Mathematics for Economics 1
Professor. Debarshi Das
Department of Humanities and Social Sciences
Indian Institute of Technology, Guwahati
Lecture No. 09
Differentiation - Preliminaries

Welcome to the ninth lecture of this course called Mathematics for Economics Part 1. So, today we are going to start with a new topic this topic is called differentiation. Earlier what we have discussed? We have discussed about functions and this differentiation is related to the idea of functions and differentiation is a very old tool used in mathematics and many other scientific inquiries.

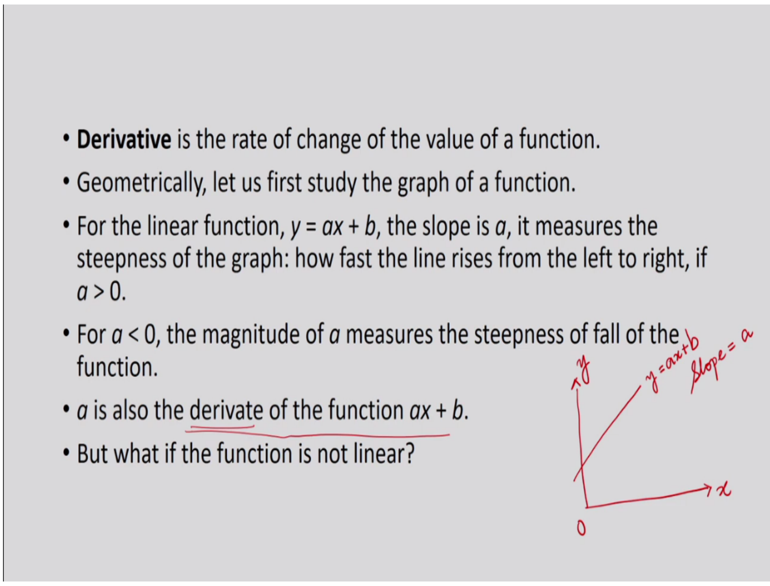
And it is a commonly used tool in many fields of science such as physics, mechanics, mathematics of course and it was discovered this tool of differentiation and calculus as such was discovered by two prominent mathematicians Isaac Newton and Leibniz separately and independently. So, what it looks at this idea of differentiation. The differentiation looks at the rate of change of a function.

And so in economics also that idea is often used we want to find out how a particular function changes maybe population for example changes over period of time, how is it changing, at what rate, what will be a total amount of population after a period of time or for example national income so that is also changing with respect to time. So, those are the things that one can look at.

Here time is the independent variable. We are talking about changes of population with respect to time or national income or per capita income with respect to time, but it is not necessary that we always take time to be the independent variable one can think about change in the production of a firm with respect to change in the inputs. In that case the inputs are the independent variables.

Or you can think about change in the consumption of all the consumers of the economy with respect to change in the national income. In that case the national income is the independent variable. So, in each case one tries to find out what is the rate of change of the dependent variable.

(Refer Slide Time: 3:27)



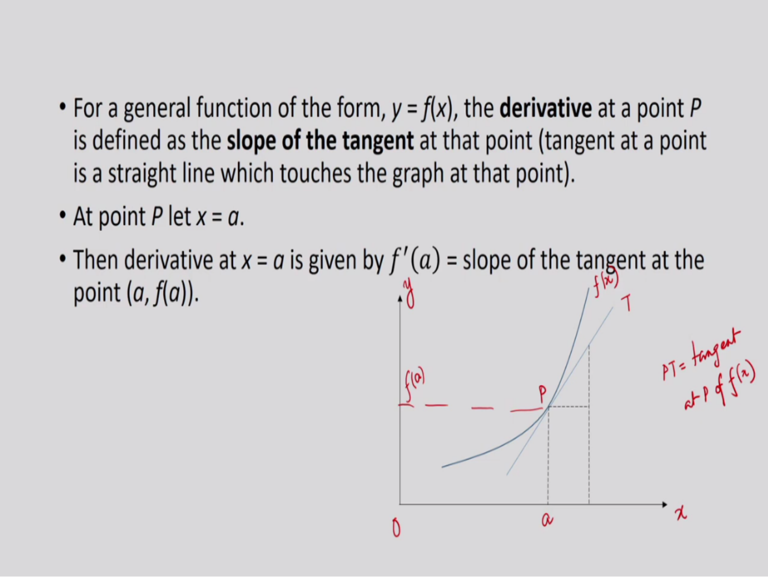
Let me start with this idea of differentiation. Derivative or this is closely linked with idea of differentiation as we shall see. Derivative is the rate of change of the value of a function. Geometrically, let us first study the graph of a function. So, before going into the mathematics of it let us try to see how it can be interpreted in a geometric manner. We have defined what is the graph of a function?

So, suppose you have a graph like this so $y = f(x)$ suppose and suppose you have a linear function $y = ax + b$. Now, here the slope of this function we know is given by small a and what was the interpretation of this small a the slope? It is the steepness of the graph, how fast the line rises from the left to right if $a > 0$ and if $a < 0$, then this a measures the steepness of the fall of the function because we know that if a is negative, then the function will be downward sloping.

But in that case also the magnitude of a will measure how steeply the function is declining. Now a is also called the derivative of this function ax plus b. So, this is basically the slope of this function if we have a linear function, then the slope is the derivative of that function. So, it measures how much the y is changing with respect to x and that is the idea of slope and that is also the idea of derivative.

Now this was the case where the function is a linear function of the form $ax + b$, but as we know this is a very simple sort of function. In general function may not be linear and if you do not have a linear function if it is not of the form $ax + b$, then how do you define the derivative of a function.

