Mathematics for Economics 1
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Module: Functions
Lecture 8: Exponential Functions, et cetera.

Welcome to the eighth lecture of this course called Mathematics for Economics part 1. So, we are talking about different kinds of Functions. We started with linear functions, where the general form is y= ax+b , then we explored some different more complicated forms of functions.

We talked about quadratic functions and we also talked about cubic functions and discussed certain properties of them. I also discussed that in applications of these functions and now, we are talking about a general what is known as a polynomial function.

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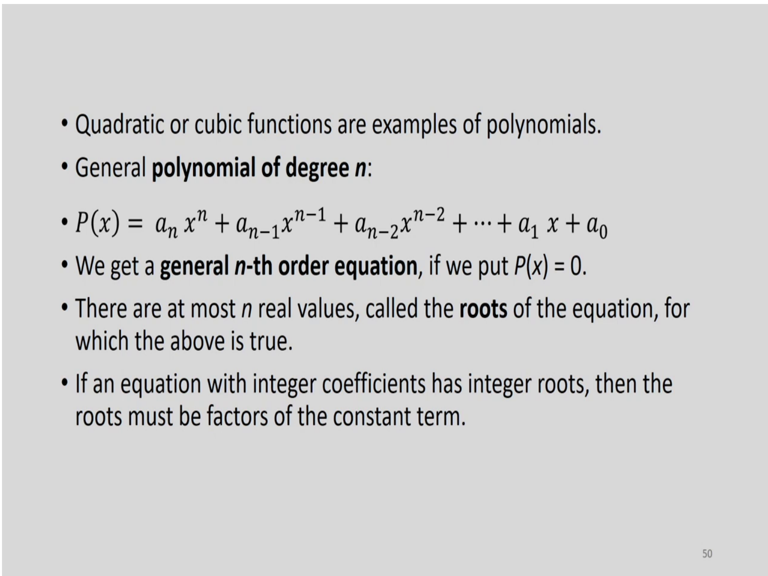

So, we have this general form P(x)= $a_n x^n + a_{n-1}x^{n-1}$ …….+ $a_1 x + a_0$. So, this is a polynomial and from this polynomial you can define a function which is like y = P(x) and then we get the rest of these terms.

Now, if we consider this to be an equation, rather than a function that is P(x)= 0, then we have an equation, then there are at most n real values called the roots of the equation for which the above is true. So, this is the definition of roots of the equation, the number of the roots can be at most n, if they are real values. If an equation with integer coefficients as integer roots, then the roots must be factors of the constant term. So, let us look at this idea.

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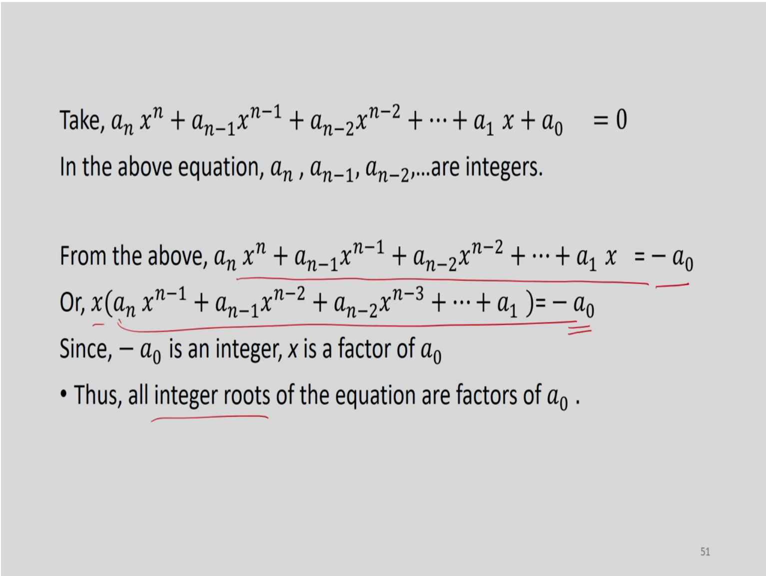

So, we have this equation . P(x)= $a_n x^n + a_{n-1}x^{n-1}$ …….+ $a_1 x + a_0$, this is an equation and in the above equation, suppose all the coefficients are integers, a_n, $a_{n-1,}$ a_{n-2} all are integers. Now, from the above, we can just subtract a 0 from both sides.

So, we get this expression . $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2}$…….+ $a_1 x = - a_0$Now, from the left hand side we can take x common then, x($a_n x^{n-1} + a_{n-1}x^{n-2} + a_{n-2}x^{n-3} +$ …….+ $a_1 = - a_0$. Now, we know that a 0 is an integer, because it is the constant term and all the coefficients, constant term is actually a coefficient in some sense. So, -a_0 is an integer. Therefore, x is a factor of a_0, because you know we are talking about the roots which are integers. So, x is a factor of a_0.

Thus all integer roots of the equation are factors of a_0. So, take any integer root obviously, this will not hold for non-integer roots, but if we are talking about only integer roots, then all the integer roots will satisfy this relation and therefore, all integer roots will be factors of $-a_0$ and therefore, they will be factors of a_0.

