Mathematics for Economics-I
Professor Debarshi Das
Humanities and Social Sciences Department,
Indian Institute of Technology, Guwahati
Module Name: Tutorials - 2
Lecture 33: Tutorials-2a

Hello and welcome to another lecture of this course Mathematics for Economics Part-I. Today what we shall do we shall do the tutorial and in this tutorial we are going to cover some of the topics of optimization that is we have talked about how to find a maximum and minimum of a function and the economic applications of such problems of maximization and minimization. So today, we shall discuss some of the problems related to that.

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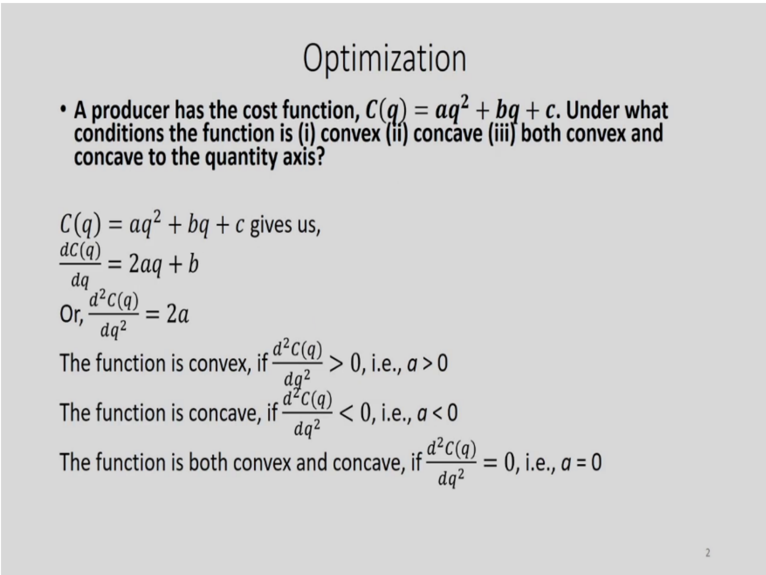

So as you can see on your screen this is **tutorial second** part. So this batch of tutorials is focused towards optimization and other topics. So we start with the topic of optimization.

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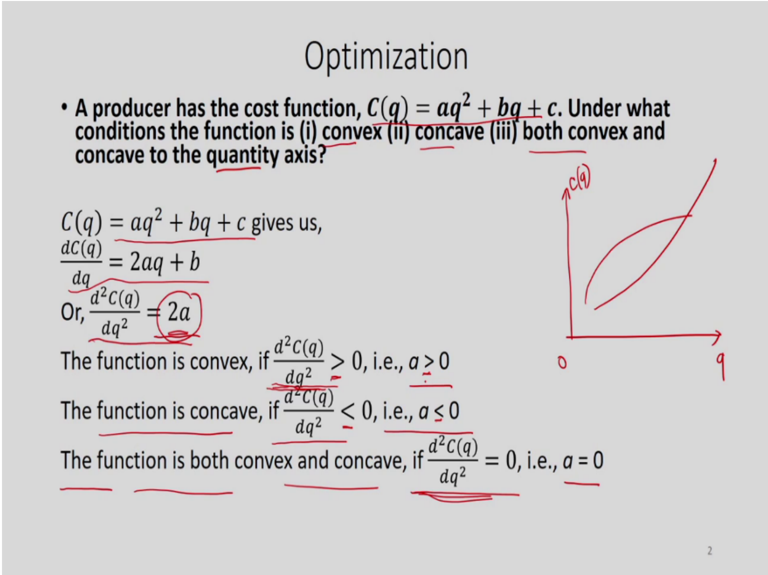

So this is the first question that you can see on your screen. A producer has a cost function. $C(q) = aq^2 + bq + c$. a,b and c are small letters. Under what conditions, the function is convex. This is the first part. The second part, the function is concave and the third part both convex and concave to the quantity axis.

Just to clarify what is meant by the quantity axis, if you recall this is how a cost function might look like. So q, that is the quantity of output is represented along the horizontal axis and $C(q)$, that is, costs associated with the different quantities are represented along the vertical axis. So this function might look like this or it could be something like this. So it might have different kinds of shapes.

We want to find out given this particular form, $C(q) = aq^2 + bq + c$. This is a polynomial function. Under what condition this polynomial function represents the convex cost function that is part 1. In Part 2, under what conditions, this polynomial function represents a concave function and in the third part both convex and concave.

So the function has to be specified as convex or concave with respect to the horizontal axis that is the quantity axis. All right so this is the function that is given to us $C(q) = aq^2 + bq + c$. Now from this I can find out the derivative of the function with respect to q. The first derivative gives us $2aq + b$. I am just using the power rule.

a and b are constants and from the first derivative I can differentiate this once more and I will get the second derivative $\frac{d^2C(q)}{dq^2} = 2a$. So this is something we have found out. Now the conclusion will come. Under what conditions the function is convex. Now here we use the property that a function is convex if the second derivative is positive.

The second derivative $\frac{d^2C(q)}{dq^2} > 0$. Now when will this second derivative be greater than 0? It will be greater than 0 if $2a$, $a > 0$ and $2a > 0$ if $a > 0$. So this is the required condition. The function is convex if a, a is positive. Second part, concave the function is concave and here like before we use the property that the function is concave if the second derivative $\frac{d^2C(q)}{dq^2} < 0$ and like before here $\frac{d^2C(q)}{dq^2} = 2a$.

So 2a is negative if a is negative that is what I have written here. So the second part is done. The third part is both convex and concave. So this is a bit tricky, but it is not actually a big deal because a function is both convex and concave if the second derivative is equal to 0. In that case it can be both convex and concave.

The idea that is being applied here is that in this case, what I have written here is strict concavity so in case of weak convexity it is greater than or equal to 0 and for weak concavity it is less than or equal to 0. So if we have weak convexity and weak concavity then the relationship of the second derivative with respect to 0 is also weak. So in that case, the function is both convex and concave in a weak sense if the second derivative is equal to 0.

So actually this should be the answer that the function is convex if $a \geq 0$, the function is concave if $a \leq 0$ and the function is both convex and concave if $a = 0$. So we are not talking about strict convexity or concavity but convexity and concavity in a weak sense and therefore we are getting these results.

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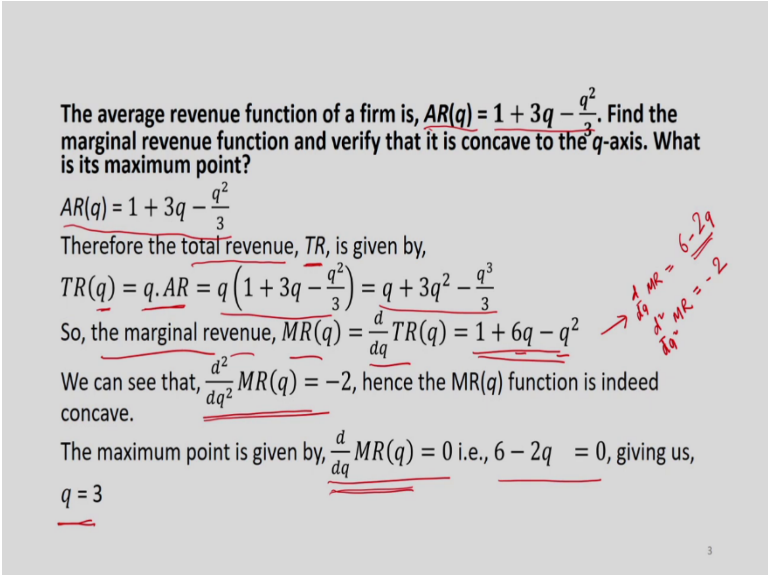

The average revenue function of a firm is given to be this $AR(q)$. It is given to be $AR(q) = 1 + 3q - \frac{q^2}{3}$. Find the marginal revenue function and verify that it is concave to the q axis. And what is its maximum point? So the revenue function is given to us. The average revenue function is given to us. We have to figure out the marginal revenue function. Average revenue function is AR.

Let us suppose $AR(q) = 1 + 3q - \frac{q^2}{3}$. From average revenue, you can go to total revenue let us call that TR. Like average revenue, total revenue will also be of a function of quantity q and that is found out by multiplying the average revenue with q quantity and it turns out to be $TR(q) = q + 3q^2 - \frac{q^3}{3}$. So this is the total revenue function and it is a function of quantity.

From this I can easily find out the marginal revenue, the marginal revenue which we are calling MR. $MR(q)$ can be found out from the total revenue by taking the first derivative. So it is calculated to be $MR(q) = 1 + 6q - q^2$.

So we have applied the power rule in this case. $MR(q) = 1 + 6q - q^2$. Now find the marginal revenue function and verify that it is concave to the q axis. We have found out the marginal revenue function. Now how to show that it is concave to the quantity axis? Well that we have seen in the previous problem.

I take the second derivative. Here what is the first derivative, the first derivative is $\frac{dMR(q)}{dq}$ and that will be if you differentiate this with respect to quantity, it becomes $\frac{dMR(q)}{dq} = 6 - 2q$. And you take the second derivative, so differentiating this with respect to quantity, $\frac{d^2MR(q)}{dq^2} = - 2$. So that is what I have written here. You can see that the second derivative is $- 2$ which is a negative quantity. Hence the marginal revenue function is indeed concave.

This is the property that the second derivative negative means the function is concave. And the last part, what is its maximum point? So if it is a concave function, it might have a maximum point. So how to find that? I take the first derivative of the marginal revenue function, set that equal to 0 and from that I will get the stationary point and first derivative I have already found out, it is $6 - 2q$.

So $6 - 2q = 0$ that is what I am getting from this condition and if I solve that for q, I will get $q = 3$. And this point is actually at the maximum point. It is not the minimum point or an inflection point because I have already seen that the MR function is concave.

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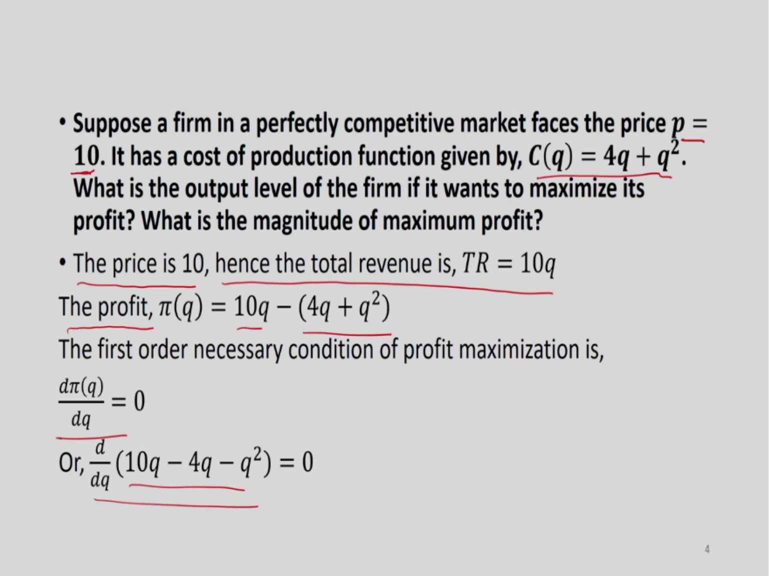

Here is a problem from profit maximization and we are talking about a firm, which is in a perfectly competitive market. This firm faces a price p given by 10. So as it is known in a perfect competition market the price is constant. So the producer cannot influence the price it is given. The cost of the production function is given by $C(q) = 4q + q^2$. Question is what is the output level of the firm if it wants to maximize its profit?

Secondly, what is the magnitude of maximum profit? So that is the question. What is given to us the price is given, price is 10. So from this I can figure out the total revenue TR. TR will be price multiplied by quantity so this is 10q and therefore the profit will be total revenue minus the cost, cost is known to us, $C(q) = 4q + q^2$. So the profit function has been found out. I have to now find out the output level where the profit is maximized.

To do that, I have to depend on the first and the second order conditions. The first order necessary condition of profit maximization is the first derivative of the profit with respect to quantity is equal to 0 so that was down to this, $\frac{d}{dq}(10q - 4q - q^2) = 0$ and this is actually $\frac{d}{dq}(6q - q^2) = 0$.