(Refer Slide Time: 6:07)



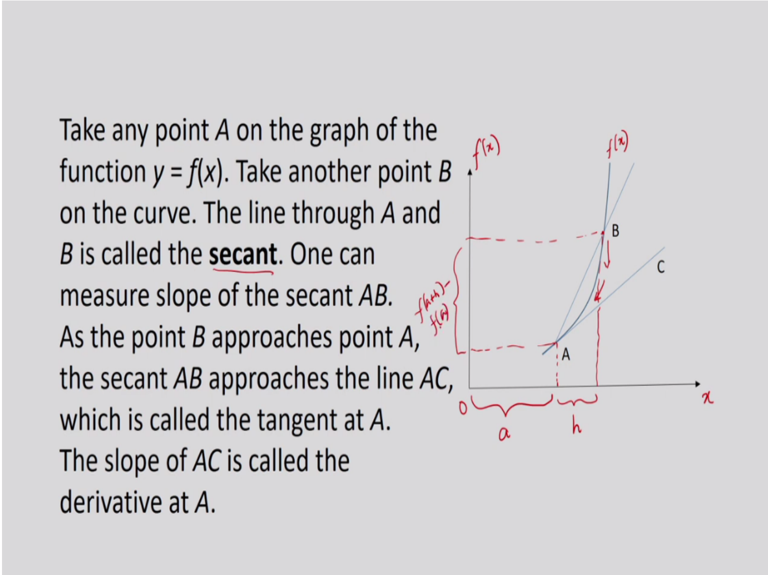
For a general function of the form $y = f(x)$ the derivative at a point P is defined as the slope of the tangent at that point. So, suppose you have this function which is given in the diagram here and here is that function $f(x)$ and here is the point P. Now we want to find out what is the derivative at point P we basically draw a tangent to the graph at that point P and then we say that the derivative of the function at point P is the slope of the tangent.

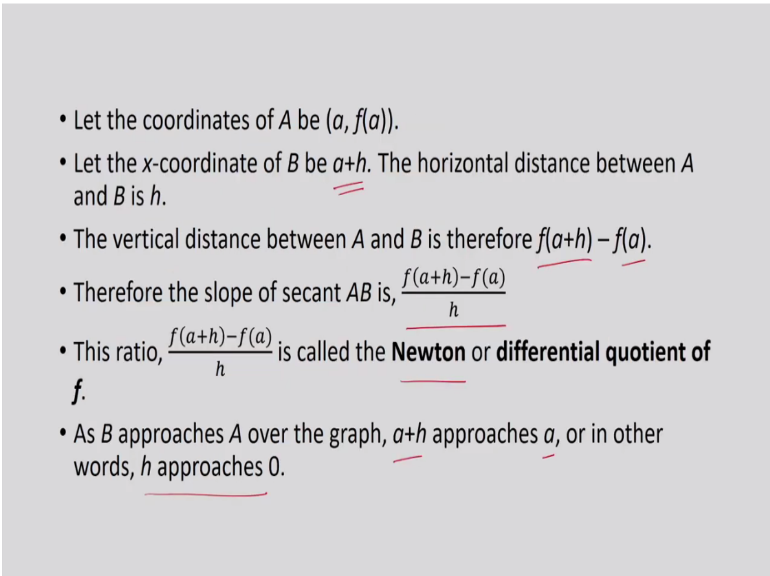
And at this point we wondering what is the tangent? Tangent at a point of a function is a straight line which touches the function or the graph at that point. So, here if I have to draw the tangent at P I draw a straight line which is touching the graph at point P. So, PT is here the tangent. PT is a straight line and which is touching the graph at point P so therefore PT is the tangent to the function at point P.

At point P suppose your value of the argument is given by a, then the derivative at point $x = a$ is given by $f'(a)$ which is the slope of the tangent at the point $(a, f(a))$. So, here at P suppose the value of x is a so what is the value of y this is given by $f(a)$. So, the coordinates of the point P is a I am sorry this is $f(a)$. $(a, f(a))$ coordinate of the point is a and $f(a)$ and therefore if we want to find the slope at point P we draw the tangent at P which is PT and then we find out the slope of that line which is PT.

Now this is kind of heuristic way to explain what is the slope and what is the derivative of any function at a point. It is a geometric sort of exponential all right, but we are not exactly defining it in a very rigorous manner.

(Refer Slide Time: 9:18)





So, let us try to do that and here you have another diagram and here you have this dark line as fx and I have taken a particular point A. So, point A is taken on the graph of the function $y = f(x)$ and on the same graph I take another point here B and I draw a straight line through A and B this line which is passing through A and B two points on the graph is called a secant. So, it is a Latin name it is like the idea of a cord.

Now, one can measure the slope of the secant AB, it is a straight line you can just take two points and this two points are A and B and you can measure the slope of this line AB. Now suppose point B approaches point A the secant AB, then we will approach the line AC which is called the tangent at A. The slope of AC is called the derivative of the function at A.

So, here I have taken another point B which is quite close to A and I am assuming that point B is approaching point A and it is staying on the graph. Now as it is approaching point A then this line the secant line AB in the limit it becomes the line AC and so AC will be called the tangent of the curve at point A and the slope of AB will become the slope of AC and the slope of AC is called the derivative of the function at point A.

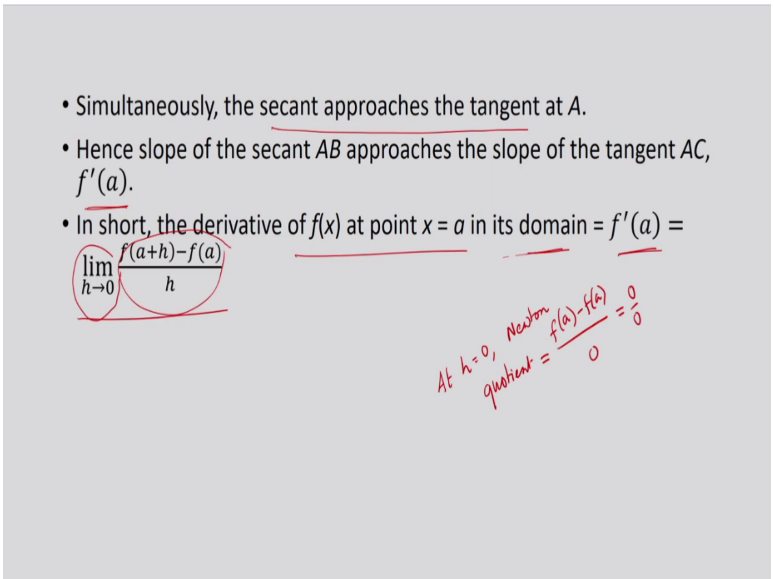
So, let me do it a little bit mathematically. Now let the coordinates of A be a and $f(a)$ because A point is on the function so $f(a)$ will be the value of the function and let the x coordinate of point B be $a + h$, h is a small number and since x coordinate is a and $a + h$ is the coordinate of B that means that the horizontal distance between A and B is h.