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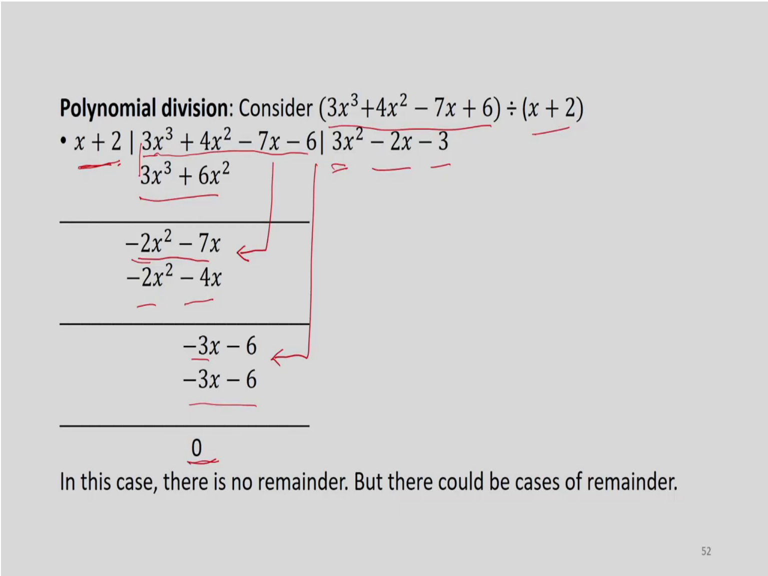

We come to another operation of polynomials, this is called polynomial division. So, consider that we are dividing this polynomial, $3x^3 + 4x^2 - 7x + 6$.. by x + 2, x + 2 is also a polynomial in some sense. Now how do we do that, we do it in the following manner. So, here we are going to divide $3x^3 + 4x^2 - 7x - 6$.

So this is going inside and on the left hand side, I am going to divide by x + 2. So the first thing I will look for is how can I make x, similar to the first term here, the first term here is $3x^3$. So, if I multiply x by $3x^2$, I get $3x^3$. So, I write $3x^2$here. Now $3x^2$is getting multiplied by the entire term, which is x + 2. So if I do that, I get $3x^3$+ $6x^2$.. So that is coming here and I do the subtraction.

Now, obviously, this the first element, if I considered them, they will get cancelled, that is by construction and from the second element, if I subtract $6x^2$from $4x^2$, I get -$2x^2$ and then I take -7x from above and so the term that I have to divide is now -$2x^2 - 7x$.

Now again, with the same strategy, I will try to find out that factor, which if multiplied with x gives me -$2x^2$ and the term is -2x, -2x multiplied by x gives me -$2x^2$, but there is + 2 also, if 2 is multiplied with -2x, you get - 4x. So, from - 7x, you subtract - 4x, you get -3x, then I take 6 from above, I get - 6 here and therefore, again, I have to think about a number which if I multiply with x, I get -3x and that is very simple - 3.

So, you get - 3x - 6 and if you subtract this you get 0. So, in this particular method, I can divide one's polynomial by another polynomial. In this case, there is no remainder here, there is a remainder here, it is 0. But this is not automatic. In many divisions, there will be some nonzero remainders. Let us take one such example.

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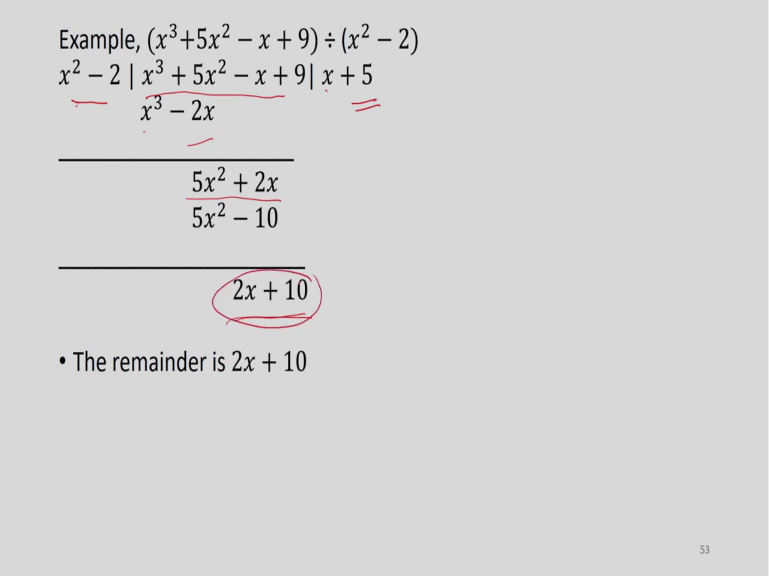

So, here you have $x^3 + 5x^2 - x + 9$, this has to be divided by the divisor, which is $x^2 - 2$. So, $x^2 - 2$ is coming on the left hand side, then inside, this $x^3 + 5x^2 - x + 9$ and I deploy the same strategy which is that I think about that thing, which if multiplied by x^2 gives me this first term x^3 and this is very simple, it will be x.

So, x multiplied by x^2 gives me x^3, but then there - 2 term. So, - 2 multiplied by x is - 2x and I subtract -2x from $5x^2$, I get this $5x^2$ + 2x and again, I think of the number which has to multiply with x^2 to give you $5x^2$ and that is just plus 5. So, you get $5x^2 - 10$.. So, you have 2x+ 10. Now, you cannot do anything else with 2x+10, because the powers of x here is 1, whereas the power of the divisor is 2. So, this will be the remainder here 2x+10.

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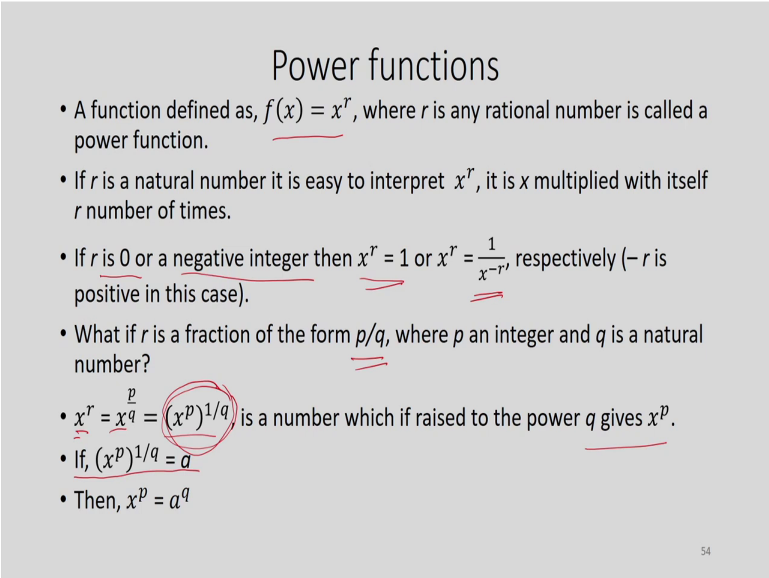

Now we talk about other kinds of functions after talking about polynomials, we talk about something called the power functions. So, a function which is defined as the following, f (x) = x^r is called a power function, where r is any rational number. So, r could be expressed in the form of p/q and that was the definition of a rational number.