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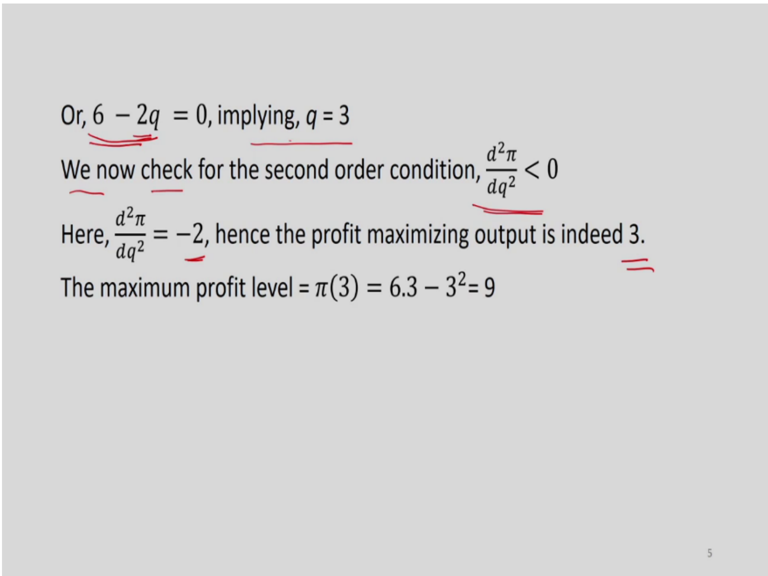

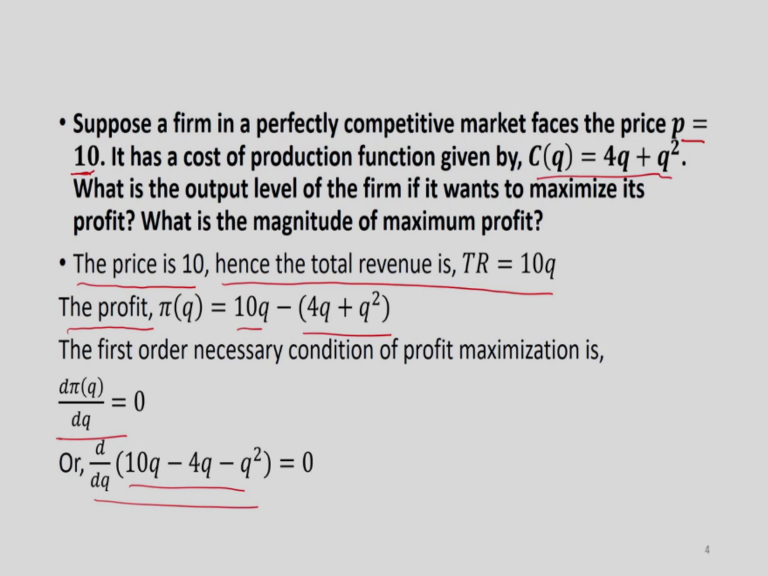

And if I take the $\frac{d}{dq}(6q - q^2)$ I get $6 - 2q$ and that gives us the value of q. $q = 3$. But we now also need to check the second order condition. So here the second derivative of the profit function that we have to check, it turns out to be - 2. This was the first derivative of the profit function. If you differentiate this with respect to q you get - 2 so that is our second order condition and it is being satisfied.

So the maximum profit is going to occur at output level 3. And what is the maximum profit level? So that was the second part. For that I have to plug in this q that is the maximum point in the profit function. What is the profit function? It is $6q - q^2$ so I just have to put $q = 3$ here. So if I do that I will get the maximum profit to be 9, so that is the answer.

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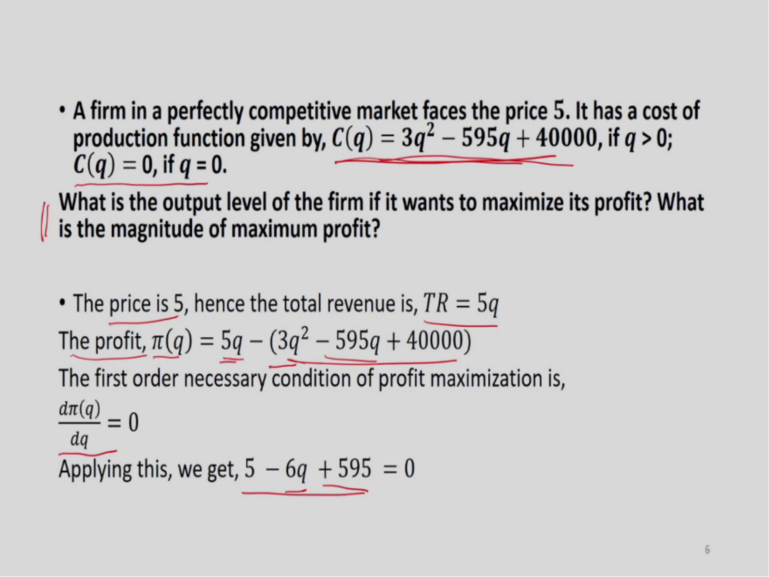

Another problem related to perfect competition. A firm in a perfectly competitive market faces the price 5. It has a cost of production given by this. $C(q) = 3q^2 - 595q + 40000$ if $q > 0$. And this $C(q) = 0$ if $q = 0$. So this is the scenario. The cost function is not a smooth function because if you put $q = 0$ in this form then you get $C(0) = 40000$.

But actually, it is specified that if $q = 0$, $C(q)$ is not 40,000, it is 0. So you can see that the cost function is having a jump at $q = 0$. It is not a continuous function. Now the question is this – what is the output level of the firm if it wants to maximize its profit? What is the magnitude of maximum profit? So in that sense, as far as the questions are concerned, the questions are like they were in the previous problem.

However, the form of the cost function is very different in this problem. So we adopt the strategy that we had adopted earlier. The price is given. Total revenue can be found out by multiplying the price with quantity so it is 5q. So the profit will be total revenue minus total cost and the first order condition of profit maximization as we know, I differentiate the profit function with respect to q set that equal to 0.

And here if I do this calculation, it comes out to be $\frac{d\pi(q)}{dq} = 5 - 6q + 595 = 0$.

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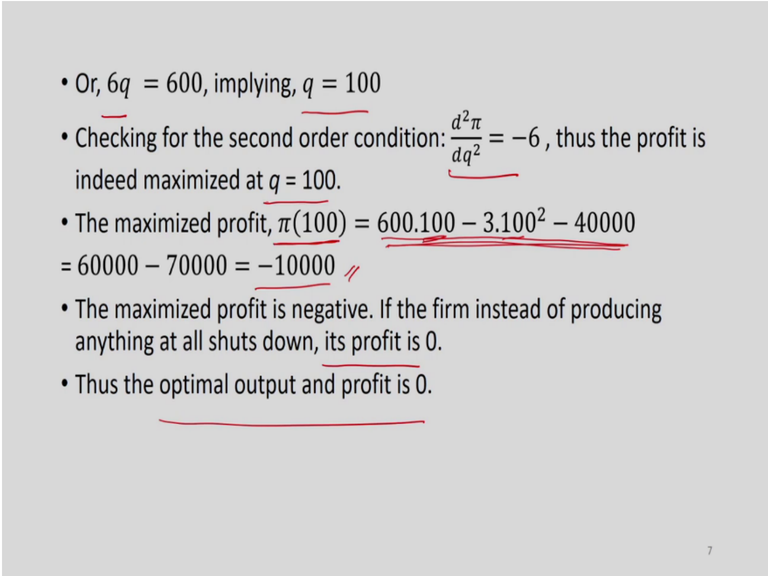

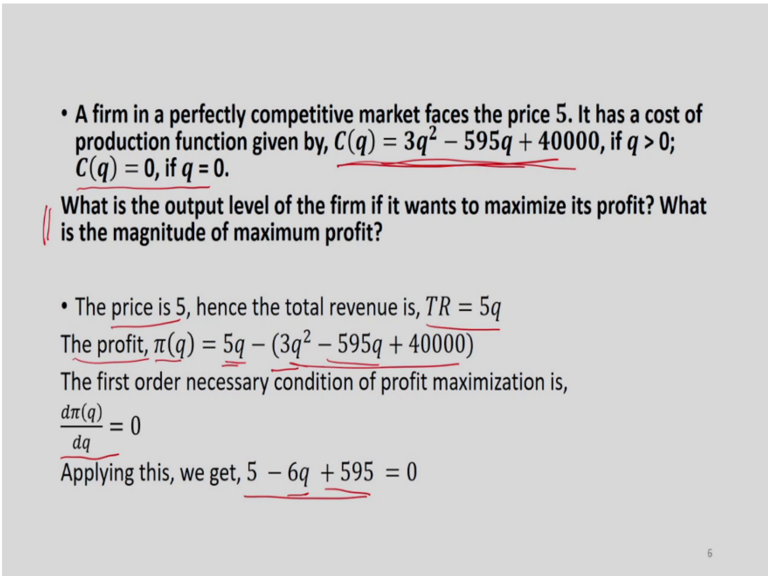

And if I solve that I get $6q = 600$ implying $q = 100$. We also need to check the second order condition. So I have to differentiate this part once again with respect to q and that will give me -6. So the second derivative is -6 thus the profit is indeed maximized at $q = 100$ because the profit function is concave. What about the maximized profit?

So that is an important question and as we shall see this is going to be a very critical question when the producer has maximized his profit, then what is the volume of that maximized profit. Here the maximum profit is occurring at $q = 100$. So I put the $q = 100$ in the profit function, so this is the profit function, $\pi(100) = 600 * 100 - 3.100^2 - 40000$.

So I just solved this, I just simplified this and it comes out to be -10,000. So what it basically means is that if the producer has to produce some goods. If the producer produces $q > 0$ then the highest amount of profit that he can get is -10,000. It is negative. If the firm instead of producing anything at all shuts down then what happens.

If the firm shuts down, it does not produce anything. So, TR= 0. Remember $q = 0$, cost is not minus 40,000. It is 0 actually. That is specified in the question so therefore $\pi = 0 - 0 = 0$. Therefore the optimal output will be 0 and as a result optimal profit will also be 0. So that is why I said that we need to figure out what is the maximized profit.

If it is turning out to be negative, as it is happening here then there is no point for the producer to produce anything at all. It will produce 0 amount of output.

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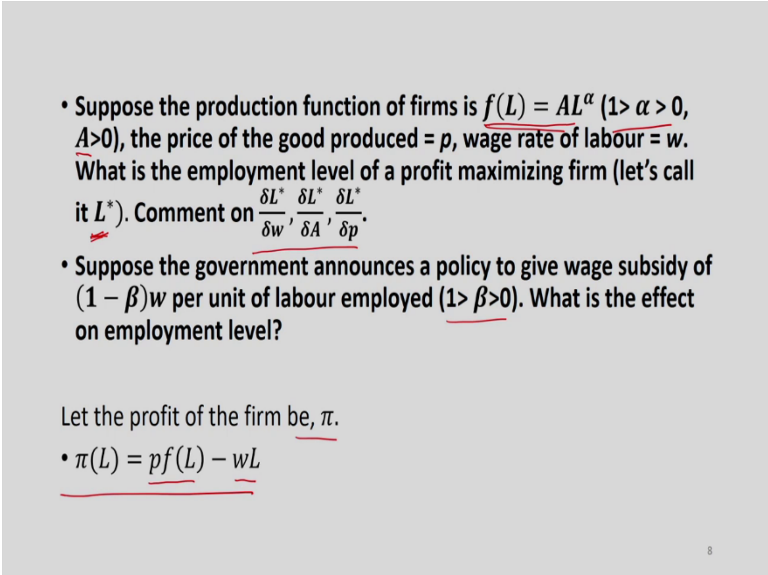

Here is a somewhat different question. Suppose the production function of firms is given by this $f(L)$ is the production function, $f(L) = AL^{\alpha}$ $where,$ $0 < \alpha < 1$, $A > 0$. The price of the good produced is given by small p. The wage rate of labor is given by small w, what is the employment level of a profit-maximizing firm? And let us call that the employment level is L^*.

Comment on $\frac{\delta L^*}{\delta w}, \frac{\delta L^*}{\delta A}, \frac{\delta L^*}{\delta p}$ so these are the 3 partial derivatives that we have to find out. Secondly suppose the government announces a policy to give wage subsidy of $(1 - \beta)w$, per unit of labor employed where $0 < \beta < 1$. What is the effect on the employment level? What is the effect of this wage subsidy? This is called wage subsidy on the employment level.

Earlier the employment level was L^* without subsidy. So how is that going to change with subsidy? So that is the question. Now we are going to follow the basic rule. We first find out the profit function and try to see at what level of employment L. So the L is the decision variable at what level of employment the producer is going to maximize the profit.

So here is the profit function $\pi(L) = pq$, q is nothing but the production function minus the cost, $\pi(L) = pf(L) - wL$. What is the cost here? There is only 1 input that we can see so cost is -wL. The wage rate multiplied by labor employment.