So, here if you draw a perpendicular this is h and this is a. So, I have taken a point on the curve and the horizontal distance between the original point and the new point is given by h. Now what is the vertical distance between A and B? It is given by $f(a + h) - f(a)$ So, in terms of this diagram so this is $f(a + h) - f(a)$.

Therefore, the slope of the secant AB is the familiar expression for slope which is $(y_1 - y_2)/(x_1 - x_2)$ or perpendicular by base. So, here the perpendicular is $f(a + h) - f(a)$ and the base is h. So, therefore this is the slope of AB. This ratio which is $[f(a + h) - f(a)]/h$ is called the Newton or the differential quotient of f, obviously at point A.

So, this is named after Isaac Newton because he was one of the proponents of differential calculus. Now as B approaches A over the graph $(a + h)$ approaches small a or in other words h approaches 0. So, one is thinking about this h becoming 0 which means B is approaching A over the graph over the function.

(Refer Slide Time: 14:49)



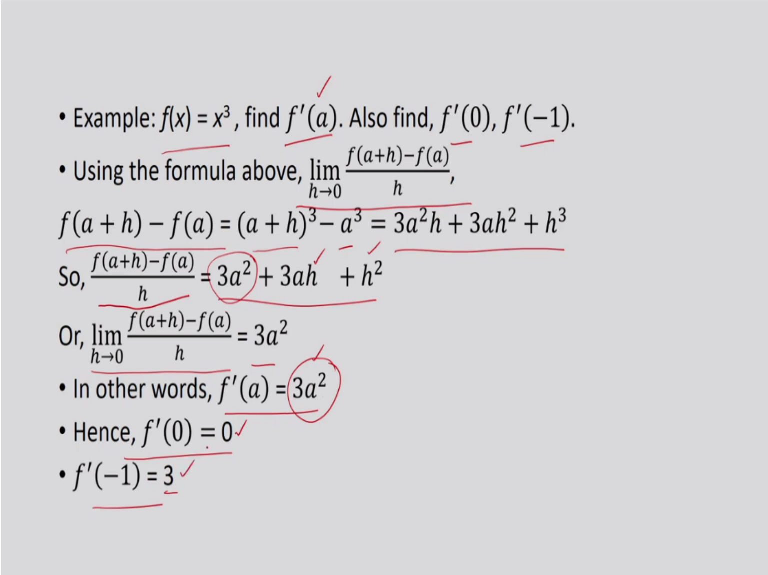
Simultaneously the secant approaches the tangent at A as we have just discussed that the secant was a chord. A and B were points on the curve. Now as h approaches 0, B is approaching A and the secant no longer remains a secant it becomes the tangent to the curve at point A that is why I wrote that secant approaches the tangent at A. Hence the slope of the secant AB approaches the slope of the tangent AC and which we denote by $f^{'}(a)$,$f^{'}(a)$ is the derivative of the function evaluated at point A.

In short the derivative of $f(x)$ at point a in its domain that we have to be careful about, this $x = a$ should belong to the domain of the function. This is given by $f^{'}(a) = \lim_{h \to 0} [f(a + h) - f(a)]/h$.

Now here I have used this new symbol limit h goes to 0 and this is a new symbol I understand it basically is trying to capture the fact that in this quotient we are taking h to be as close to 0 as possible. It does not mean that h is equal to 0. This is very important to understand because if you take $h = 0$ then what happens to the Newton quotient it becomes equal to 0 because $h = 0$ and in the numerator you have $f(a) - f(a)$ because h is equal to 0, 0 divided by 0 and which is undefined.

So, we are not saying that h is equal to 0 because that will give us a quantity which is not defined. What we are saying is that B is approaching A. So, in the limit h is tending to 0 we are not saying h is equal to 0.

(Refer Slide Time: 17:35)



We are going to talk about limits a little bit in detail later on. Now we are just introducing the idea of derivative. So, here is an example suppose the form of the function is given $f(x) = x^3$ we have to find $f'(a)$, a is any constant which belongs to the domain of the function and then we have to find out $f'(0)$ and $f'(-1)$. So, once we can find $f'(a)$, then I will just substitute 0 for a and (-1) for a and we will get these two other quantities.

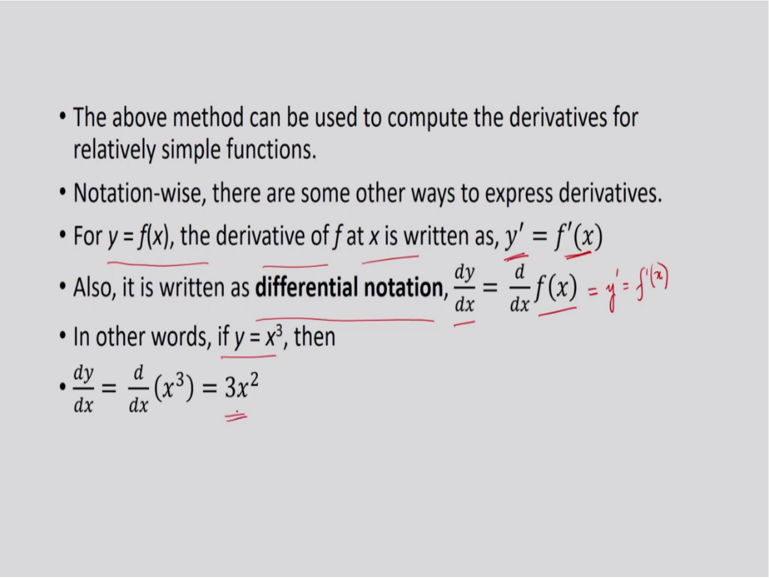
So, what do we do we go by that formula. This is the formula of derivative $\lim_{h\to 0} [f(a+h) - f(a)]/h$. Now let us first concentrate on the numerator. Now, what is the numerator in this case? So, the form of the function $f(x) = x^3$. So, instead of x I am writing $f(a+h) - f(a) = (a+h)^3 - a^3$. So, $(a+h)^3 - a^3$.