Now, there could be some difficulty in interpreting this x^r, if r is not a natural number. If r is a natural number then it is easy to interpret x^r, if r is any natural number, like 2 or 3 or 4. These are natural numbers, the next to the power r is easy to interpret, so x to the power let us say 5 that is equal to x multiplied by itself, five times that is x^5.

So, as long as r is a natural number, then there is no problem of interpreting x^r.But if r is 0 or a negative integer, then what happens? If r =0, then again it is not very difficult, x^0 is we know it is equal to 1 and if we are talking about negative power, so x to the power let us say minus 2, then I can write it as 1 / $x^{-(-2)}$that is what I have written here, - (-2) is 2.

So, here you have 1 divided by x^2 and x^2 is easy to interpret, this x multiplied with x. So, as long as r is 0, a negative integer, a positive integer there are not many difficulties. What if r is a fraction of the form p/q, where p is an integer and q is a natural number. So, here we are basically approaching the idea of rational numbers.

So, I can write $x^r = x^{p/q}$. This can be written as, , $(x^p)^{1/q}$ So, then what happens is that p gets multiplied with 1/q and you get p /q, but what is the interpretation of this number $(x^p)^{1/q}$

This is a number which is raised to the power q will give me x^p that is easy to see, you just raise this number to the power q, then q and q will get cancelled and you will get x^p.. So, this is what we have done step by step. So, I have to interpret this number. Suppose this number is equal to a, which is a constant suppose.

So, $(x^p)^{1/q}$= a and then I raised the power of both sides by q, then I get $x^p = a^q$and this is what I meant by this, that if it is a, if I can raise the power of a by q then I should get x^p.

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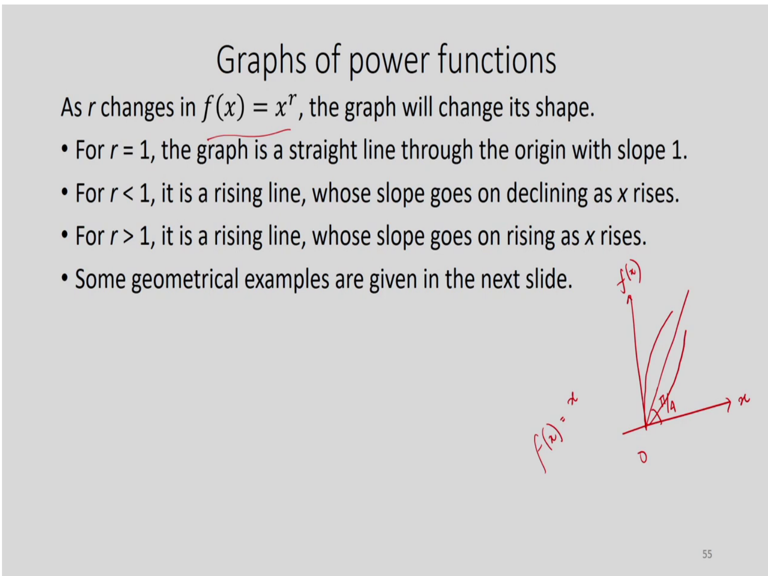

Graphs are Power Functions. Now, in this form f(x) = x^r. As r changes, the graph changes its shape. If r=1, then what you have is f (x) = x. So, what does this look like? It is a straight line through the origin making a 45 degree angle with the x axis, y is equal to x.

So, this is a slope of 1. If r is less than 1, then also it is a rising line, but it is no longer a straight line, whose slope goes on declining as it rises. So, this is like this and if r is greater than 1. So, the power is greater than 1, then it is a rising line like before, but now the shape is different, the slope goes on rising instead of declining as x rises. So, this will be like this, the slope is going on rising, it is getting steeper and steeper.

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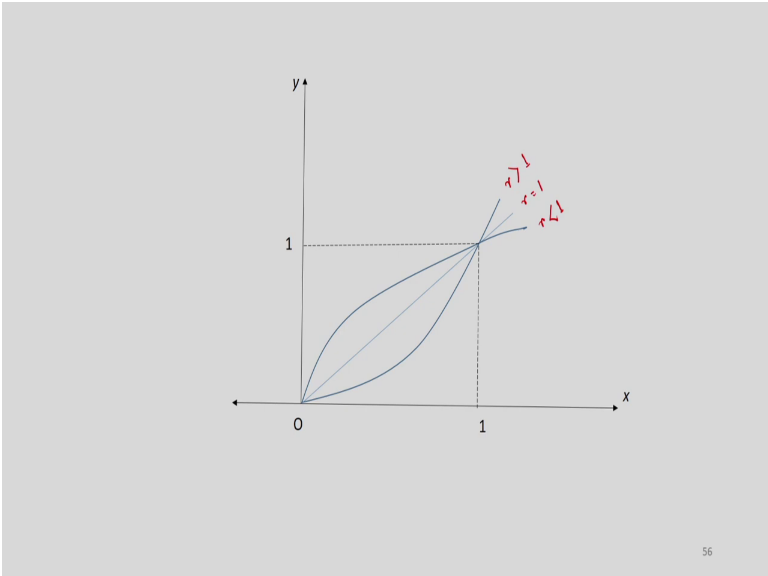

So, here is an example of how the lines could look like in the first case you have r =1, here you have r less than1 and here you have r greater than 1, but when I am saying r is less than 1, I mean r is greater than 0. So, but all these lines you can see are intersecting at the same point and that point is 1 and 1.