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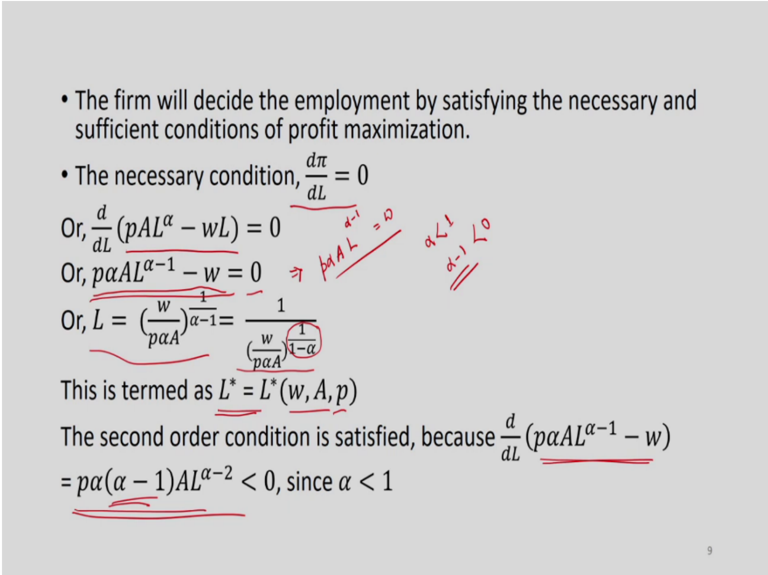

The firm will decide the employment by satisfying the necessary and sufficient conditions of profit maximization. This is standard. The necessary condition is this. $\frac{\delta\pi}{\delta L} = 0$, so $\frac{d}{dL}\pi = \frac{d}{dL}(p.f(L) - wL) = \frac{d}{dL}(p.AL^{\alpha} - wL)$. I take the derivative of this so I get $p\alpha AL^{\alpha-1} - w = 0$. This can be actually re-written as $p\alpha AL^{\alpha-1} = w$. Just adding w to both sides and I will get this.

And this gives me this, I am expressing L in terms of the rest of the parameters and so $L = (\frac{w}{p\alpha A})^{\frac{1}{\alpha-1}}$. Now we know $\alpha < 1$, so $(\alpha - 1) < 0$ so therefore I have taken the reciprocal of this so that the power here is positive, so this is the positive number. So what I am getting is $L = \frac{1}{(\frac{w}{p\alpha A})^{\frac{1}{1-\alpha}}}$. And let us call this L^*.

This is the optimal employment level of the producer as you can see it is a function of many parameters principally w, A and p. So these are the parameters of interest. But is the second order condition satisfied that we need to check. And it is actually satisfied because if I take the derivative of this part, this was the first derivative I take the derivative of the first derivative, so I will get the second derivative, second derivative is turning out to be this $p\alpha(\alpha - 1)AL^{\alpha-2} < 0$.

And this is less than 0 because $(\alpha - 1) < 0$ and $(\alpha - 1)$ is appearing here. So this is negative and that is the second order condition that the profit function should be concave and the profit function here is concave if you have the second derivative to be negative. So we have found out L^* the optimal labor employment.

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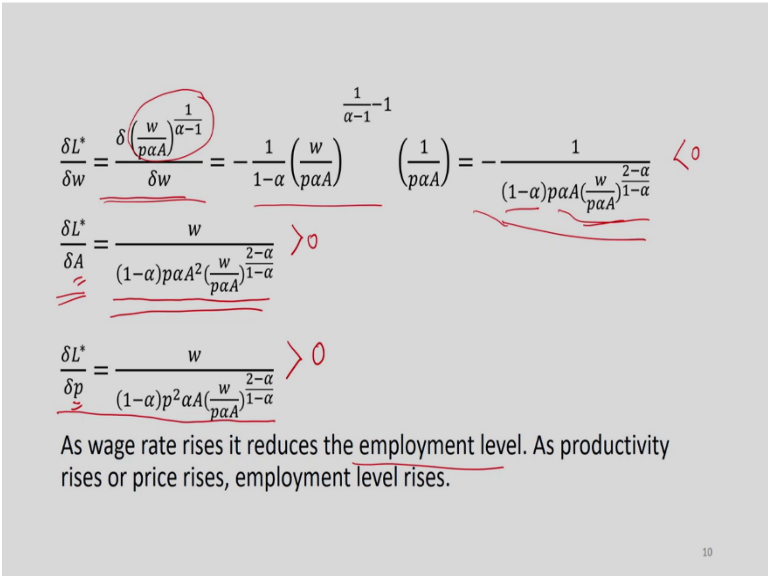

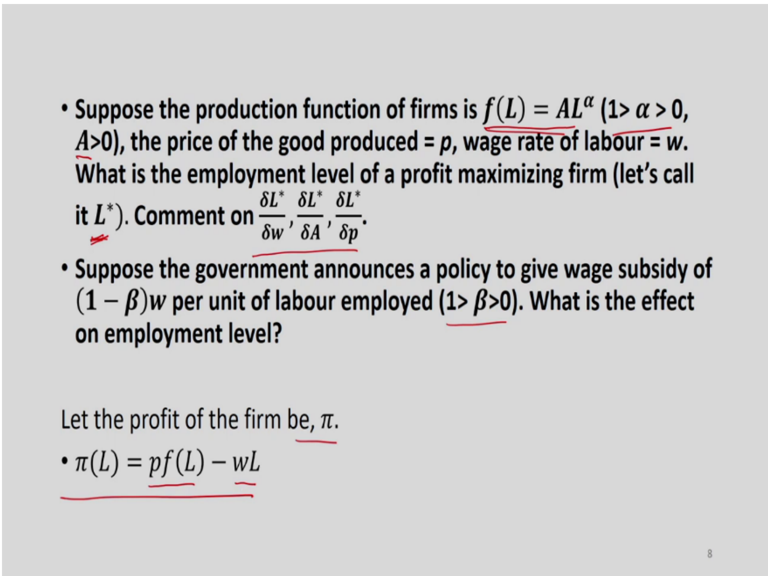

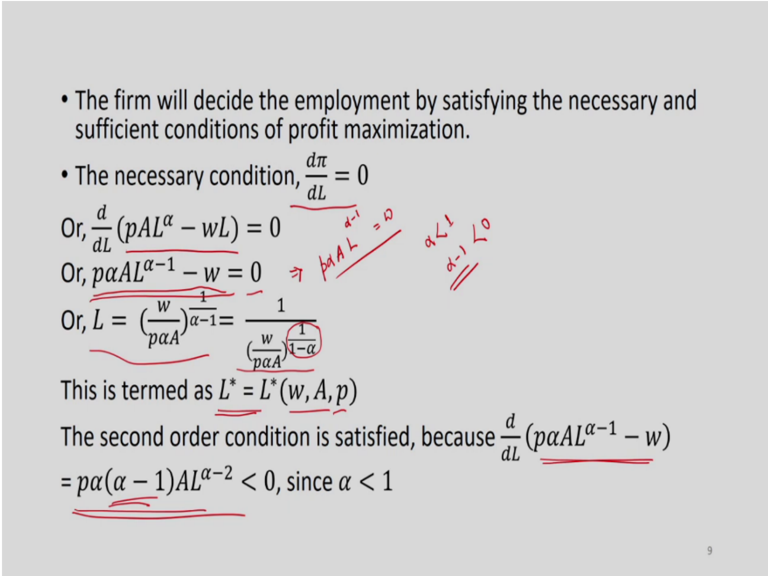

Now I have to talk about the partial derivative of the L^*. With respect to these parameters, w A and p, and this is simply manipulation. I have to take the derivative with respect to w of this function, this function and that after some steps, it turns out to be this $-1/(1-\alpha)p\alpha A(\frac{w}{p\alpha A})^{\frac{2-\alpha}{1-\alpha}}$.

What about the sign of this entire thing, it is negative because $(1-\alpha) > 0$ and the rest of the terms are also positive. So $\frac{\delta L^*}{\delta w} < 0$ That is my conclusion. What about the partial derivative of L^* with respect to A and similarly I am not specifying the steps involved here. Here it is turning out to be $\frac{\delta L^*}{\delta A} = w/(1-\alpha)p\alpha A^2(\frac{w}{p\alpha A})^{\frac{2-\alpha}{1-\alpha}}$ and as you can see by inspection that this is a positive quantity because $\alpha < 1$.

So $\frac{\delta L^*}{\delta A} > 0$. Similarly, $\frac{\delta L^*}{\delta p}$ is found out to be $\frac{\delta L^*}{\delta p} = w/(1-\alpha)p^2\alpha A(\frac{w}{p\alpha A})^{\frac{2-\alpha}{1-\alpha}} > 0$ and this is also positive. So this is negative, this is positive, this is positive. What it means is that as the wage rate rises, it reduces the employment level. This employment level, L^* is going down.

As productivity level rises or price rises, employment level rises. A can be thought of as a parameter representing productivity because look at this here A is occurring. And if A is high, that means the same amount of labor will be capable of producing more output. So that is why one can think of A as a marker of productivity.

So as productivity rises, or as price rises, p is the price as we can see L^* goes on rising because the partial derivatives are positive.

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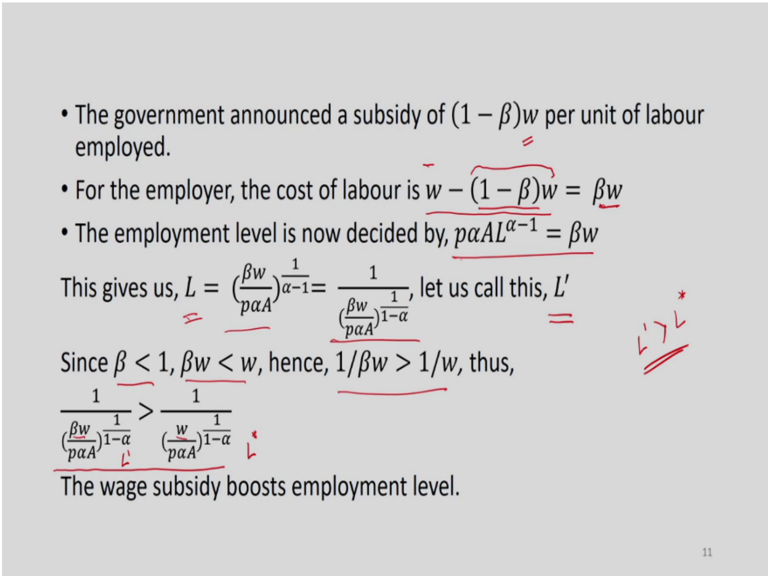

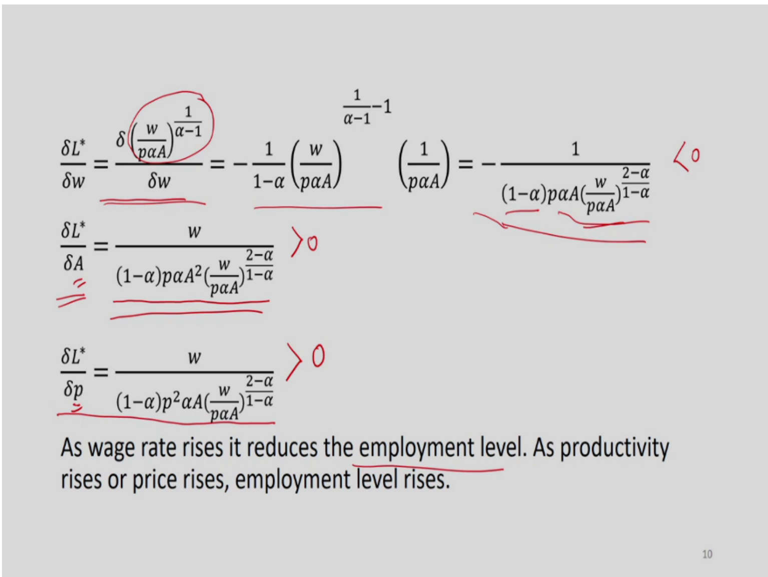

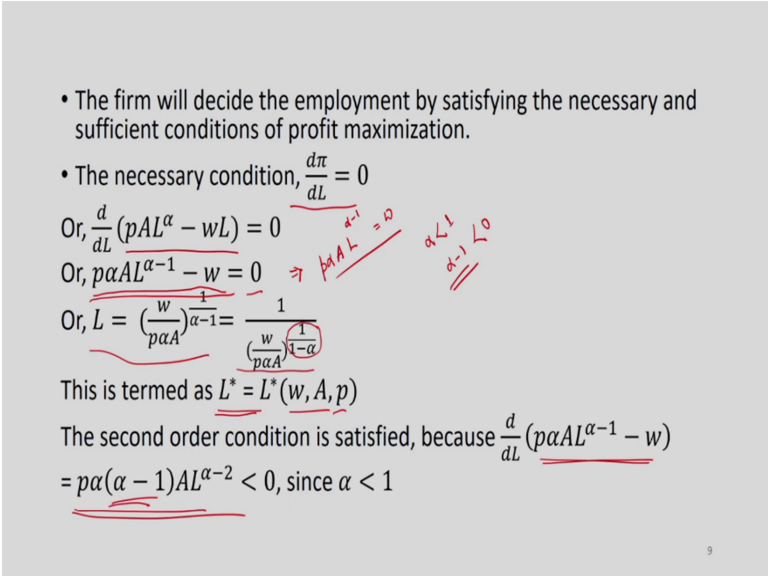

Now we come to the second part the government announced a subsidy of $(1 - \beta)w$ per unit of labor employed. For the employer the cost of labor after the subsidy is announced is given by this, $w - (1 - \beta)w$ because this part is given by the government.