And if I just expand $(a+h)^3$, then I will get a term a^3 which will get cancelled with this minus a^3. So, I will be left with this part $3a^2h + 3ah^2 + h^3$. Now the Newton quotient is this thing divided by h and so if I divide this thing by h, I am left with this expression $3a^2 + 3ah + h^2$ and then I take limit h goes to 0 because this is the derivative and if I do so then this part will vanish, this part will vanish.

So, I will be left with only $3a^2$. So, $3a^2$ is the answer $f'(a) = 3a^2$. So, the first part is answered. Now we have to find out what is $f'(0)$ and $f'(-1)$ for that I just have to substitute 0 for a in this expression. So, if I put $a = 0$ I get 3 multiplied by 0 which is 0 and

$f^{'}(-1)$ so here again I put $a = -1$, $(-1)^{2} = +1$. So, this becomes plus 3 so these are the answers this one, this one and this one $3a^{2}$, 0 and 3.

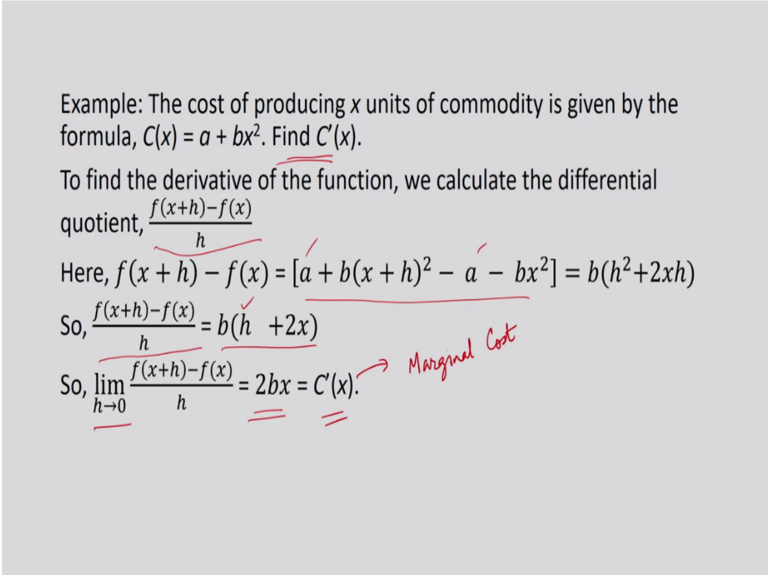
(Refer Slide Time: 20:51)



Now this above method of finding the Newton quotient first and taking h limit h going to 0 this method can be used to compute the derivatives for relatively simple functions. Notation wise there are some other ways to express derivatives. So, what we have used is $f^{'}$ so that is the notation to denote derivatives, but there are other notations also. So, if $y = f(x)$ then the derivative of f at x is written as $y^{'} = f^{'}(x)$.

It is also written as differential notation which is written as $\frac{dy}{dx}$. So this was used by Leibniz and he used this particular notation $\frac{dy}{dx}$. $\frac{dy}{dx}$ in this case will be written as $\frac{dy}{dx} = \frac{d}{dx} f(x)$ because $y = f(x)$. So, all these things are same things they mean the same thing. So, in other words you are given a function let us suppose $y = x^{3}$, then $\frac{dy}{dx} = \frac{d}{dx}(x^{3}) = 3x^{2}$.

(Refer Slide Time: 22:29)



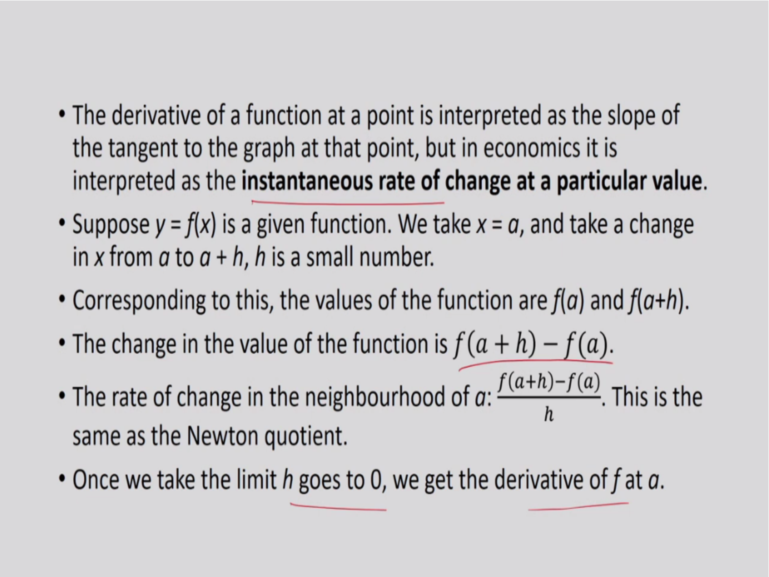
Here is an example from economics. The cost of producing x units of a commodity is given by the formula $C(x) = a + bx^2$. Find $C'(x)$. $C'(x)$is what this is the derivative of the cost function, this is the cost function with respect to x, what is the x? This x is the units of the commodity that is being produced. To find the derivative of the function we calculate the differential quotient the Newton quotient or differential quotient.

Now what is the numerator, $f(x + h) - f(x)$. So, that will be if I use the formula that $f(x) = a + bx^2$ it becomes this much $f(x + h) - f(x) = [a + b(x + h)^2 - a - bx^2] = b(h^2 + 2xh)$ and I have to divide it by h to get the Newton quotient or the differential quotient.

And so we are getting this term $b(h\ + 2x)$ and then I take h limit going to 0 so this term will drop out and we will be left with $2bx$ and this is our $C'(x)$. This $C'(x)$ is also called the marginal cost. Now, if I go back to the idea of derivative what is the notion of derivative, then how do we interpret the marginal cost? So, derivative is the rate of change of the dependent variable.