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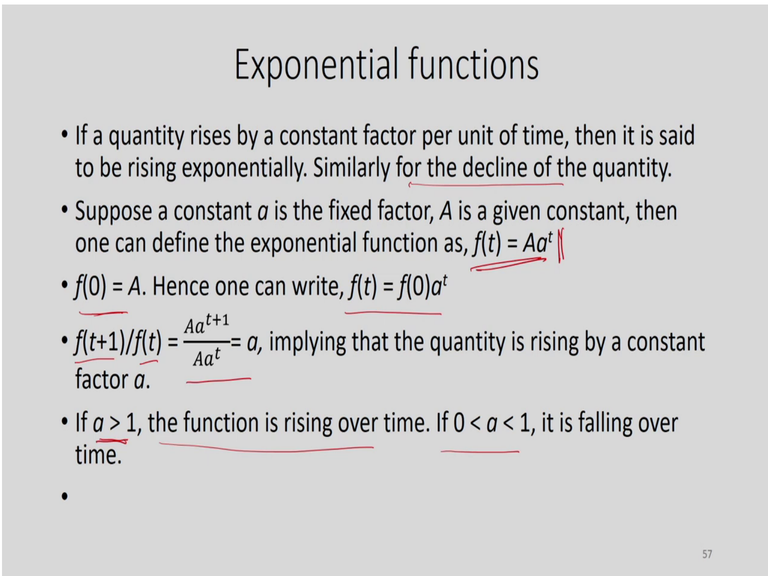

After power functions, we come to a kind of similar idea of a function which is called an exponential function. If a quantity rises by a constant factor per unit of time, then it is said to be rising exponentially. Similarly, for the decline of the quantity So, suppose any quantity is rising with time.

So, time here is changing, time is the independent variable, but the quantity in question is rising with a constant factor per unit of time. So, maybe it is getting doubled every year, that will be like rising by a constant factors or it could be going down by a constant factor also, it can decline and but the decline is also by a constant factor, if that is the case, then we can say that this quantity is rising or declining exponentially.

Suppose, a is the fixed factor and capital A is a given constant, then one can define the exponential function as f (t) = Aa^t.. So, again just to repeat here, I am taking t as the independent variable, it is rising and therefore, this quantity is changing and if it is changing by a constant factor, we are calling that to be an exponential function.

Here, in this case, in this particular example, a is that constant factors, this I will show it in a second, at first look at this form and put t = 0, if you put t = 0, then you get f(0)and that is equal to A, because a^0. is 1. Now, A = f (0), then I can substitute that here and I get f(t)= f(0) a^r and how do I know that indeed for this function the factor is equal to a.

I said that it rises or declines by a factors and that factor is a , what is the proof of that, you take two values of the function at two successive times, one is f of t +1 and divide that by f (t) and if you do that, you will get a and since a is constant, so that means from one period to the next, the rise is also by a constant factor.

What about the rise and fall of this function, if small a is greater than 1, then the function is rising over time, that is easy to understand. So, a is the factor with which the value of the function is getting multiplied in one period to get the value the next period. So if a is just equal to 1, then the value will not change. It will remain the same, which means that if a slightly greater than 1 even, then in the next period the value will rise.

So, that is why I say that if a is greater than one then the function is rising over time. On the other hand, if a is strictly less than one and suppose it is greater than 0, then it is falling over time. So, think about this: a number is getting multiplied by 0.5. Now, if you multiply any number by 0.5, then it becomes half of the number. So and again, if you multiply it by 0.5, it becomes one fourth. So, it is going down over time.

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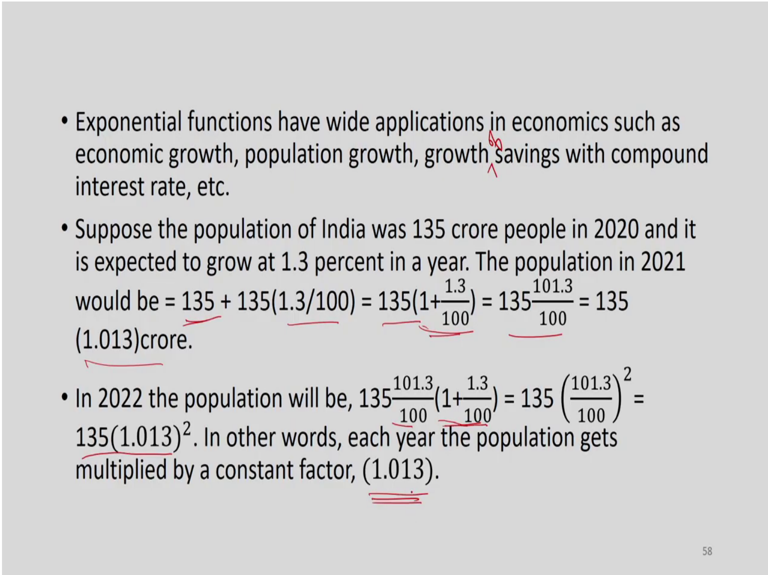

Exponential functions have wide applications in economics, such as economic growth, population growth, growth of savings, this should be of, growth of savings with compound interest rate etc. So, economic growth means, the income of a country grows over time, how do you analyse that.

To analyse that, you might be using the exponential functions or population growth, population also goes on rising over time and generally it is seen that over a, not so long a time frame the growth rate of population could be exponential with a constant factor. And similarly, growth of savings with the compound interest rate, we will see some examples of that.

So, here is an example, suppose the population of India was 135 crore people in 2020 and it is expected to grow at 1.3 percent in a year. So, what will be the population in the next year, in the next year means 2021. Now, that can be calculated, it will be the population in the initial year that is 2020, which is 135 plus you multiply 135 with the factor.

