 2 is equal to 2.25, the payoff for both of them is 81 by 8 which is strictly greater than 9.

Therefore, both of them can strictly improve their payoff, which means the Nash equilibrium is not worry to obtain. And therefore, again we have an example of a game which is similar to the prisoner's dilemma, since there is one Nash equilibrium and the Nash equilibrium is not parry to optimal. So, therefore, this game the Cournot duopoly in the game in that sense similar to the prisoner’s dilemma. And now there might be an interesting question that might be in your mind that is why cannot both these firms agree to produce this quantity 2.25 to improve their payoff, which is a reasonable question.

First as we have already reason they cannot explicitly collude to produce the quantity 2.25 to artificially inflict the price. Because, that is illegal as we already discussed in the beginning that is not allowed by the anti-collision it is let say anti-collision, loss let say they implicitly agree to produce this quantity 2.25. Let us go back and look at our best response back ground.

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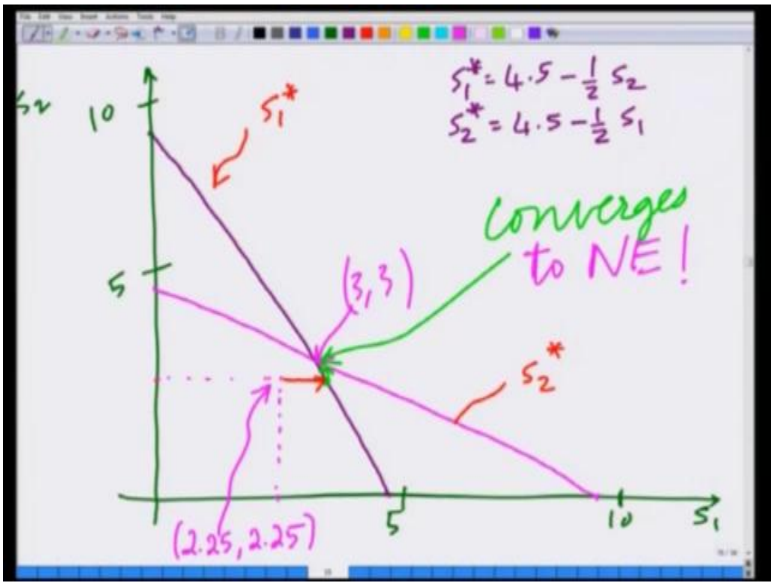

Once again try to understand this dynamic better, we again have or best response dynamic we are on the x axis I have s 1 and on the y axis I have s 2 and remember I am going to draw m y s 1 star equals 4.5 minus half s 2 and also s 2 star equals 4.5 minus half s 1 and I can draw that over here. And now you can see the Nash equilibrium is 3 comma 3, but there is a point 2 comma 2 or 2.25 comma 2.25 which yields the Nash equilibrium is 3 comma 3, but there is a point 2.25 comma 2.25 that is s 1 equal to 2.25 s 2 equal to 2.25 which produces a higher payoff for both the firms.

Let say the implicitly agree to work at this point that is firm 1 and firm 2 make implicit agreement to produce the quantity 2.25 each. What is going to happen in that scenario, is that agreement going to sustain is that the self-enforcing agreement and you can see it is clearly not. Because, the movement s 2 agrees to produce 2.5, s 1 this is the curve s 1 star, this is the curve s 2 star, s 1 will deviate to it is best response that is s 1 will deviate to this point which is it is best response.

The movement s 1 deviates to it is best response s 2 will deviate to it is best response, which is less and intern s 1 will deviate to it is best response and you can see this again converges to the Nash equilibrium converges to N E. Therefore, this is not a self-sustaining equilibrium or a self-sustaining outcome or a self-enforcing agreement and the reason for that is even though the implicit implicitly agree to produce the quantities 2.25 the movement, there is no reason to strict to the agreement.

Because, back in their firm 1 can play it is best response to 2.25, because it wants to maximize it is payoff. Seen firm 1 deviating from that implicit agreement 2.25 firm 2 will deviate to it is best response and in term for one will again deviates two. So, each will intern update their strategy by playing the best response to the other strategy and this is the best response dynamically. So, each will update each firm or each agent will updated strategy by playing the best response to the strategy of other and this will subsequently converts to the Nash equilibrium.

And therefore, the Nash equilibrium is the only self-sustaining outcome, because at that point each is playing the best response to other by which I mean that firm 1 is producing s 1 is going to 3, firm 2 is producing a s 2 equal to 3 and each is playing the best response to the other, like s 1 star is the best response our 3 is the best response the quantity 3 produce by firm 2, s 2 is equal to 3 is the best response to s 1 is equal to 3 produce by firm 1 at which point neither firm 1 firm 2 have no incentive to unilaterally deviate to another quantity and therefore, that is the only self-sustaining or self-enforcing agreement.

So, I hope that has clarified a lot of the dress ting aspects about these games, first we are looked at the Cournot duopoly from the context of a dominated strategies that is we have looked at what happens when we progressively eliminate dominated strategy in the Cournot game and we found that we converge to the Nash equilibrium.

And the other point we looked at is what happens if there have make an implicit agreement to implicitly collude and produce a lower quantity that is 2.25 each was to artificially in flip the prices we found that, that is not an equilibrium that is not a self-enforcing agreement since both have an incentive to deviate at once the start deviating they will indeed intern converge to the Nash equilibrium, which is the only self-enforcing agreement from which no one has an incentive to unilaterally deviate. So, I hope this interesting example as clarified a lot of these certain details about the Nash equilibrium and parry to optimality etcetera and games, please go through it and try to understand it better thoroughly.

Thank you, thank you very much and we look at other games in the next modules, thank you.