So, here marginal cost is the rate at which the cost is changing with respect to change in the output level because here the independent variable is the output. So, this is the rate of change of the cost function with respect to change in the output level.

(Refer Slide Time: 25:15)



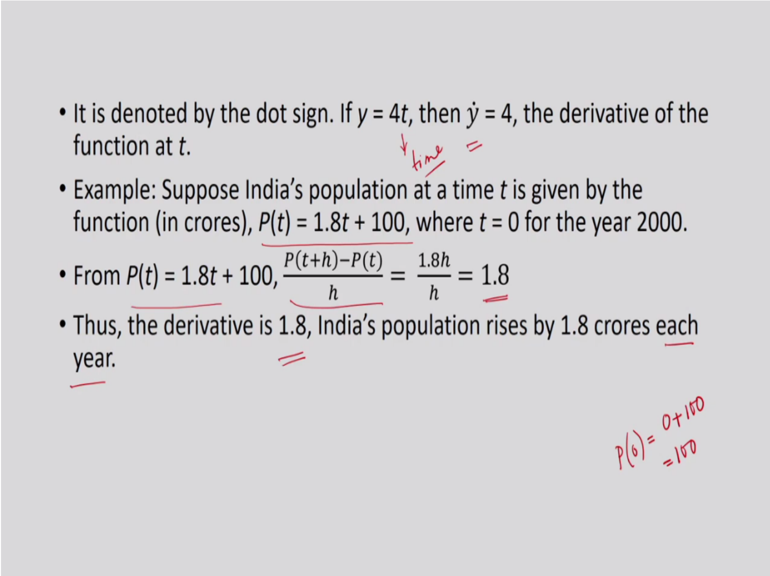
The derivative of a function at a point is interpreted as the slope of the tangent to the graph at that point. So, this was a geometric interpretation. In economics, it is interpreted as the instantaneous rate of change at a particular value. This is important instantaneous rate of change because we are considering B to be in the immediate neighborhood of A and then we are assuming that h is going to 0.

So, that basically means that a and b are very close together. So, one is considering very small change of the independent variable and then looking at the change of the dependent variable which is why this is called the instantaneous change at a particular value. Suppose, $y = f(x)$ is a given function we take $x = a$ and take a change of x from a to $a + h$, h is a small number.

Corresponding to this, the values of the function are $f(a)$ and $f(a + h)$. The change in the value of the function is given by $f(a + h) - f(a)$a. The rate of the change in the neighborhood of a is on the numerator you have the change of the value of the function and in the denominator you have the change in the value of the independent variable which is h. This is the same as the Newton quotient.

Once we take the limit h goes to 0 we get the derivative of $f(a)$ so this is what I just explained that derivative of a function at a particular point a is the change instantaneous change rate of change at a particular value.

(Refer Slide Time: 27:24)



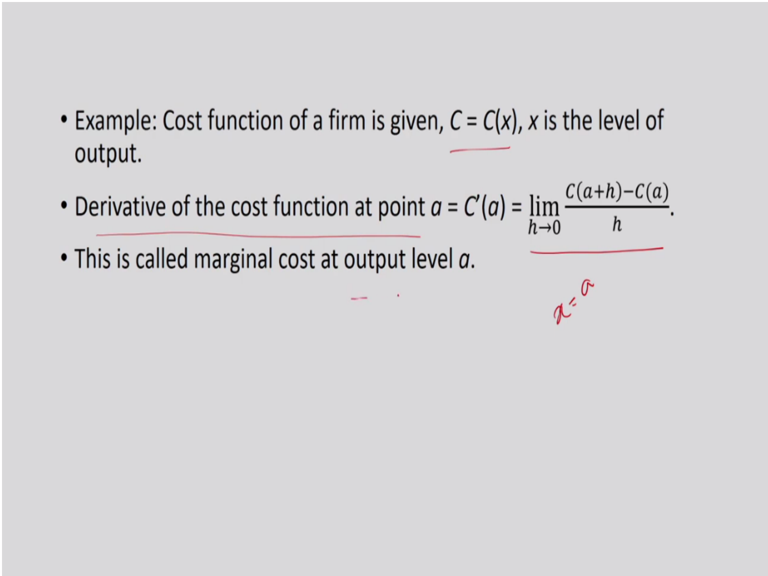
It is sometimes denoted by the dot sign, especially if we take the independent variable to be time. So, t is time so here in this particular function $y = 4t$,that means as time is changing the dependent variable y is changing by this formula 4t, $y = 4t$. Now here I can find out the derivative of this function $\frac{dy}{dt}$and this $\frac{dy}{dt}$ is also written in this fashion ẏ and we can verify that here the derivative will be 4 is constant number.

So, here is an example suppose India's population at a time t is given by the function $P(t) = 1.8t + 100$ and suppose $P(t)$ is the population which is in crores and we are given this information that at $t = 0$ we are considering the year 2000. So, this is called the origin so if you put$P(t)$, then from this function what you get you get $P(0) = 1.8.0 + 100 = 100$.

So, in the year 2000 the population was 100 crore which is actually not of the mark India's population was around 100 crore in the year 2000 and then we are assuming that the growth of population or the population function is given by this formula $1.8t + 100$. Now from this formula we can find out what is the Newton quotient or the differential quotient and it turns out to be 1.8.