Here the constant factor is 1.3 percent. Now, what is 1.3 percent? It is 1.3/ 100 and it will be 135 (1+1.3/100) and if you simplify this, it becomes 135*101.3 /100 and this can be written as 135 multiplied by 1.013 crore.

So, this is the population in the year 2021. The next question is what about the population of 2022 if the same exponential growth continues. So this will be the population of the year 2021 multiplied by this factor. Because of this factor, I am just using the same formula. So, this becomes 135 multiplied by this and this and this are exactly the same and so, you have $(135 * 101.3 / 100)^2$ and this can be written as $135 * 1.013^2$.

In other words, each year the population gets multiplied by a constant factor which is 1.013. So, this is the important thing, remember when we talked about exponential growth, I say that each year the value is getting multiplied by a constant factor that is a, here that a is this number 1.013.

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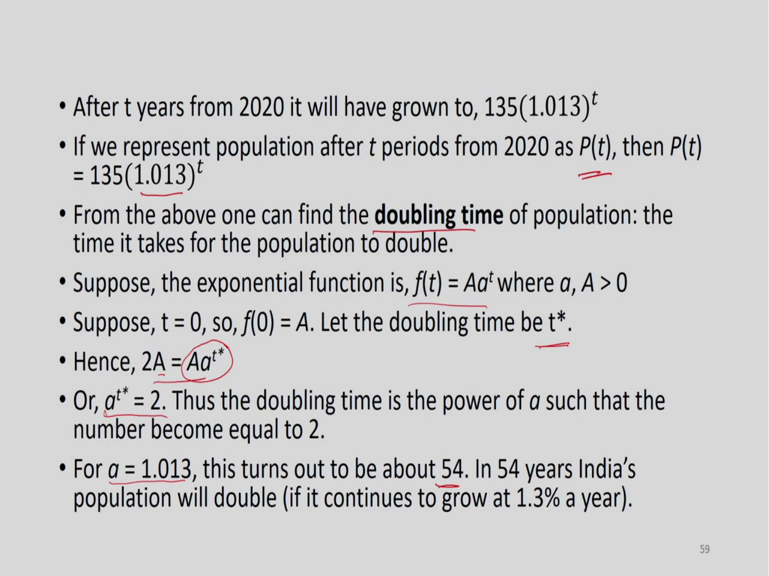

Afters t years from 2020, it will have grown to $135 * 1.013^t$.. So, that was the general form, it was A multiplied by a^t. Here small a is equal to 1.013, if we represent population after t periods from 2020 as P(t), therefore P(t) will be the same thing, which is $135 * 1.013^t$.

From the average one can find the doubling time. So, this important concept is often used to know about doubling time of population. What does it mean? It is the time it takes for the population to double So, population is growing, let us suppose a is positive and greater than 1, then population is growing. Now growing by a constant factor.

Question is, how long does it take for the population to exactly double. We can call that time the doubling time, suppose the exponential function is f (t) = Aa^t., where A is greater than 0. Suppose t is equal to 0. So, therefore, f (0) is going to capital A.

Let the doubling time t*, we are assuming that the doubling time is t* which means , after t* the population will double. So, what does it mean? It means that the initial population was A and after t* the population will be what? It will be this. So, this Aa^{t^*}= 2A, because A was the original population.

Now, if I use this then A will get cancelled from both sides. So, we are left with a^{t^*}= 2 . The doubling time is the power of A, such that the number becomes equal 2. This you can see from here. So, suppose a is given, I have to find out the power r such that the number is equal to 2.

So, this is an example, a is 1.013, I am using this 1.013 from here. Basically it means that the population per year is growing at 1.3 percent. Now, if A is taken to be 1.013, it turns out to be 54, it turns out to be means t* is equal to 54. So, in 54 years India's population will double, if it continues to grow at 1.3 percent per year. So, that is the meaning of doubling time.

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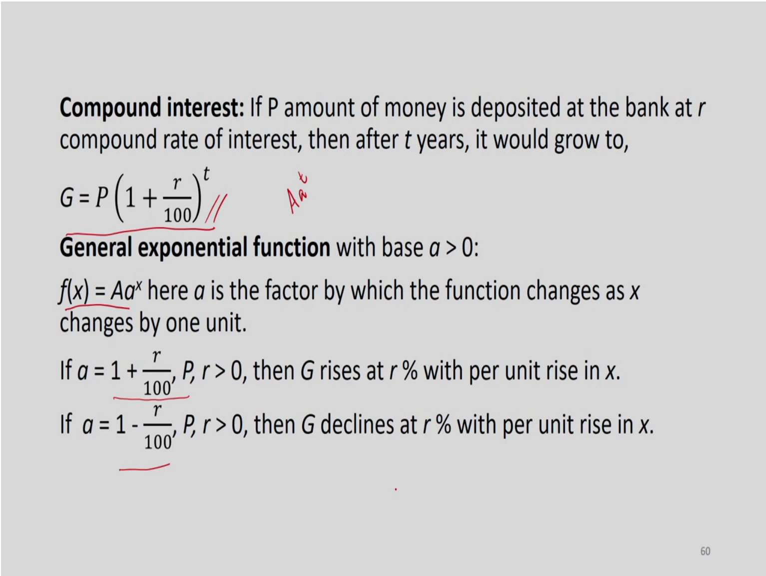

Compound interest - So, this is another application of exponential functions, suppose P amount of money is deposited at the bank at r compound rate of interest, then after t years it would grow to this number, G is equal to P, P is the amount of money that was deposited initially multiplied by $\{(1 + r)/ 100\}^{2}$. If you compare this with this form, Aa^t, then small a is really (1 + r)/ 100 and A is equal to P..