So the producer will only have to bear the wage that is paid to the worker minus the power that is provided by the government and this is simply βw. Now this is the wage that is actually paid by the producer and as we have seen earlier, the optimal employment level is decided by this rule, the first order condition.

This was the condition. The marginal product on the left hand side you have, price multiplied by the marginal product and on the right hand side, the wage that is paid by the producer. Now what has changed now is that the left hand side has remained the same. The right hand side is no longer w. Right now the producer is paying only β part of the w. So the producer is paying βw.

So the new condition of profit maximization is this, $p\alpha AL^{\alpha-1} = \beta w$. From this we can solve for L optimal labor employment and that like before is turning out to be $1/(\frac{\beta w}{p\alpha A})^{\frac{1}{1-\alpha}}$. And let us call this $L^{'}$. Now I am going to claim that $L^{'} > L^{*}$. On what basis am I saying this?

Remember $\beta < 1$, that means $\beta w < w$. Hence if I take the reciprocal $\frac{1}{\beta w} > \frac{1}{w}$. And that helps us conclude that this is going to be correct. What you see here is thatβw is occurring in the denominator and here w is occurring in the denominator. And what is the left hand side, left hand side is $L^{'}$, the right hand side is L^{*}.

So we have just proved that $L^{'} > L^{*}$ which means that if the government provides wage subsidy to the producer then the producer actually will raise the employment level. So this is the kind of policy that is often advocated by many people if the government wants to improve.

The employment level in the economy then what the government can do is that it can tell the producers, look out of the 100 rupees wage that has to be paid to the laborer, you pay maybe 70 percent of that, 70 rupees and I will pay 30 percent that I will pay 30 rupees. In that case the cost per labor to the producer in that case falls, it falls to 70 instead of 100 and the effect is going to be that the producers will raise the employment level.

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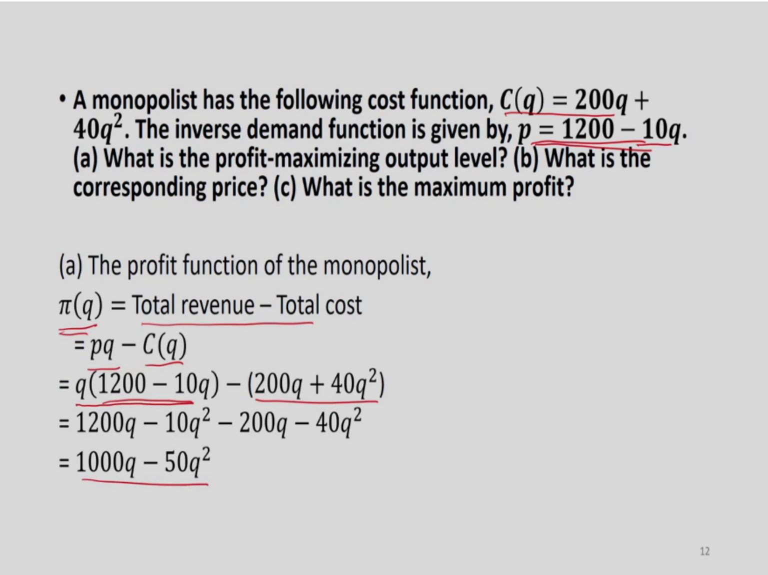

Now we come to a different topic, which is suppose you have a monopoly market, then what happens to profit maximization? A monopolist has the following cost function. $C(q) = 200q + 40q^2$. The inverse demand function is given. $p = 1200 - 10q$. First question – what is the profit maximizing output level? Second question – what is the corresponding price and third – what is the maximum profit?

This is the case where you do not have perfect competition but there is a monopolist and the monopolist wants to maximize the profit then how does he do so. Now like before the $\pi(q)$ = Total revenue - Total cost. What is total revenue? $p.q$, cost is $C(q)$. So pq that is total revenue can be found out by multiplying the quantity with price. Price is obtained from the inverse demand function that is $p = 1200 - 10q$.

We substitute that here minus the cost and this is simplified as $\pi(q) = 1000q - 50q^2$. This is the profit function.

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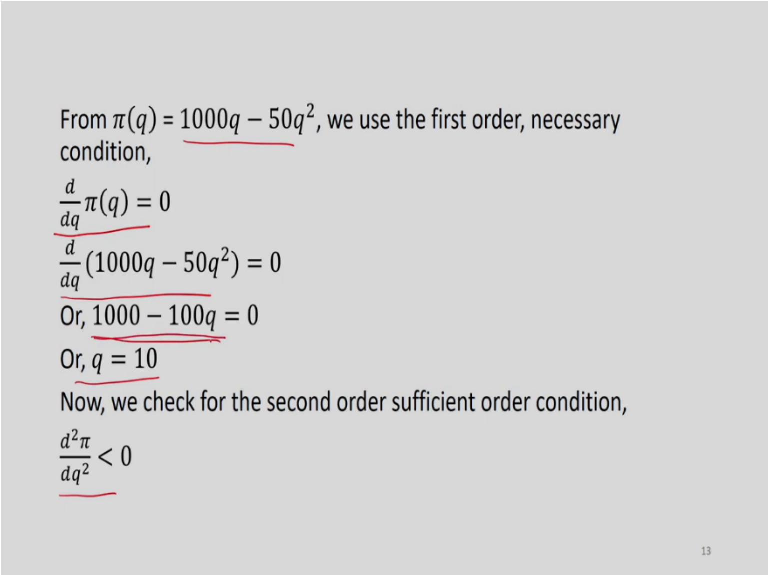

And as usual we use the first order necessary condition. We take the first derivative of the profit function set that equal to 0 that is $\frac{d}{dq}\pi(q) = \frac{d}{dq}(1000q - 50q^2) = 0$. That gives me $1000 - 100q = 0$ that is $q = 10$. What about the second order condition? We have to have the second derivative of the profit function to be negative. Here actually it is negative because this is the first derivative and if you take the second derivative you just get $\frac{d^2}{dq^2}\pi(q) = -100$which is negative.

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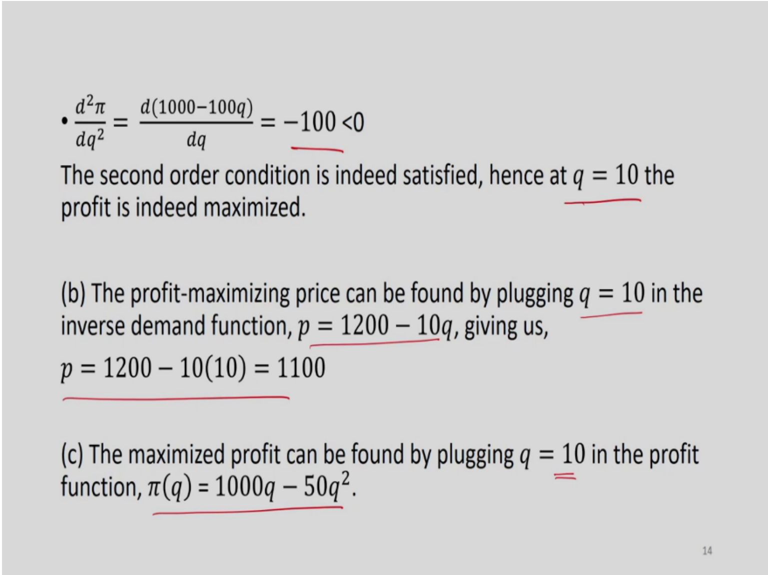

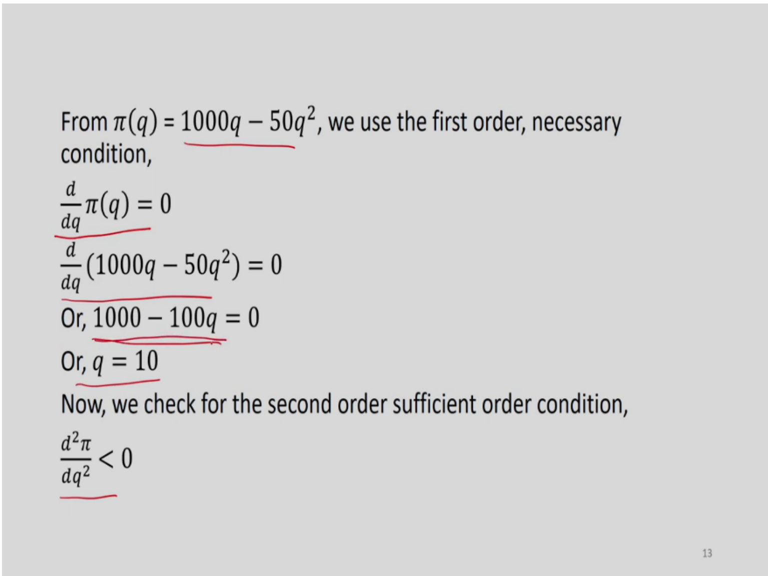

So the second order condition is indeed satisfied hence $q = 10$ so that is what we got from the first order condition $q = 10$ is the level of output where profit is maximized. Now we check for the price. The price at which the producer is going to sell his goods. The profit maximizing price can be found by plugging $q = 10$ in the inverse demand function.

This is the inverse demand function $p = 1200 - 10q$ that is the inverse demand function. I put $q = 10$ here. I get $p = 1100$. So this is the optimal price. The maximized profit can be found by plugging $q = 10$ in the profit function. Profit function was $\pi(q) = 1000q - 50q^2$. I will put $q = 10$ here.

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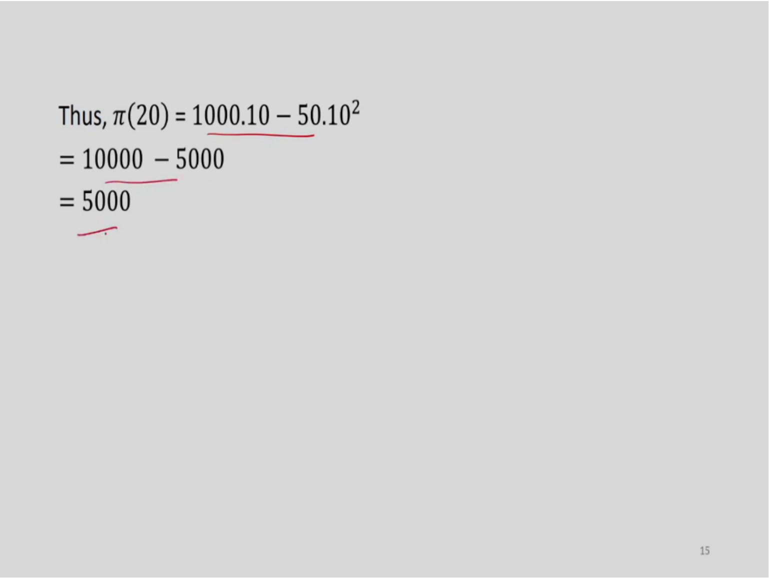

And if I do so it turns out to be 5000. So the profit is 5000 when the producer is operating optimally.

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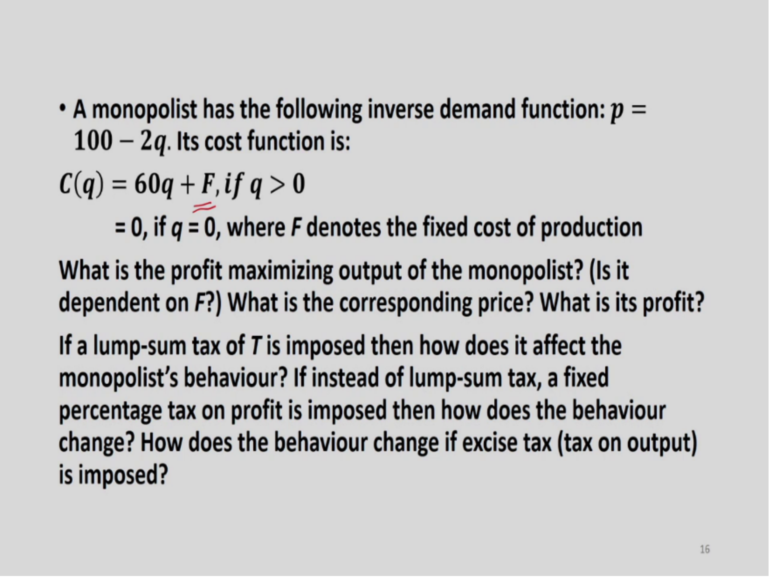

Another problem related to a monopoly market but this has something to do with the government as well. A monopolist has the following inverse demand function. $p = 100 - 2q$. The cost function is $C(q) = 60q + F$. if $q > 0$ and is equal to 0 if $q = 0$ where F denotes the fixed cost of production. What is the profit maximizing output of the monopolist? Is it dependent on F?