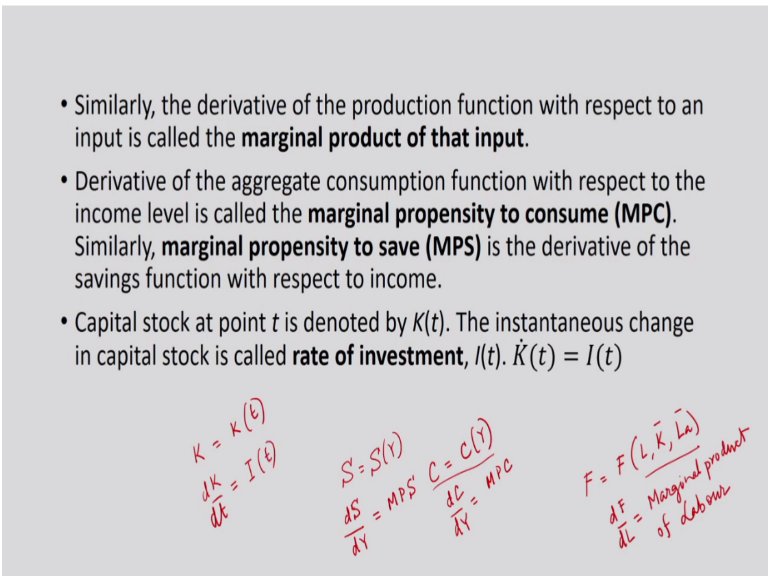
And obviously if you take the limit h going to 0 this 1.8 remains 1.8 because there is no h term here. So, that means that the derivative of this function with respect to t is 1.8. So, India's population rises at this rate 1.8 crore each year. I am going to talk about why we should interpret we can interpret this as per year each year means in 1 year. So, why we say so? Why you are not saying that this is the instantaneous rate of change?

(Refer Slide Time: 30:29)



Another example is here cost function we have already talked this in a particular context so cost function of a firm is given by $C = C(x)$. x is the level of output. Now we can take the derivative of this function and evaluate this at a particular point at a point suppose $x = a$ and then this become $C'(a)$ and this is the definition of that. This is called the marginal cost at output level $x = a$. So, I do not want to spend more time on this because you have talked about this before.

(Refer Slide Time: 31:17)



So, I am going through some of the examples where this idea of derivative is useful in economics. These are some other examples. Similarly, the derivative of the production function with respect to an input is called the marginal product of that input. So, here is the production function suppose F so you have labor, capital, land etcetera, $F = F(L, K, L_a)$ and suppose you take the derivative.

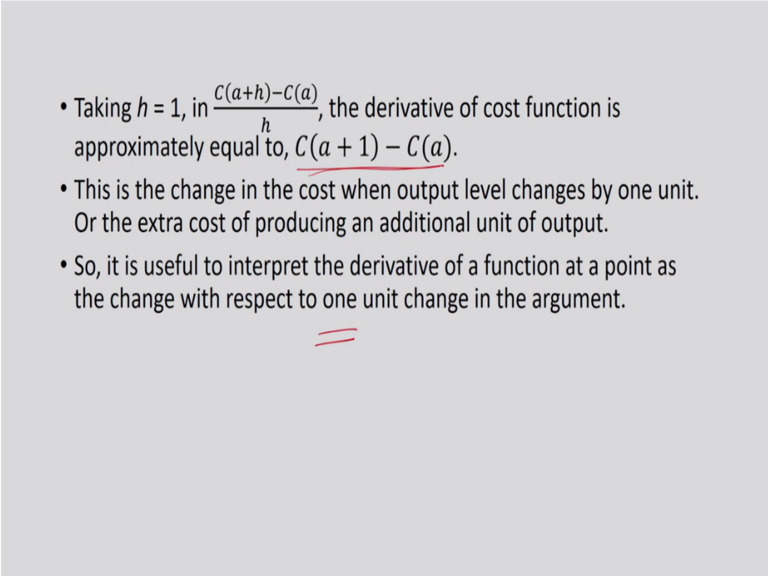
Here I have taken a function which is a function of many variables and we are going to talk about that also we have multiple variables, but for the time being let us suppose these are constant so these are parameters, capital and land are fixed. So, this becomes a function of only one variable which is L labour. So, you can now find out what is $\frac{dF}{dL}$ this will be called the marginal product of this particular input which is labour.

So, once we do economics we shall see plenty of examples of this marginal product. Derivative of the aggregate consumption function with respect to the income level is called the marginal propensity to consume. We have again seen some examples of this. So, here is this consumption function, $C = C(Y)$ C is consumption Y is the income. Now we can talk about the derivative of this. So, $\frac{dC}{dY}$ this is called the marginal propensity to consume MPC.

Similarly, marginal propensity to save is the derivative of the savings function with respect to income. So, just as consumption of people depends on their income so that is why you have $C = C(Y)$, savings also depend on the income level of people. The more rich people are ,the more they save. So, you have S which is the savings is equal to $S = S(Y)$ and so you can now talk about the derivative this is $\frac{dS}{dY}$ and this is called the MPS marginal propensity to save.

Capital stock again this is an example from macroeconomics capital stock at point t is denoted by $K(t)$. The instantaneous change in the capital stock is called the rate of investment and which is denoted by $I(t)$. So, K is capital stock $K(t)$ the more time passes it is conceivable that more capital is being accumulated. So, capital stock is a function of time. If one takes the derivative of this, this becomes the instantaneous rate of change of capital stock and this is denoted by I, $K(t) = I(t)$. This itself can be a function of t.

(Refer Slide Time: 35:04)

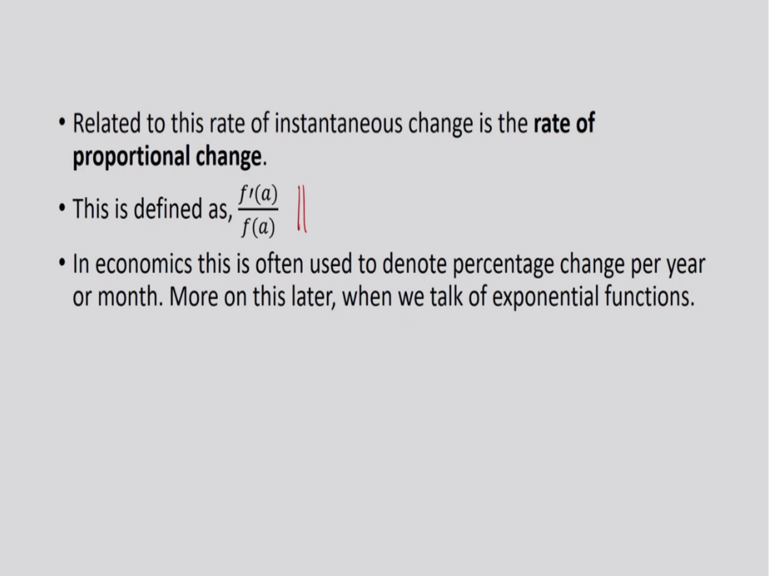


So, I come to that point why we are talking about change in the output level by one unit. Suppose I take $h = 1$ then this Newton quotient becomes equal to this. So, $C(a + 1) - C(a)/1$ divided by 1 so it becomes $C(a + 1) - C(a)$. So, derivative becomes something very close to this amount and what is $C(a + 1) - C(a)$. This is the change in the cost when output level changes by one unit or the extra cost of producing additional unit of output.

