Mathematics for Economics -1
Difference Equations
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Lecture – 31
Market model with inventory, Phase diagrams, Higher order equations

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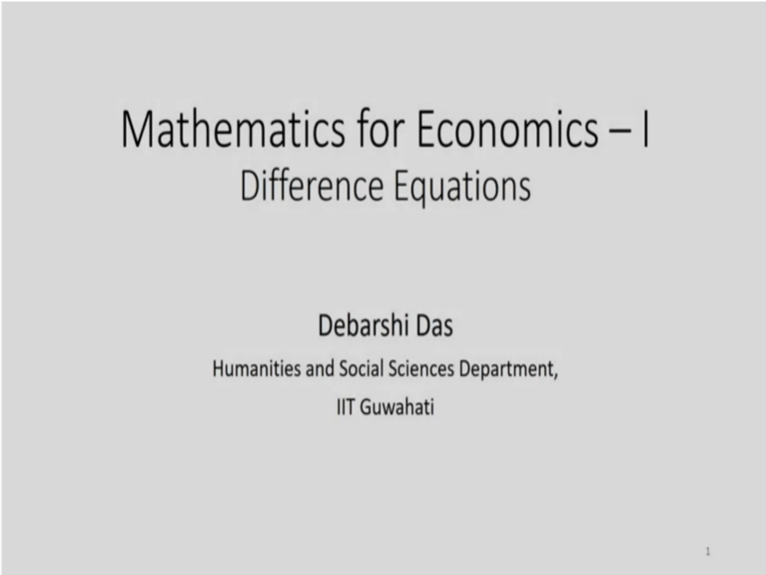

Hello and welcome to another lecture of this course, Mathematics for Economics. The module that we are covering right now is about Difference Equations. We have seen in the last few lectures how to solve a first order difference equation, which is non-homogeneous and right now we are talking about certain examples and applications where such difference equations are used.

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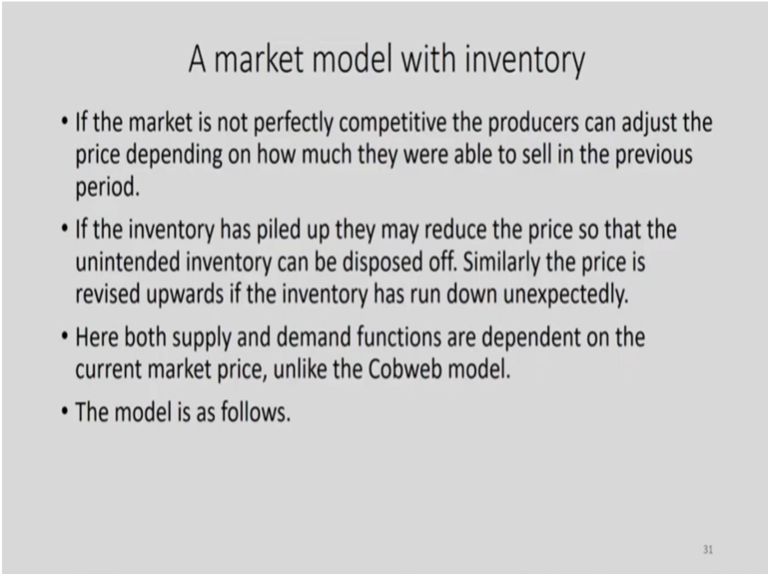

Now this is the model that I introduced in the last lecture, a market model with inventory. So here there is a demand side and there is a supply side like in the Cobweb model. However, unlike the Cobweb model we are not assuming that the market clears in every period. Here it is possible that there is some excess demand or extra supply in the market, and that affects the inventory of the producers.

And because of this inventory and change of this inventory the producers might change their prices in the next period, in particular, as you can see on your screen the second point that I have written here, if the inventory has piled up, they may reduce the price so that the unintended inventory can be disposed of. Similarly, the price is revised upwards, if the inventory has run down unexpectedly.

The idea is that in any period suppose the inventory is piling up, then the producers will like to give some encouragement to the buyers to buy this unintentional rise in inventory, so in the next period they will reduce the price so as to give incentive to the buyers, and similarly if the inventory is running down unexpectedly in that case they will like to stop that rundown and they will raise the price so as to discourage the buyers. So this is how the model is working.