What is the corresponding price and what is its profit? So this is the first part. You can see that there is a fixed cost involved here. So something similar to this we have seen before where fixed cost was involved and now we come to the second part, if a lump-sum tax of T is imposed, then how does it affect the monopolist's behavior.

Monopolist's behavior means what is the quantity that it is going to produce, price it is going to charge etc. if instead of a lump-sum tax a fixed percentage tax on profit is imposed then how does the behavior change. How does the behavior change if excise tax that is tax on output is imposed. So it is quite a long question. We have to do it carefully and slowly to get the answer. But it is not very difficult.

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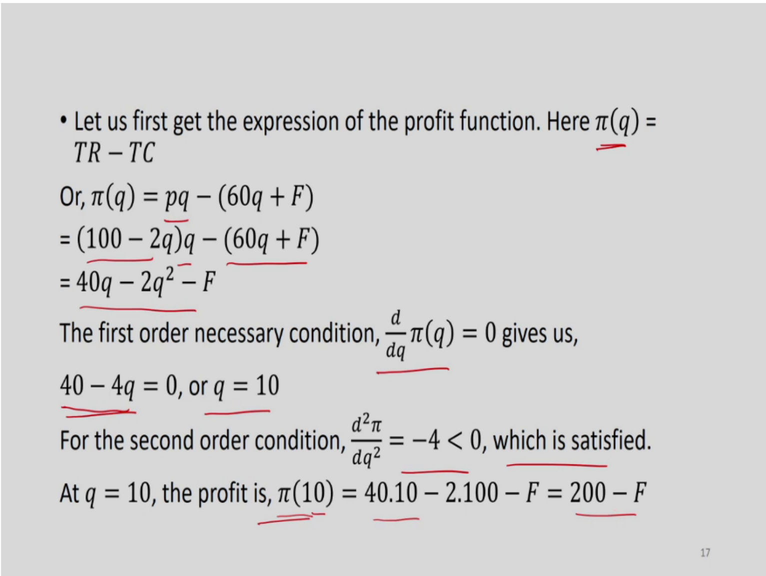

Let us first get the expression of profit function. So this is the standard method. The profit function is $\pi(q)$ is given by total revenue − total cost. We know the inverse demand function, we put it here that multiplied by q − the cost function, which is simply $C(q) = 60q + F$. So this is the profit function $\pi(q) = 40q - 2q^2 - F$.

We know the first order necessary condition is that the derivative of this profit function should be equal to 0 and that will give me $\frac{d}{dq}\pi(q) = 40 - 4q = 0$ or $q = 10$. Second order condition I take the derivative of this once again I will get -4 .which is negative. So indeed the profit is getting maximized here at $q = 10$.

Now what is the amount of profit? The profit is given by $\pi(q)$, which is a function of 10 now. So I put $q = 10$ here that actually simplifies to be $\pi(10) = 200 - F$. This is the maximized profit.

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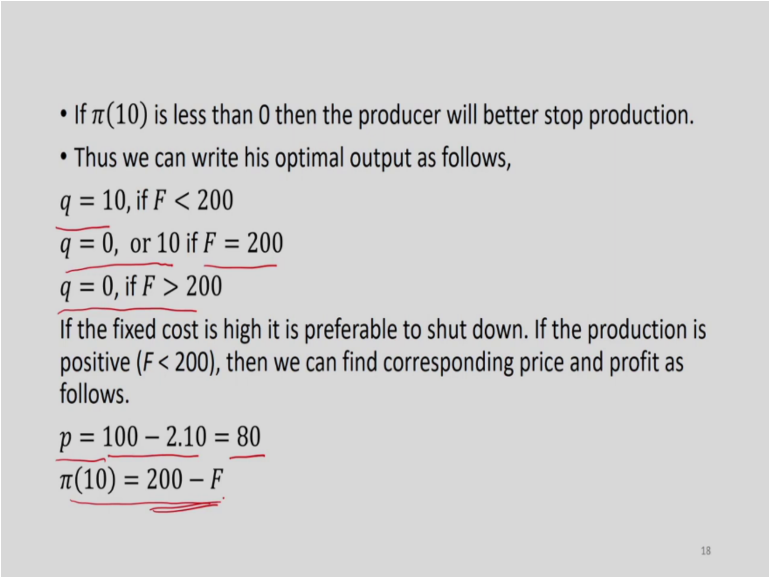

Now as we have seen earlier, this maximized profit if it is less than 0 then the producer will better stop production because if he does not produce then the cost is 0 and the revenue is also 0 so the profit will be 0 if he does not produce. So the maximized profit if it is less than 0 then it is better to stop production. Thus we can write the optimal output as follows.

It is $q = 10$ if $F < 200$ because what is the maximized profit, it is this $\pi(10) = 200 - F$. So as long as $F < 200$ then the maximized profit is positive. So in that case, it is worthwhile to produce. On the other hand, if $F = 200$ then the producer will get 0 profit whether he produces 10 or he produces 0.

So both are optimal in this case. And finally, if the fixed cost is $F > 200$ then the maximized profit is actually negative. In that case, it is best not to produce so $q = 0$. So the summary is this. If the fixed cost is high, it is preferable to shut down. If the production is positive that is $F < 200$ then we can find the corresponding price and profit as follows.

So the price will be just plugging in $q = 10$ in the inverse demand function so this comes out to be $p = 100 - 2.10 = 80$. So this is the optimal price and as we have seen earlier the maximized profit is $\pi(10) = 200 - F$.

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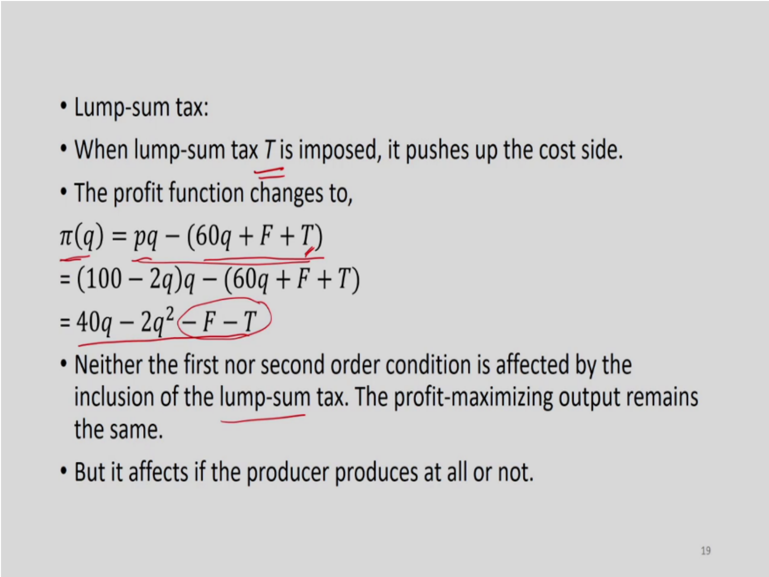

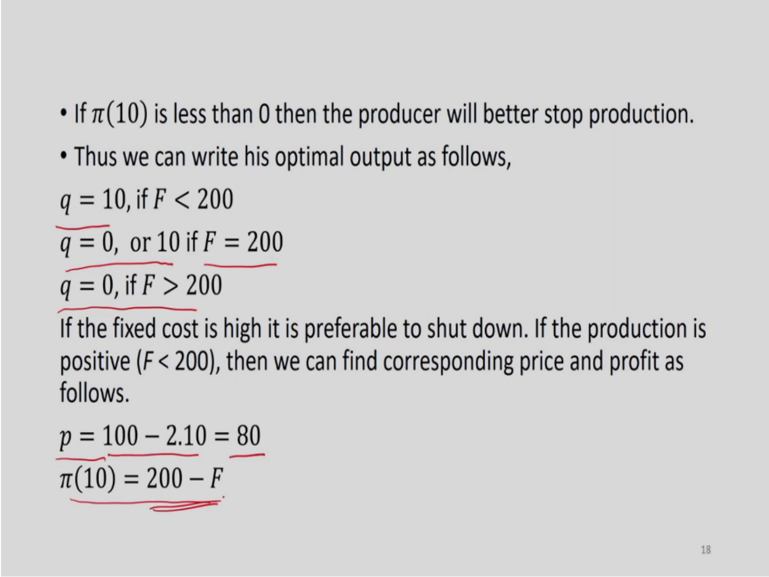

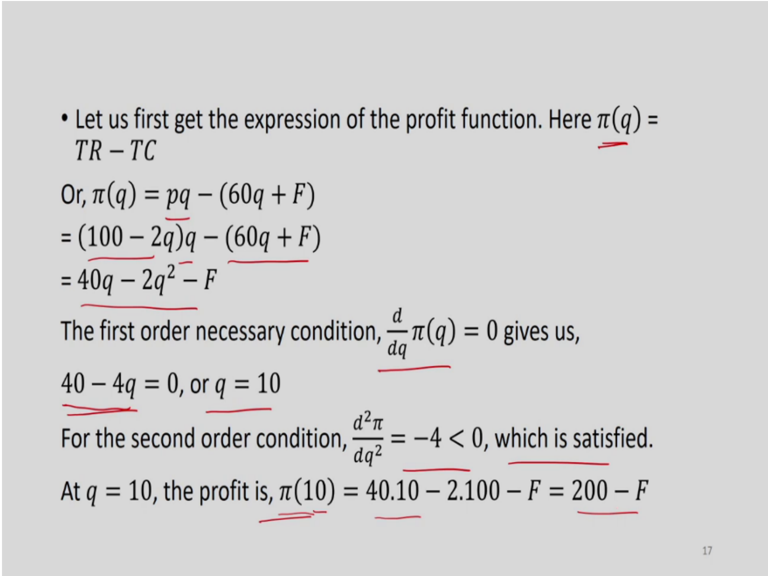

Now we come to the tax part. So the government has imposed a lump-sum tax. What is a lump sum tax? It basically pushed up the cost side. So the producer has to pay a fixed amount of money to the government that is called a lump-sum tax. And that amount of money is given by T. So in this case the profit function has changed. It has changed to this.

The total revenue is pq − cost and you can see in the cost there is an additional term. It is +T. So earlier it was just this much $C(q) = 60q + F$. Now it is $C(q) = 60q + F + T$. And I can simplify this to this expression $\pi(q) = 40q - 2q^2 - F - T$. Now what is to be noted is neither the first or the second order condition is affected by the inclusion of the lump sum tax. The profit maximizing output remains the same.

Why am I saying this is that if you differentiate the profit function with respect to quantity that will give you the first order condition and if you differentiate it twice then it will get the second order condition but if you differentiate it once then these terms will simply drop out because these are constants. So therefore the first order and the second order conditions are not going to be affected by the inclusion of this tax.

However, it affects if the producer produces at all or not. So since the first order and the second order conditions are not going to be affected, therefore the optimal quantity and the optimal price are going to remain the same as long as $q > 0$. But whether the producer produces at all or not that is going to be affected by the inclusion of this lump-sum tax.

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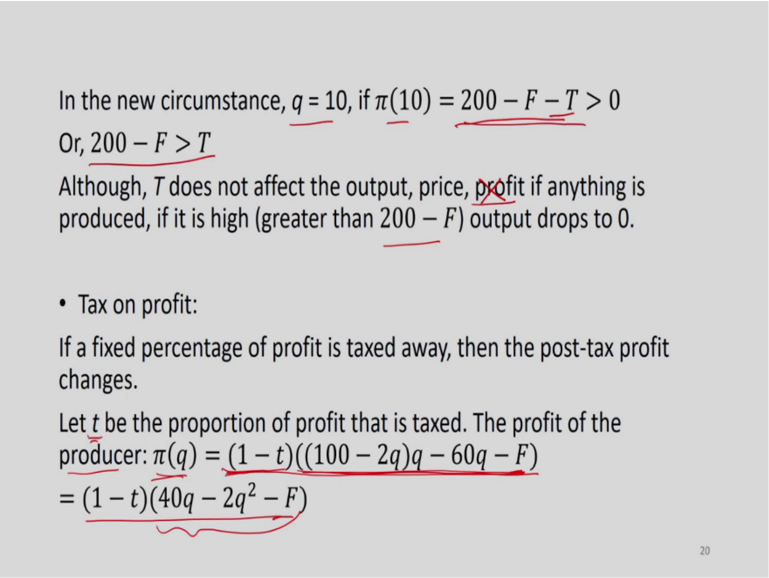

And here is the reason. In the new circumstance if $q = 10$, that is the optimal quantity that has remained the same. The profit is given by this, $\pi(q) = 200 - F - T > 0$. So this has to be greater than 0 that we have seen. The maximized profit or the optimal profit cannot be negative and that boils down to $200 - F > T$.