So, this is how we can calculate the deposit in a bank. If we know the rate of interest, if we know the time that has elapsed and if we know the initial deposit. Now, a general exponential function with base small a which is greater than 0, can be written as the following, $f(x) = Aa^x$

Remember here I am not using t anymore, I am using x because t was time, but there could be some other variables also that one could be considering, which is changing and which might be affecting the value of the variable. So, here x is that independent variable, x could be equal to t, it could be something else.

So, that is why I have written here small a is the factor by which the function changes, as x changes by 1 unit. If a = (1 + r)/ 100 and suppose P and r are positive, then G rises at r percent with per unit rise in x, so just imagine comparing this with this formula, then it will become very clear.

On the other hand, if a is equal to (1- r) / 100. So, here I am considering basically it could be that r is negative. So, instead of rising if it is declining, if you are considering a negative interest rate, then A is expressed as this (1- r) / 100. and here r is positive P is positive, then G is declining at r percent per unit rise in x.

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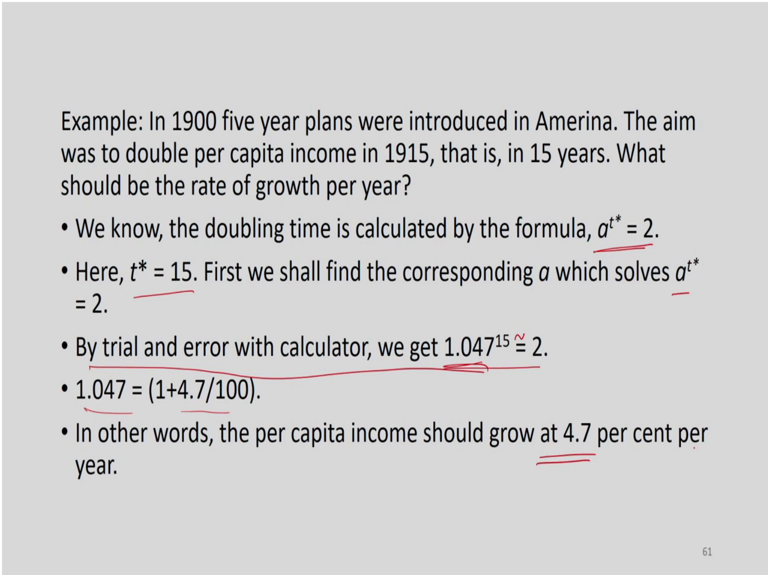

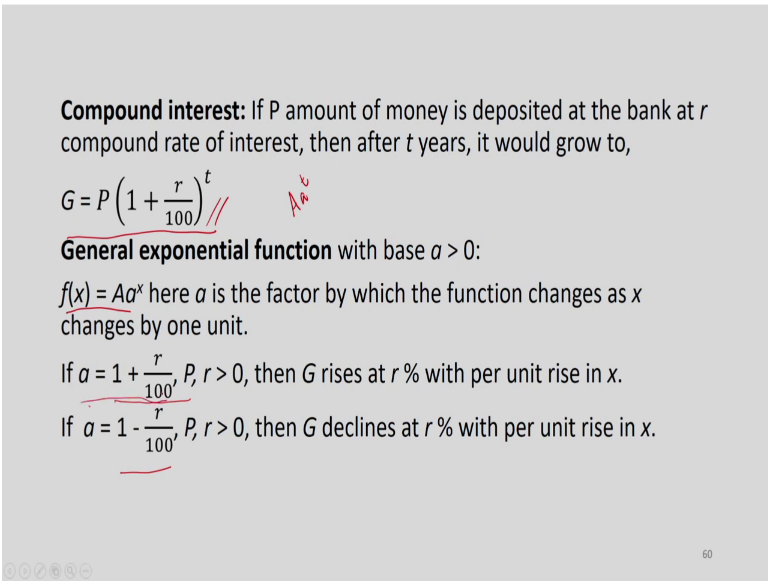

An example is given in 1900, 5 year plans were introduced in this country called Amerina. The aim was to double per capita income in 1915 that is in 15 years, what should be the rate of growth per year. So, what are we given here, we are given the time frame, there are 15 years that have lapsed, we are given the information that in these 15 years the per capita income should double.

So, we have to find out what should be the growth rate per year. So, basically we have to find out the value of the small a, we know the doubling time is calculated by the formula at^* = 2, we have just seen that, the doubling time. Here small a is the factor, constant factor. Now, in this particular example, we know that t^* is 15, doubling time is given 15. First we shall find the corresponding small a, which solves this one, this equation.

So, t^* is given which is 15, on the right hand side 2 is there, which is a number. So, we have to find out small a. Now, this can be done by trial and error if you have a calculator, very simple calculator which allows you to find the power of a number, then you can find out that if you take small a is equal to 1.047, then 1.047^{15} is very close to 2, the number 2, not exactly 2, but approximately equal to 2. So, our small a is this number, we have solved that.

Now, remember, we have just seen in the previous slide that this relationship is there, a is equal to 1 + r / 100. So, I can write this in this form a is equal to 1 + 4.7/100, 4.7 is therefore, the rate of interest, not rate of the interest in this case it is the growth rate of the per capita income per year. So, this is our solution, which means that this country which adopted the five year plans from 1900 onwards and wants to double its per capita income in 1915 in 15 years, it must grow at 4.7 percent per year.