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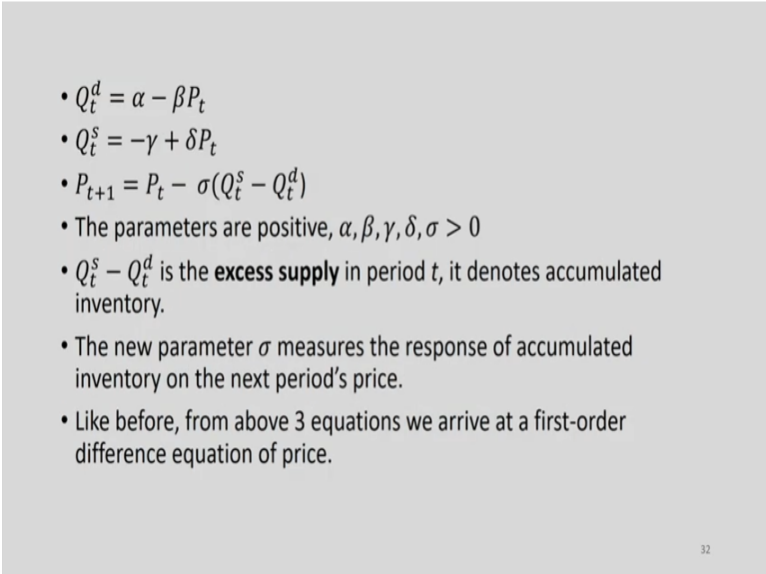

Mathematically, we represent this in terms of these three equations, so like the Cobweb model, we are assuming the demand function and the supply function, they are linear. But, unlike the Cobweb model, the supply function is a function of the current price, and this third equation is the new thing. You do not have market clearance in each period, but what you have is adjustment of the price, and presumably this adjustment is done by the producers.

So $P_{t+1} = P_t - \sigma(Q_t^s - Q_t^d)$. The parameters are all positive: $\alpha, \beta, \gamma, \delta, \sigma > 0$. $Q_t^s - Q_t^d$ is the excess supply in period t, this denotes the accumulated inventory. The part of the production that is not sold, that will go to the inventory. So this is the accumulation of inventory and the new parameter that we have introduced here is σ.

It measures the response of accumulated inventory on the next period's price. So this is where the lag is coming. So if there is accumulated inventory, $Q_t^s - Q_t^d > 0$. Then that is going to affect the price in the next period, and how much does it affect that response is measured by σ. In particular, $Q_t^s - Q_t^d$ affects the next year's price in a negative manner, because it measures the excess supply.

That is an accumulation of inventory and that reduces the price. Now these three equations if we put them together then that will give us the first order difference equation, and this is how we are getting it, so we are basically substituting the demand function and the supply function here in the P_{t+1} equation, and that will reduce everything to functions of P_t only.

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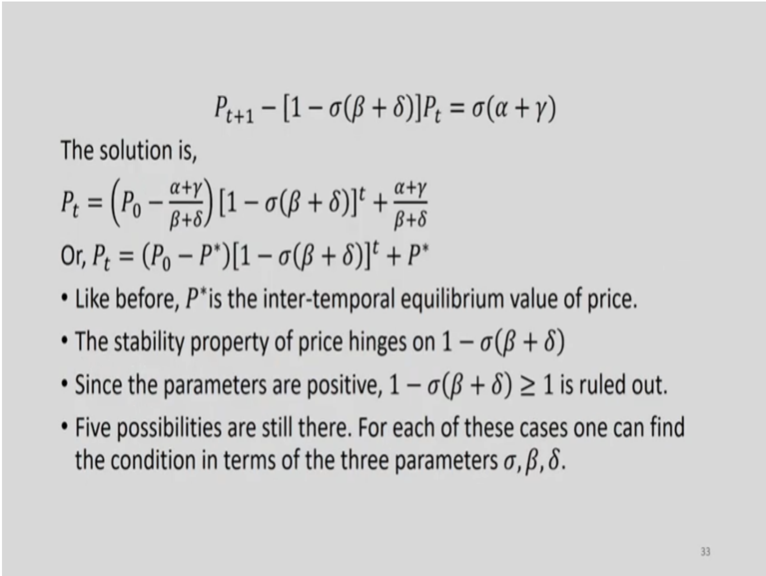

So it is a difference equation of first order in P_t. $P_{t+1} - [1 - \sigma(\beta + \delta)]P_t = \sigma(\alpha + \gamma)$. Now, I am not going into the steps of the solution, so we have to find out the particular integral and the complementary function. The particular integral turns out to be this, like in the Cobweb model, $\frac{\alpha+\gamma}{\beta+\delta}$.

And the complementary function is this, this is the complementary function, Ab^t, where if I know that in period 0, the price was P_0, I use that information. So this becomes the complete solution. $P_t = (P_0 - \frac{\alpha+\gamma}{\beta+\delta})[1 - \sigma(\beta + \delta)]^t + \frac{\alpha+\gamma}{\beta+\delta}$.

And to make it look simple I assume that this P^* is the particular integral, $P^* = \frac{\alpha+\gamma}{\beta+\delta}$, I assume that, so I get rid of this cumbersome term. I only write P^* and, as you know, P^* is the intertemporal equilibrium value of price in this case. Now, let us talk about stability. As we know that in this solution the stability of the equilibrium depends on this term the **b term**, which is $[1 - \sigma(\beta + \delta)]$.