Earlier it was $200 - F > 0$, although T does not affect the output price, profit if anything is produced if it is high (greater than $200 - F$) output drops to 0. Actually this is not correct. Profit is going to be affected. The maximized profit is going to be affected if the producer is producing anything positive because as you can see the maximized profit has this minus T component.

The optimal output and price remains the same in case of lump-sum tax provided the producer is producing something positive but if the producer is producing anything positive or not, that is going to be affected by this lump-sum tax. The second question is tax on profit. If a fixed percentage of profit is taxed away, then the post-tax profit changes.

So suppose t be the proportion of profit that is tax. The profit of the producer therefore is equal to let us suppose it is $\pi(q) = (1 - t)((100 - 2q)q - 60q - F)$. Here there is no lump sum tax. Fixed cost is there.

However, a portion of the profit that is t percentage of the profit has been taxed so therefore the producer is left with $(1 - t)$multiplied by the profit. And this is simply this, $\pi(q) = (1 - t)(40q - 2q^2 - F)$. You might recall that this expression was the profit expression earlier. Earlier means when there was no tax.

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- The first order condition will give the same solution of q as before, as long as tax rate t is a constant. The second order condition will be like before.
- The optimal output and price remain the same, there is no change of behaviour of the producer.
- The condition to produce:

$(1-t)(200-F)>0$ which boils down to,

$(200-F)>0$, like in the previous case.

- The maximized profit will be less than without the tax.

In the new circumstance, $q = 10$, if $\pi(10) = 200 - F - T > 0$

Or, $200 - F > T$

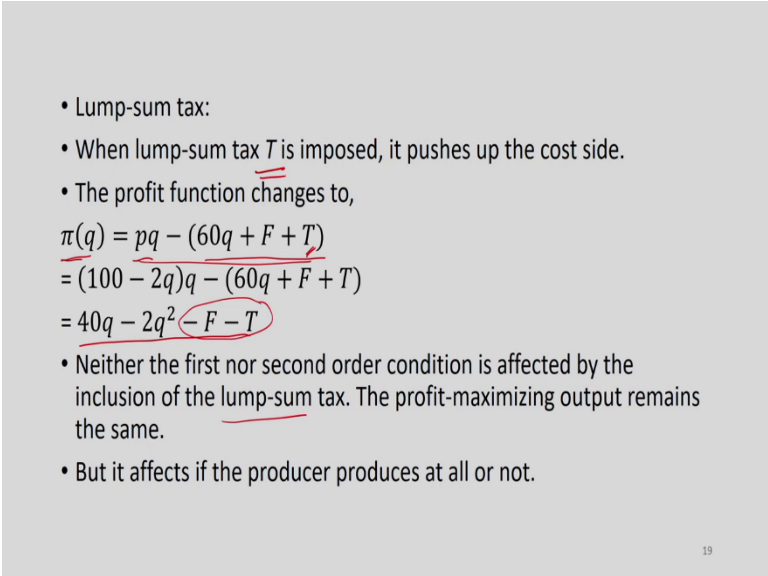
Although, T does not affect the output, price, profit if anything is produced, if it is high (greater than $200 - F$) output drops to 0.

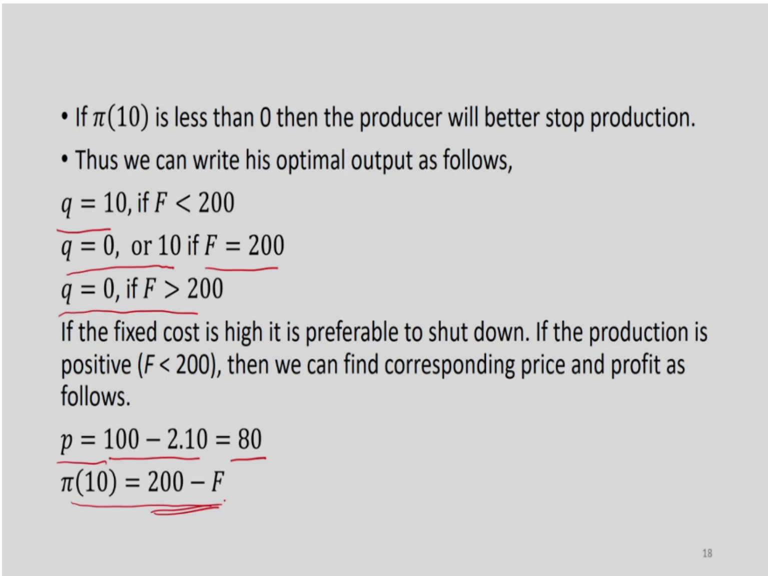
- Tax on profit:

If a fixed percentage of profit is taxed away, then the post-tax profit changes.

Let t be the proportion of profit that is taxed. The profit of the producer: $\pi(q) = (1-t)((100-2q)q - 60q - F)$

$= (1-t)(40q - 2q^2 - F)$





The first order condition will give the same solution of q as before as long as tax rate T is a constant. The second order condition will be like before. So what is being said is that if you set the derivative of this with respect to $q = 0$ then that will give you just the derivative of this with respect to $q = 0$. Because this is just a constant. T is a constant, so this is a constant.

So you are going to get the same first order condition that you got when there was no tax. And similar to the second order condition will be like before. It will be satisfied. Therefore the optimal output and price remain the same. There is no change in the behavior of the producer. The optimal price and output we got earlier where what $q = 10$and $p = 80$. So 10 and 80 are optimal quantity and price.

Those things will hold here also but there was remember a condition whether the producer produce will be producing at all or not that depends on what about the profit. It has to be positive. So this is the maximized profit $(1 - t)(200 - F) > 0$and that is actually the same condition because $(1 - t)$ is just a constant and this is positive, so what matters is this part.

This has to be greater than 0. So in case of tax on profit also the behavior of the producer does not change due to the tax. He will demand the same price. He will be producing the same output as before the tax. One thing will change that the maximized profit will be less because a part of the profit has been taxed away.

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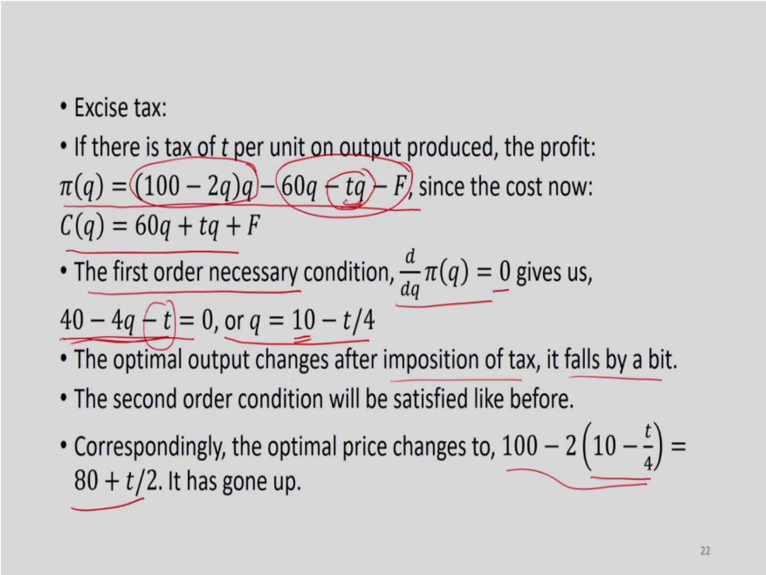

Now we come to the third category of taxes – the excise tax. If there is tax of t per unit on output produced then the profit turns out to be this. $\pi(q) = (100 - 2q)q - 60q - tq - F$. So this is the revenue part and this is the cost part as you can see that this is a new term here $-tq$, t is the rate of tax and in case of excise tax, the quantity is being taxed. How much quantity the producer is producing based on that the producer has to pay taxes to the government.

So the total amount of tax it has to pay, which is tq it is a product of the tax rate multiplied by the quantity, it has decided to produce. So this is the cost function, new cost function. Now again I apply the first order condition that is the derivative of the profit function with respect to quantity set that equal to 0 and that gives me this $\frac{d\pi}{dq} = 40 - 4q - t = 0$. And you get the solution of $q = 10 - t/4$.

Just for comparison earlier, when there was no tax, no excise tax, the optimal quantity was simply 10. Now that optimal quantity is $q = 10 - t/4$. So the optimal output changes after the imposition of tax. It falls by a bit. The second order condition will be satisfied like before because if you take the derivative of this with respect to q then this part simply drops out so you get -4 so the second order condition is satisfied.

Correspondingly, the optimal price also changes. So I put the optimal quantity in the inverse demand function so $p = 100 - 2(10 - t/4) = 80 + t/2$. So the price actually has gone up due to the tax. Earlier it was 80, now it is $80 + t/2$.

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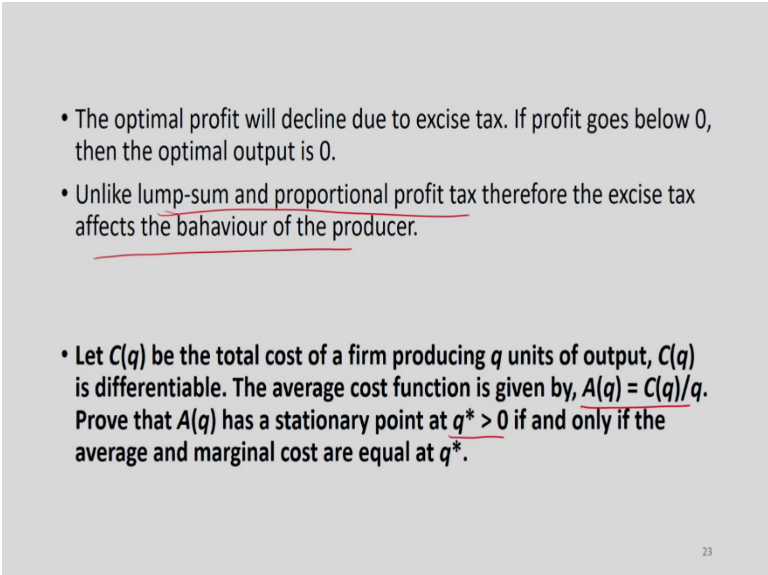

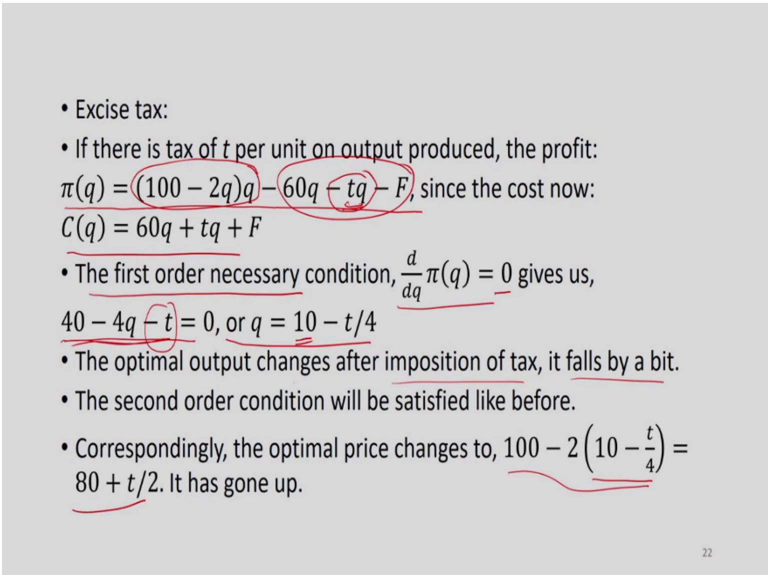

The optimal profit will decline due to the excise tax and as we know if the profit goes below 0 then the optimal output is 0. Optimal profit will decline because of the tax because the quantity is being produced less the price is charged more so obviously the profit will decline. What is the conclusion? Unlike lump sum and proportional profit tax, the excise tax affects the behavior of the producer.

In lump sum tax or proportional profit tax, there was no effect on the optimal quantity, the producer was producing or the optimal price except for the fact that whether the producing is going to be producing at all or not that was getting affected by the tax. Other than that there was no change on the optimal quantity and price but here in case of excise tax the optimal quantity and price is getting affected. Here is another problem from optimization.

Let $C(q)$ be the total cost of a firm producing q units of output $C(q)$ is differentiable. The average cost function is given by $A(q) = C(q)/q$ prove that $A(q)$ has a stationary point at $q^* > 0$ if and only if the average and marginal costs are equal at q^*.