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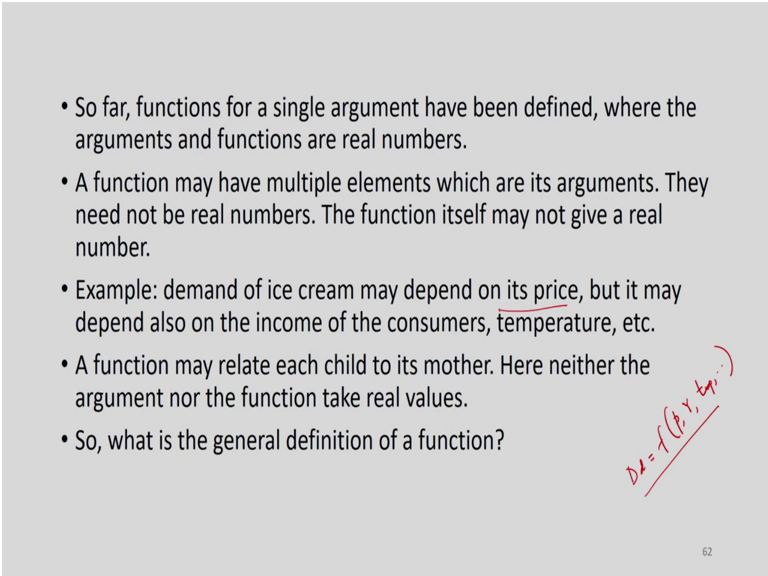

So far functions for a single argument have been defined, where the arguments and the functions are real numbers. A function may have multiple elements which are its arguments, they need not be real numbers. The function itself may not give real numbers. So, there could be variations from the original function or original example of a function that we have been talking about.

Firstly, the functions may not be real numbers, they may be something else, some object may be, some person maybe and secondly, the arguments may be multiple, it is not necessary that there is a single argument or single independent variable. So, there are some examples that I have given here. The demand for ice cream may depend on its price and that is called the demand function, if you remember, demand is a function of its price, but it may also depend on the income of the consumer, temperature etc.

So, there are multiple factors that might be affecting the demand for ice cream. So, I can write it like this, d which is demand is a function of price, it could be a function of income, it could be a function of temperatures, let us say it is t,m, p, etc, etc, there are multiple independent variables, which might be affecting the demand and so this function is a function of many variables.

And secondly, a function may relate each child to its mothers. Here, neither the argument nor the function, take real values. Remember, a function is not necessarily a mathematical formula, it is just a relation. So, suppose this relation is relating a child to its mother. So, you name a child, a particular child in a class and related to its mothers and that relationship itself is a function.

Now here, the child is not a number. It is maybe a person or you can say it is a name and the mother is also not a number, it is a, it could be named a person. So, in this case, you do not get real values as arguments nor do you get real values as the value of the function. So, what is the general definition of a function? So, if you have these complications, then how do you generally define a function?

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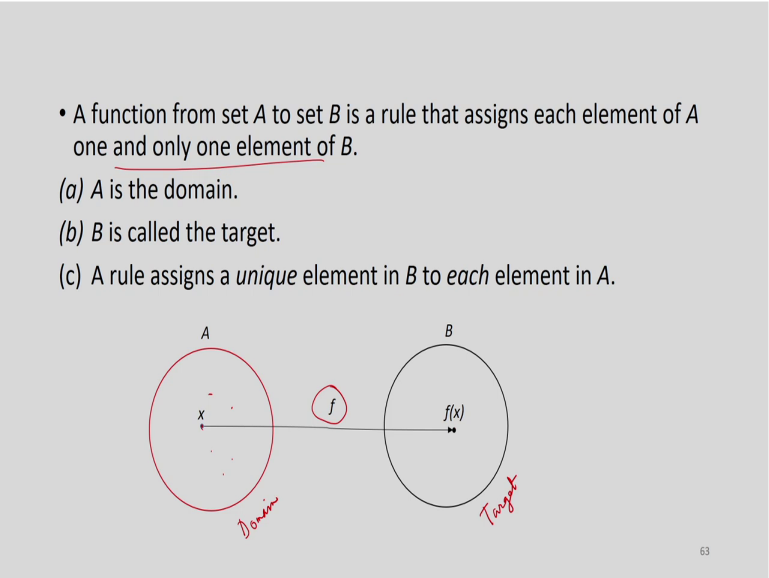

So, this is the sort of general definition of a function, a function from set A to set B is a rule that assigns each element of A one and only one element of B. So, you have two sets, this sets could be consisting of real numbers, they may not consist of real numbers, sets could consist of elements general.

But what a function does that it is a rule that assigns each element, means it should include all possible elements in the set A and it should assign each and every element to one and only one element of B, it cannot assign more than one element of B. So, A is called the domain we have seen this. So, here A is domain, B is called the target, domain set, target set, a rule assigns a unique element in B to each element in A and that rule is the function f. So, here you are taking element x from A and this function gives you an unique element f (x) in B.

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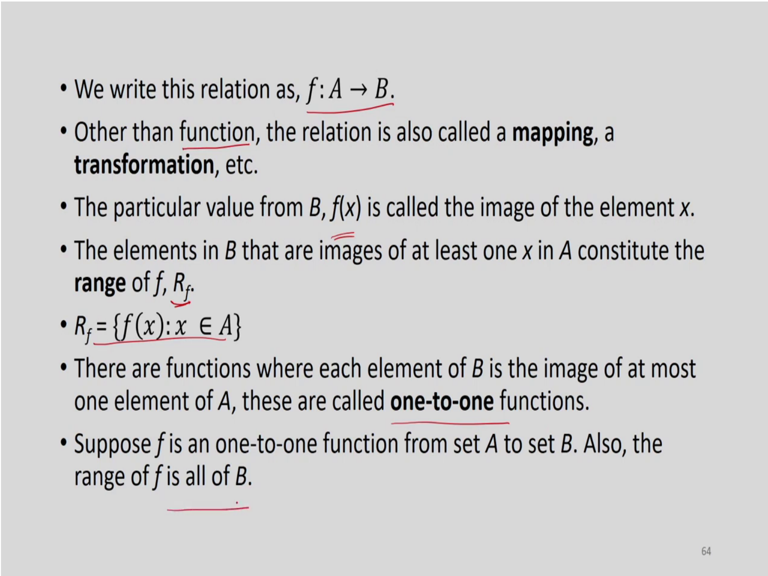

And we write this relation as f:A → B. So, it is a function from A to B, f is a function from A to B. Other than this term function, the relation is also called a mapping or transformation, etc. So, other names could also be used. A particular value from B, f(x) is called the image of the element x. So, f (x) is generally called an image or the image of the element x.