So, if the output level changes by one unit then this $C(a + 1) - C(a)$ gives you the change in the cost to produce that extra unit of output and this we have seen is approximately equal to the derivative. So, it is useful to interpret the derivative of a function at a point as the change with respect to one unit change in the argument.

So, practically speaking in economics we shall be dealing with discrete changes more often. And if we are talking about discrete changes then the minimum amount of change can be assumed to be one unit so therefore, the derivatives are often interpreted as change in the value of the function with respect to one unit change in the argument.

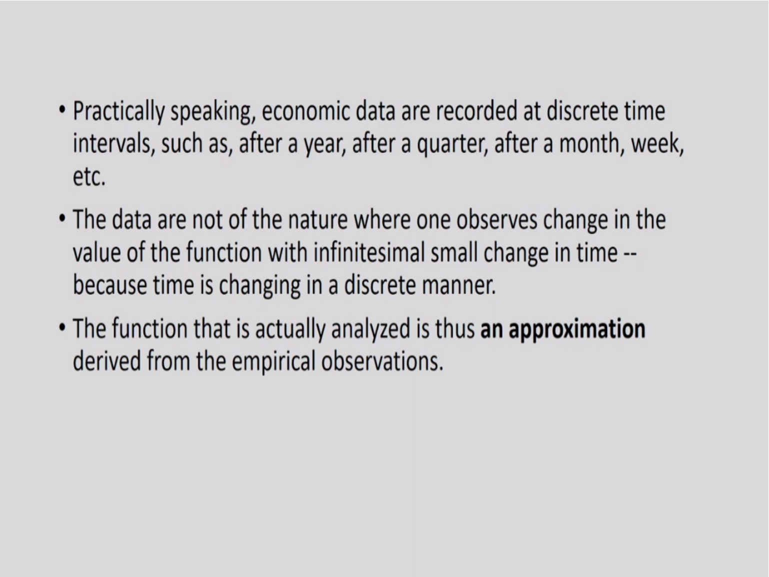
(Refer Slide Time: 36:47)



Now derivatives gives you the instantaneous change, but compare to that there is another rate of change which is called the rate of proportional change and this is defined as $\frac{f'(a)}{f(a)}$. Now on the numerator you have the derivative, but you are dividing that by the value of the function this is called the rate of proportional change proportionately how much is the value of the derivative.

In economics, this is often used to denote percentage change per year or per month. More on this later when we talk about exponential functions, so there is something called an exponential function in exponential function we shall see I think we have discussed this before, but we have not talked about the derivative of exponential functions. So, in case of exponential functions the rate of proportional change is always constant.

(Refer Slide Time: 38:00)

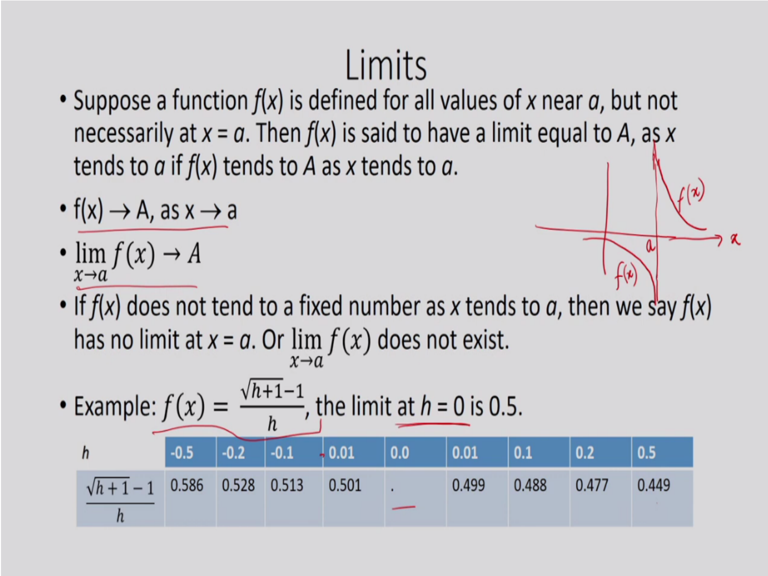


Practically speaking economic data are recorded at discrete time intervals such as after a year, after a quarter, after a month, week etcetera. So, for example the GDP of a country is generally denoted with respect to a year in a particular year what is the GDP in the next year or if we go to smaller time intervals it could be in a particular quarter. So, nonetheless it is discrete change in the time it is not continuous change.

The data are not of the nature where one observes change in the value of the function with infinitesimal small change in time because time is changing in a discrete manner. So, time is granular it cannot be broken down further to very small, small grains. The function that is actually analyzed is thus an approximation derived from the empirical observations. So, empirically what one finds is changes in the discrete manner.

There you do not find change in the continuous manner, but from those discrete data one constructs function which is an approximation and that function which one construct is then analyzed as a function of a continuous variable.

(Refer Slide Time: 39:42)



Now we come to something that we have been talking about, but did not define it in a precise manner. This is the idea of limits. We talked about the fact that h going to 0 in the limit h is going to 0, but what is the idea of limits? Suppose a function $f(x)$ is defined for all values of x near a, but not necessarily at a, then $f(x)$ is said to have a limit equal to A as x tends to a if $f(x)$ tends to A as small x tends to a.

So, you have a function $f(x)$ which is defined for all values of x near a, but it is possible that it is not defined at $x = a$. Then we say that this function has a limit and that limit is equal to A if the following thing is satisfied that $f(x)$ tends to A as x tends to a. So, this is in mathematical terms this is written as this $f(x) \to A,\ as\ x \to a$.