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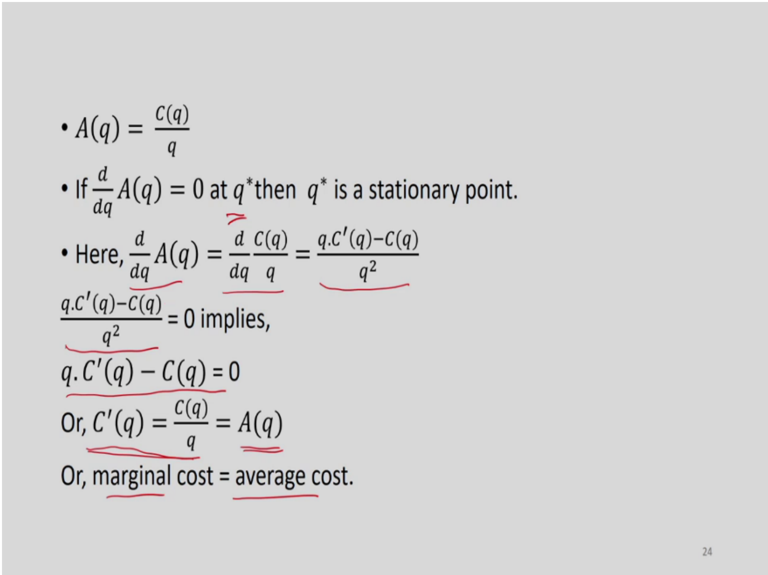

So as given in the question, $A(q) = C(q)/q$. $C(q)$ is the cost function. Now if the derivative of the $A(q)$ the average cost is equal to 0 at q^* then I can call q^* as a stationary point of the $A(q)$ function. So I take the derivative of the $A(q)$ function, the average cost function and I apply the quotient rule so I get $\frac{d}{dq}A(q) = \frac{q.C'(q)-C(q)}{q^2}$.

Now if we have a stationary point then this has to be equal to 0. The derivative has to be equal to 0 for a stationary point and that means that the numerator is equal to 0, which is equivalent to saying that $q.C'(q) - C(q) = 0$ and I can take $C(q)$ to the right hand side and divide both sides by q that will give me $C'(q) = \frac{C(q)}{q} = A(q)$ it is the average cost.

So marginal cost is equal to average cost. This is what we are getting from this condition that the first derivative is equal to 0 of the average cost function.

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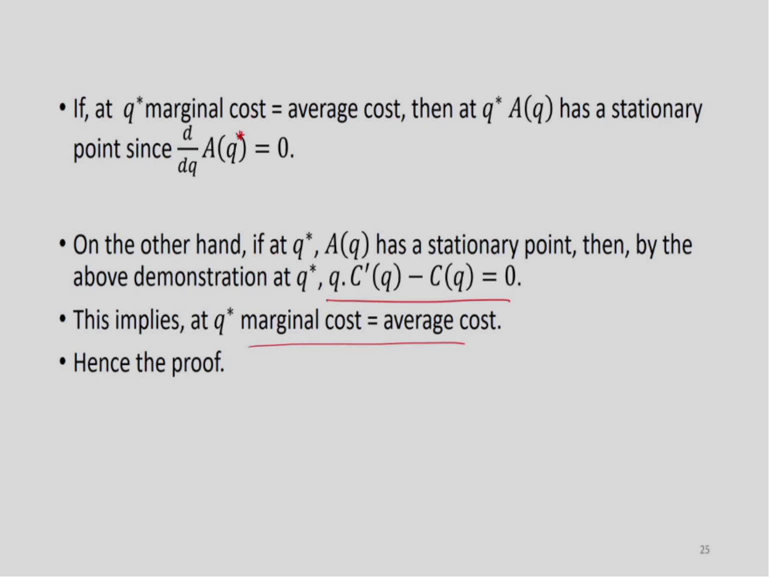

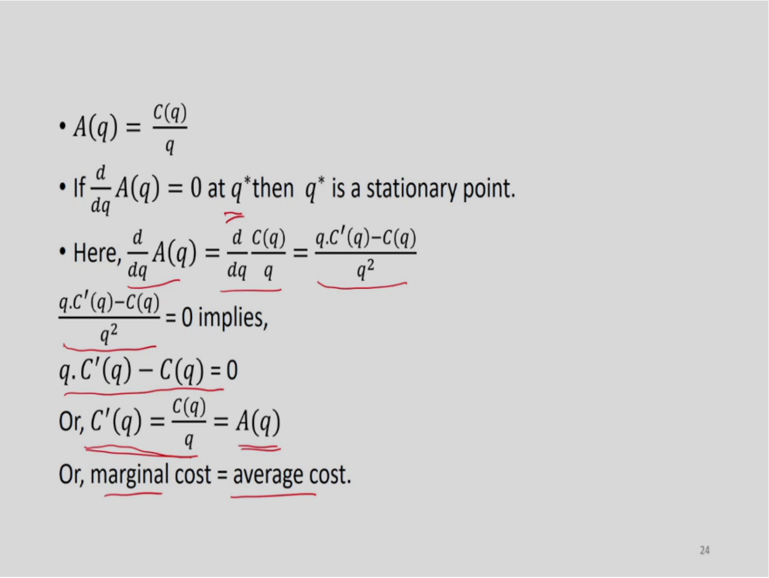

If at q^* marginal cost = average cost then at q^*, $A(q)$has a stationary point since $\frac{d}{dq}A(q) = 0$. So that is what we have just seen. If the marginal cost and average cost are equal at a particular quantity level q^* then that actually means that the first derivative of the average cost is equal to 0 which means that q^* is a stationary point.

On the other hand , if at q^* $A(q)$ has a stationary point then by the above demonstration the first derivative of the average cost is equal to 0 which means this is 0 and that actually means marginal cost = average cost, hence the proof. So it is basically if and only kind of statement. If you have a stationary point then marginal cost is equal to average cost.

On the other hand, if marginal cost is equal to average cost then you have a stationary point of the average cost function.

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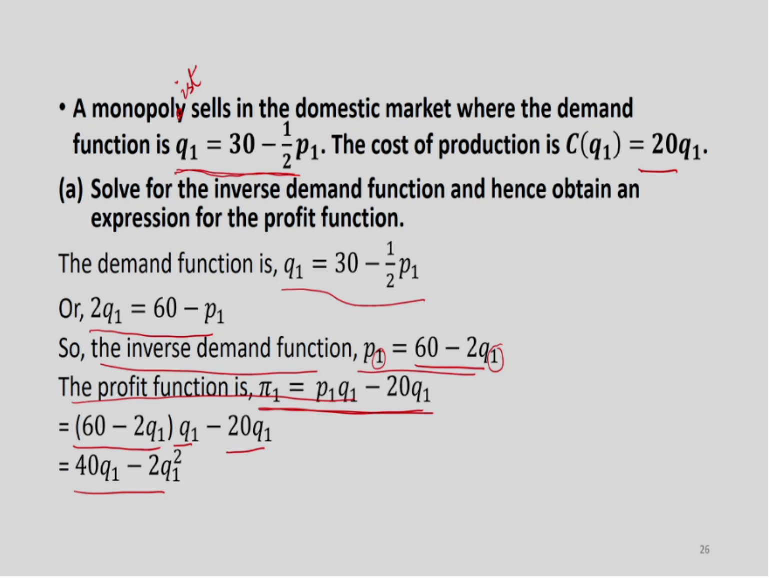

Here is another problem related to monopoly. A monopolist sells in the domestic market. So there is a monopolist firm which sells in the domestic market where the demand function is given by this $q_1 = 30 - \frac{1}{2}p_1$. The cost of production is given by $C(q_1) = 20q_1$. Solve for the inverse demand function and hence obtain an expression for the profit function. So this is the first part.

Now notice what we are given is not the inverse demand function. We are given the demand function where quantity is a function of the price. So first thing to do is to we have to find the inverse demand function and from there one can find the profit function. Now the demand function is given by this $q_1 = 30 - \frac{1}{2}p_1$.

Now that is equivalent to $2q_1 = 60 - p_1$ I have multiplied both sides by 2 and from here, I will get $p_1 = 60 - 2q_1$. Now actually I have p_1 as a function of quantity this is the inverse demand function. Now you might be wondering at this stage why we are talking about this subscript 1 all the time that will become clear later on when we introduce another market.

Now the inverse demand function has been found out. So therefore the profit function can be found out by using that π = revenue − cost and therefore I get $\pi_1 = (60 - 2q_1)q_1 - 20q_1$. So this is simplifying to be $\pi_1 = 40q_1 - 2q_1^2$, so this is the profit function π_1.

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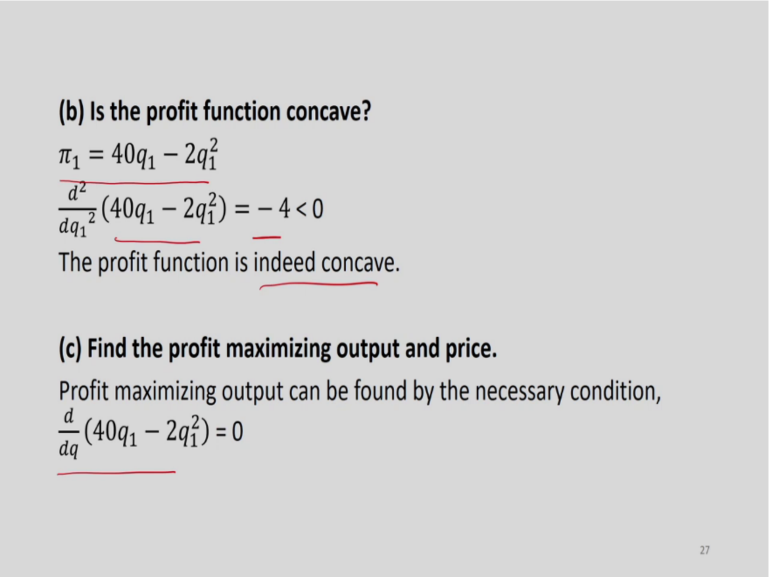

Second part is the profit function concave. Well, I can find that by taking the second derivative of the profit function and the second derivative of the profit function is -4 which is negative so the profit function is indeed concave. Third part, find the profit maximizing output and price.

So again this is standard. I maximize the profit and for that I use the necessary condition so I take the first derivative of the profit function set that is equal to 0.

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Or, $40 - 4q_1 = 0$

Or, q_1 = 10, this is the profit maximizing output. So the profit maximizing price given by the inverse demand function,

$p_1 = 60 - 2q_1$

I.e., $p_1 = 60 - 20$

= 40

(b) Is the profit function concave?

$\pi_1 = 40q_1 - 2q_1^2$

$\frac{d^2}{dq_1{}^2}(40q_1 - 2q_1^2) = -4 < 0$

The profit function is indeed concave.

(c) Find the profit maximizing output and price.

Profit maximizing output can be found by the necessary condition,

$\frac{d}{dq}(40q_1 - 2q_1^2) = 0$

- **A monopolist sells in the domestic market where the demand function is $q_1 = 30 - \frac{1}{2}p_1$. The cost of production is $C(q_1) = 20q_1$.**

(a) Solve for the inverse demand function and hence obtain an expression for the profit function.

The demand function is, $q_1 = 30 - \frac{1}{2}p_1$

Or, $2q_1 = 60 - p_1$

So, the inverse demand function, $p_1 = 60 - 2q_1$

The profit function is, $\pi_1 = p_1q_1 - 20q_1$

= $(60 - 2q_1)\,q_1 - 20q_1$

= $40q_1 - 2q_1^2$

That gives me this $\frac{d\pi_1}{dq} = 40 - 4q_1$ or $q_1 = 10$. This is the profit maximizing output so the profit maximizing price is given by the inverse demand function which was $p_1 = 60 - 2q_1$. So here I put $q_1 = 10$ so I will get $p_1 = 60 - 20$ so the optimal price is 40 so this is the answer the optimal quantity is 10. Optimal price is 40.

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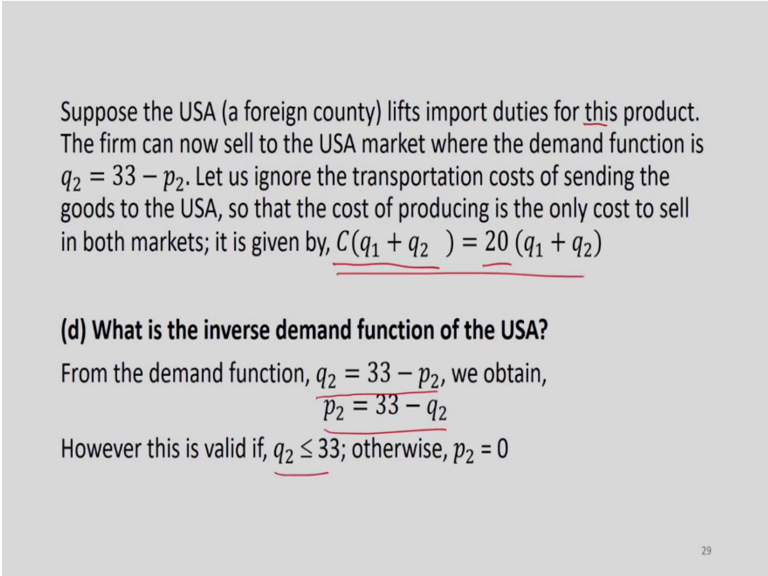

Here is the foreign market. So I will get another subscript here. Suppose the USA foreign country lifts import duties for this product. For this product means, the product that is produced by the monopolist firm. The firm can now sell to the US market where the demand function is $q_2 = 33 - p_2$.