The elements in B that are images of at least 1 x in A, constitute the range of f or R_f. We have seen this in the case of our simple, one variable function, real valued, but that range could be defined in a more general sense here. So, here R_f which is the range it consists of those values of f (x) such that x is an element of A.

So, R_f consists of only those elements of B, which are images of the elements of A. So, all elements of B may not be inside the range. Some of them could be left out. There are functions where each element of B is the image of at most one element of A. Suppose f is a one to one function from set A to set B also the range of f is all of B.

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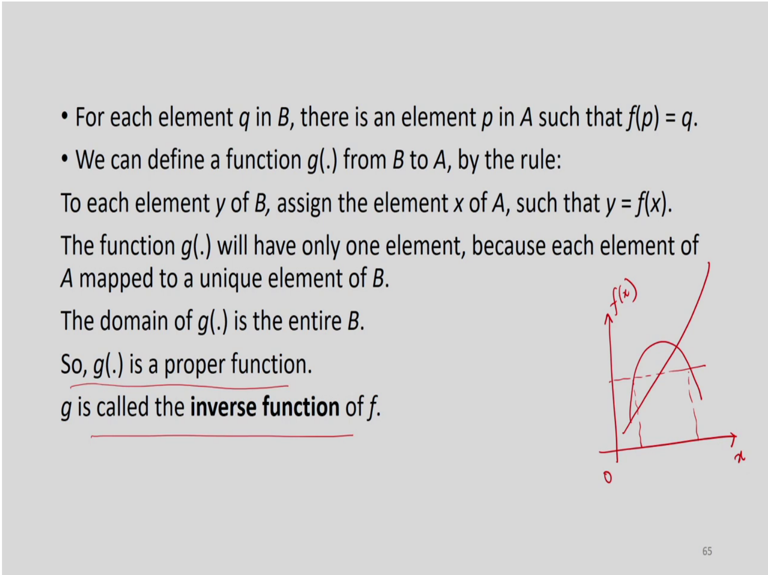

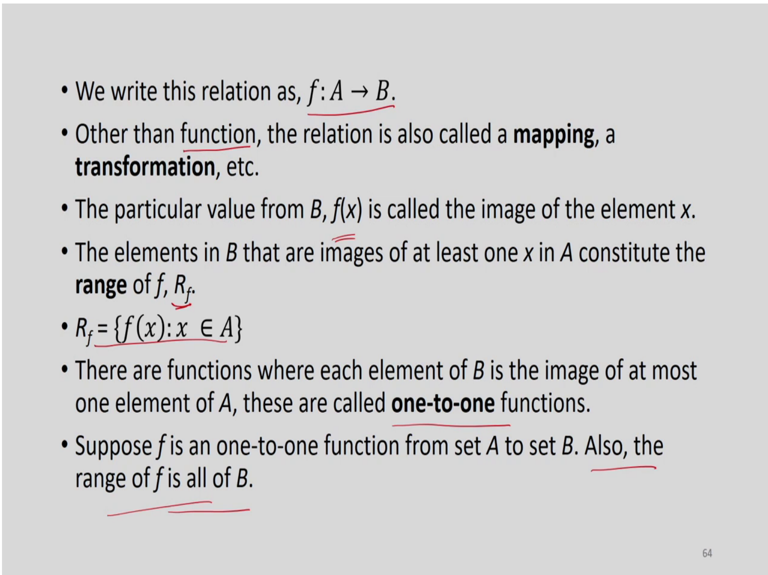

For each element q in B, there is an element p in A such that f (p)= q. So, you take any element of q and there will be one element of p in A such that f (p) = q, we can define a function g from B to A by the rule for each element y of p assigned the element x of A such that y is called f(x).

Now, there is one point that I wanted to draw your attention to that one to one functions are those functions where if you take any element of B, the corresponding element of A cannot be multiple, there should be at most one element of A, it cannot be more than one and those functions are called one to one functions.

Each element of B assign the element of x such that y is equal to f x, the function g (x) will have only one element because each element of A map to a unique element of B. So, let me give you an example of what I mean by a one to one function. So, suppose you have a function like this, then you take a particular value of y, then it maps to two values from the domain.

So, this is not a one to one function, whereas if you take a function like this, then this could be a one to one function, because here no matter what y you take, you map to a unique element of x. The function g will have only one element, because each element of A is mapped to a unique element of p, the domain of g is the entire B.

Because we have seen that in the previous slide that the range of f is all of B. So, the entire set B is covered in the range. So, therefore, it can serve as the domain. Therefore, g is a proper function and g is called the inverse function of f. So, we have defined what is an inverse of a function. So, for a function to have an inverse it needs to satisfy two conditions.

One is that it has to be a one to one function, which means for any value in the B that is the target set you get at most one corresponding value in the domain set, it cannot be more than one and secondly, the entire set of B which is the domain, has to be covered by the range. In that case, you can invert the function because every element of B will map to unique element of A and that is the definition of a function.

I think I will stop here. So, what we have covered so far in this particular module, we have covered functions and it is quite large module, we covered about four lectures in this module, we talked about the definition of a function, we talked about what are called the linear functions and then we talked about nonlinear functions as well like, quadratic functions and cubic functions, we talked about polynomials and then we talked about power functions, exponential functions.

In each case, we have discussed the applications in economics of these functions and finally, we talked about the general definition of functions where you do not have a single argument, there could be multiple arguments and we talked about the case that the values, the elements from the domain and the range may not be real numbers, but maybe something else. So, that was the general definition of a function. So, we shall end this class now and see you in the next lecture. Thank you.