And in a different manner it is also written as this $\lim_{x \to a} f(x) \to A$. If $f(x)$ does not tend to a fixed number as x tends to a then we say $f(x)$ has no limit at $x = a$ or $\lim_{x \to a} f(x)$ does not exist. So, what could be the visualization of this that you have x going to a, but $f(x)$ does not go to any finite or fixed number.

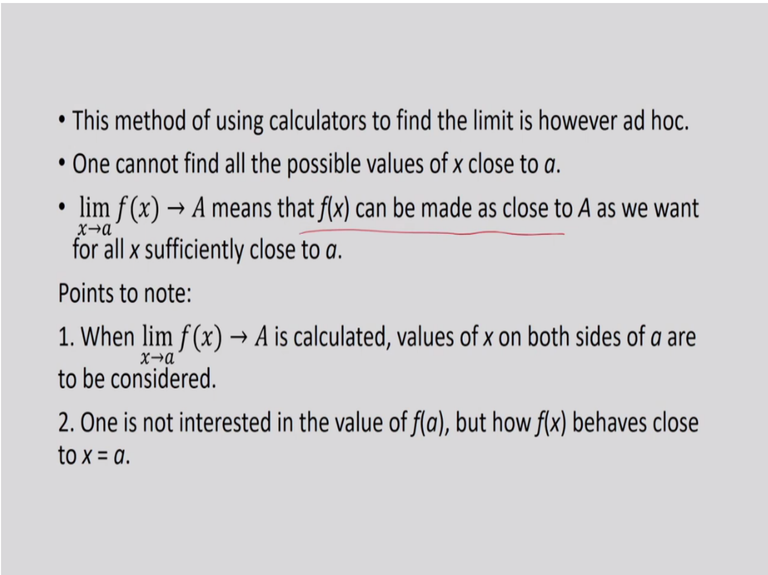
So, you can imagine a function like this that you have this x axis here you have a and the function is something like this. So, you see here the function is composed of two parts this is $f(x)$, this is also $f(x)$. As x goes to a the function from the left hand side it goes to minus infinity from the right hand side it goes to plus infinity. So, in this case this limit at $x = a$ does not exist.

Here is an example of how to find the limit. Here $f(x) = \frac{\sqrt{h+1}-1}{h}$ and we want to find out the limit at $h = 0$ so that we have to find out. So, what do we do, we take values of h very close to 0 at t $h = 0$ this function is undefined because the denominator become 0. So, here you have a dot, nothing is there.

I am not taking t $h = 0$, I am taking h very close to 0. For example I am starting h at 0.5 and I am going to t $h = 0.01$ and correspondingly I can see that the value of the function from 0.449 it goes to close and close to 0.5 it approaches 0.5 because at 0.01 it takes the value 0.499. I can approach the value $h = 0$ from the left hand side also. So, I take point $h =- 0.5$ the corresponding value of the function is 0.586.

Then I choose to take this will be - 0.1 the corresponding value of the function will be 0.501 so it is very close to the value 0.5. So, one can guess that as h approaches 0, then the value of the function $f(x)$ approaches 0.5 because from both the sides as h is approaching 0 that is the idea of limit as it approaches 0 the value of the function becomes very close to the value 0.5.

(Refer Slide Time: 45:03)



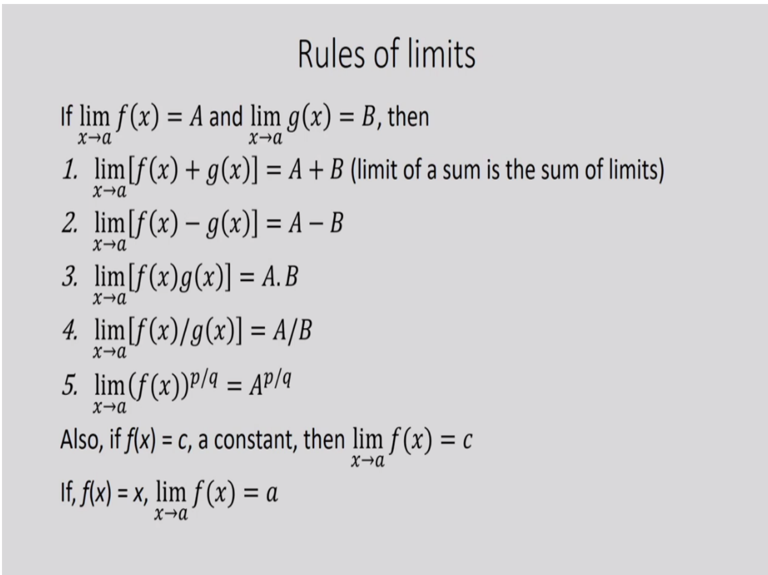
This method of using calculators to find the limit is however ad hoc. So, what we are doing is that using calculators to find out the value of the function as we are taking the value of h very close to 0. This is ad hoc why because we cannot find all the possible values of x close to a. So, in a more precise manner how do we define a function or how do we define a limit is that suppose you have $f(x)$ going to a as x goes to small a.

It means that $f(x)$ can be made as close to A as possible as we want for all x sufficiently close to small a. So, the value of the function can be made very close to the limit which is capital A. If we take the argument, the value of the argument very close to that point which is a. Now two points to note here one is firstly when we say that the limit is A when it is calculated the value of x on both sides of A are to be considered.

Just as we have done before here we are approaching the value 0 from the left hand side here and from the right hand side here from the negative and from the positive we are approaching the value 0 and the second point is more conceptual point. One is not interested in the value of $f(a)$, $f(a)$ is not interesting to us, but how $f(x)$ behaves close to $x = a$ that is the idea of limit.

So, you are approaching $x = a$ and if we are approaching $x = a$, then how is this function f of $f(x)$ behaving.

(Refer Slide Time: 47:20)



So, these are rules of limits I think will stop here and take these up in the next lecture. So, we are calling it a day today and I shall see you in the next lecture. Thank you.