Let us ignore the transportation costs of sending the goods to the USA so that the cost of producing is the only cost to sell in both the markets and it is given by this, so this is the cost function. $C(q_1 + q_2) = 20(q_1 + q_2)$. So basically, it means that it does not matter where your good is being sold. The cost of production is the same. So we are ignoring the cost of sending the goods to the foreign country.

And the cost is just 20 maybe rupees per unit and there is no fixed cost so the total cost of producing $q_1 + q_2$ total amount of output is $C(q_1 + q_2) = 20(q_1 + q_2)$. So here the fourth question, what is the inverse demand function of the USA that is in the USA market the American market, the inverse demand function is what.

Well the demand function is given $q_2 = 33 - p_2$ from here I get $p_2 = 33 - q_2$. This is the inverse demand function. However this has to be valid if $q_2 \leq 33$ otherwise if q_2 is higher than 33 I cannot say that p_2 is negative in that case $p_2 = 0$.

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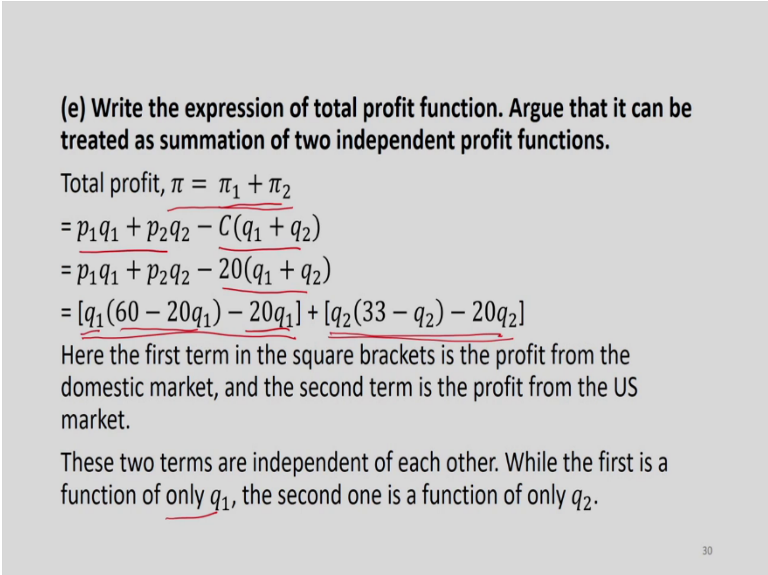

Here is the fifth question: write the expression of total profit function, argue that it can be treated as a summation of two independent profit functions. Total profit function will be the profit from the first market and the profit from the second market. But remember the cost is coming from one source.

So total cost is this whereas total revenue is this so total revenue − total cost and total cost as we have seen it is just $C(q_1 + q_2) = 20(q_1 + q_2)$ and as you can see I can take all the q_1 terms in the first bracket and all the q_2 terms in the second bracket. Here the first term in this square bracket is the profit from the domestic market and the second term is the profit from the US market that is quite evident.

This is the quantity sold in the domestic market. This is the price obtained in the domestic market. This is the cost of production for the goods sold in the domestic market and here – these are the expressions for the foreign market. So what we are getting here is that these two terms are independent of each other while the first is a function of only q_1, the second one is a function of only q_2.

So the interesting part of this problem is that total profit which the firm is obtaining by selling the goods in two different markets is a completely decomposable function. Decomposable in the sense that you can decompose the function in terms of profit from the first market and profit from the second market. There are no composite terms. There is no term involving let us suppose q_1 and q_2 together.

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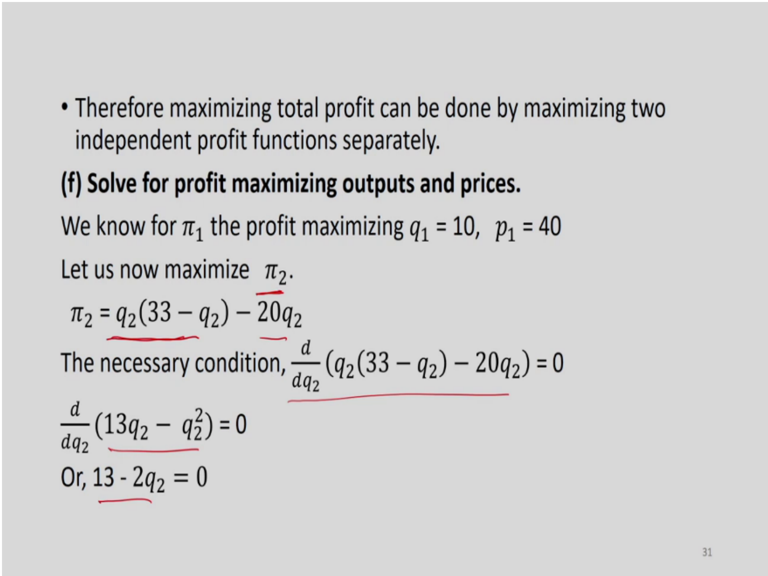

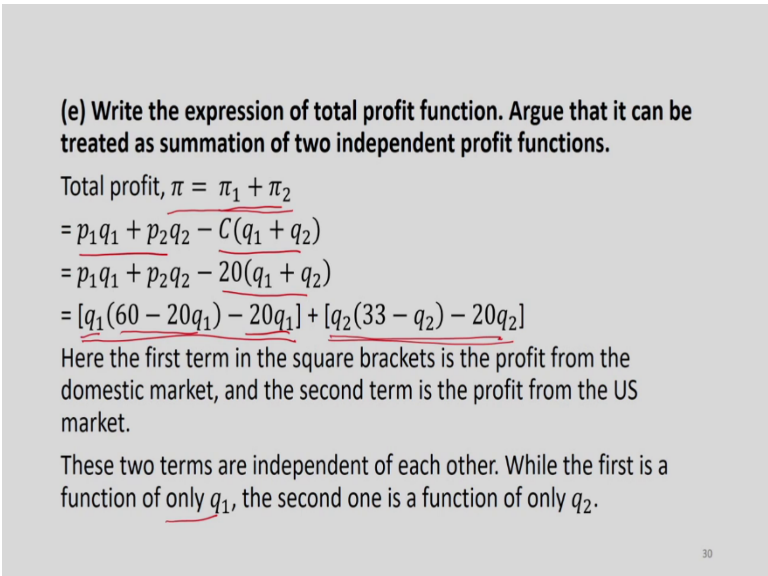

Or, $40 - 4q_1 = 0$

Or, $q_1 = 10$, this is the profit maximizing output. So the profit maximizing price given by the inverse demand function,

$p_1 = 60 - 2q_1$

I.e., $p_1 = 60 - 20$

$= 40$

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Suppose the USA (a foreign county) lifts import duties for this product. The firm can now sell to the USA market where the demand function is $q_2 = 33 - p_2$. Let us ignore the transportation costs of sending the goods to the USA, so that the cost of producing is the only cost to sell in both markets; it is given by, $C(q_1 + q_2) = 20\,(q_1 + q_2)$

(d) What is the inverse demand function of the USA?

From the demand function, $q_2 = 33 - p_2$, we obtain,

$$p_2 = 33 - q_2$$

However this is valid if, $q_2 \leq 33$; otherwise, $p_2 = 0$

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So therefore, maximizing the total profit can be done by maximizing two independent profit functions separately. There is something called maximizing function which has multiple variables. Here it is actually a function, which has multiple independent variables q_1 and q_2 but the fun part of this profit function is that it can be seen as composed of two independent functions, each function has only 1 independent variable.

And therefore, I can optimize these two functions separately. I can do that because I know how to maximize a function with a single variable and that will give me the solution. Solve for profit maximizing outputs and prices. So this is the 6th part. This is I think the last part.

Solve for profit maximizing outputs and prices. Now for the first part which is the profit maximization in the domestic market that we have already done, this part.

The quantity was 10 and the price was 40. q_1 was 10, p_1 was 40 so that was done. Now you can do the profit maximization for π_2. π_2 is total revenue from the US market minus the cost and this is standard. I take the derivative of this with respect to q_2 set that equal to 0 and it will give me $\frac{d}{dq_2}(13q_2 - q_2^2) = 0$ and that is $13 - 2q_2 = 0$.

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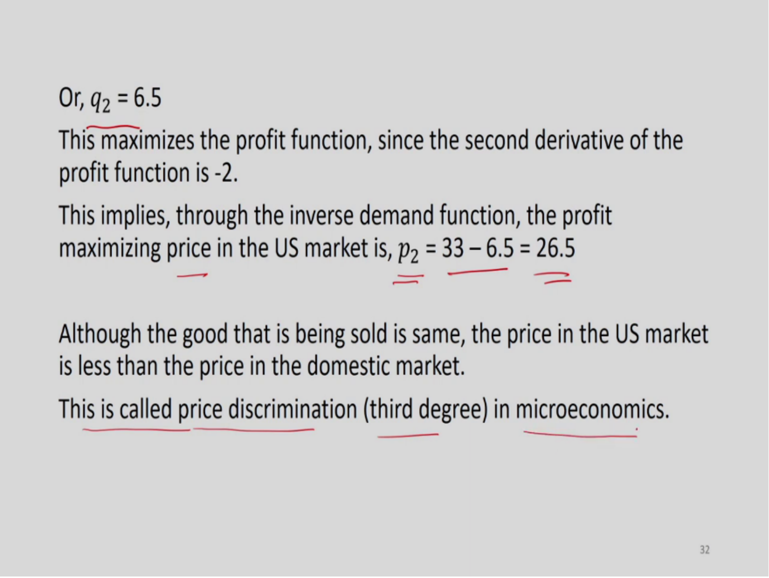

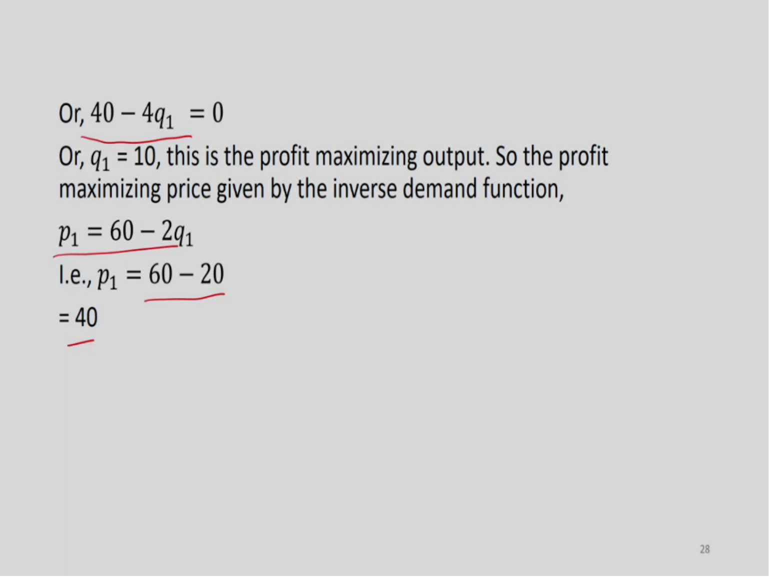

And it gives me the solution $q_2 = 6.5$. So the quantity that is going to be sold in the US market is 6.5. That is the answer and that is going to maximize the profit from the US market. 6.5 maximizes the profit function since the second derivative of the profit function is -2. So the second order condition is satisfied because if you take the second derivative it is -2.

This implies through the inverse demand function the profit maximizing price in the US market that is p2 is equal to 33 minus $p_2 = 33 - q_2$ so 6.5 is the quantity q_2, so I am getting

$p_2 = 26.5$. So, that is the price that is going to be charged in the US market, $p_2 = 26.5$. Here is an observation although the good that is being sold in both the markets is the same.

The price in the US market is less than the price in the domestic market. $p_2 = 26.5$, what was p_1? p_1 was 40, so there is a gap between the price that is being charged for the good in the domestic market which is a high price and the price that is charged in the US market p_2. Although the good is just the same but the producer while maximizing the profit is charging different prices in different markets.

This phenomenon is called price discrimination. It is the price discrimination of the third degree in microeconomics. I think I will stop here today and I will pick up the thread in the next lecture to cover more topics from the tutorials. Thank you for joining me.