Now, as we know, it can take seven possible different values or ranges of values. However, we can get rid of two such values, because we know σ, β, δ, all are positive. Therefore, this possibility that this $b \geq 1$ is ruled out, that is ruled out, because all these parameters are positive, so 1 minus these things cannot be greater than 1.

But, even if we have got rid of two possibilities, there are five possibilities that are left out. For each of these cases, one can find the condition in terms of the three parameters: σ, β, δ. So, five cases are left out and let us try to see how they turn out to be.

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Region	Value of $1-\sigma(\beta+\delta)$	Value of σ	Time path of price
1	$0<1-\sigma(\beta+\delta) < 1$	$0<\sigma<\frac{1}{\beta+\delta}$	Non-oscillatory, convergent
2	$1-\sigma(\beta+\delta) = 0$	$\sigma = \frac{1}{\beta+\delta}$	In equilibrium
3	$-1 <1-\sigma(\beta+\delta) < 0$	$\frac{1}{\beta+\delta}<\sigma<\frac{2}{\beta+\delta}$	Damped oscillation
4	$1-\sigma(\beta+\delta) = -1$	$\sigma = \frac{2}{\beta+\delta}$	Uniform oscillation
5	$1-\sigma(\beta+\delta) < -1$	$\sigma > \frac{2}{\beta+\delta}$	Explosive oscillation

So these are the five cases that I am talking about, one two, three, four five. This b can be between one and zero. It can be exactly equal to zero. It can lie between zero and minus one. It can be equal to minus one and it could be less than minus one, and in each of these cases, these conditions, if we play around with them, then they turn out to be like these.

The first region or the first condition turns out to be that $0 < \sigma < \frac{1}{\beta+\delta}$, and this case, as we know 0<b<1, it will give me a time path which is non-oscillatory and convergent. That comes from our initial results. Secondly, if this is exactly equal to 0, we know we are always in equilibrium, because b=0, and this condition turns out to be this, that $\sigma = \frac{1}{\beta+\delta}$.

The third case is $-1 < 1 - \sigma(\beta + \delta) < 0$, and this turns out to be this condition that sigma lies between two limits: $\frac{1}{\beta+\delta} < \sigma < \frac{2}{\beta+\delta}$, and as we know, this gives me a damped oscillation case. Fourth case is $1 - \sigma(\beta + \delta) = -1$.

We know this will be the case of uniform oscillation because b=-1 and this condition can be written as this $\sigma = \frac{2}{\beta+\delta}$. And finally, the fifth case is the b value that is $1 - \sigma(\beta + \delta)$ is very low. It is $1 - \sigma(\beta + \delta) < -1$, and in this case we know there is going to be explosive oscillation.

And this condition once again turns out to be, this, that $\sigma > \frac{2}{\beta+\delta}$. Now, out of these five cases, therefore, we have these first three as stable. In these cases, you have convergence or you are always in equilibrium, whereas the last two are not stable.

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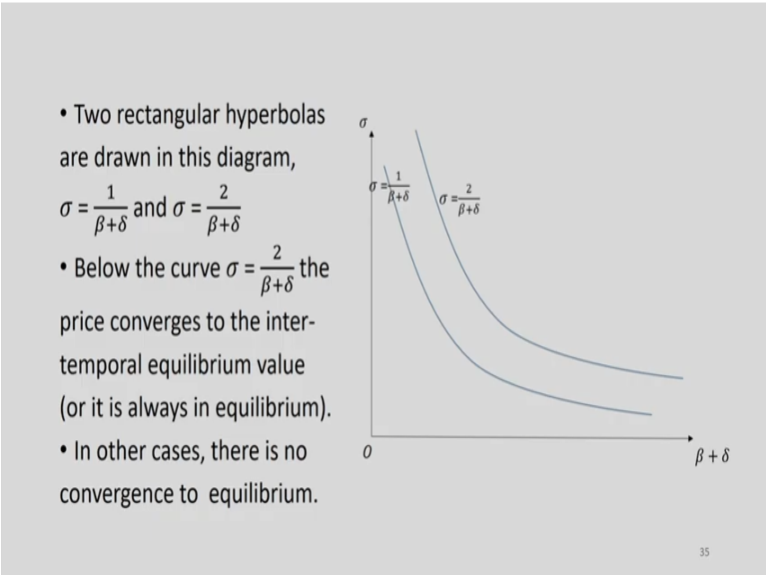

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Now these five cases can be, in fact shown in terms of a very convenient diagram. In this diagram what we have done is that, along the horizontal axis we have taken $\beta + \delta$ and on the vertical axis I have taken σ. Now, why have I done so? Because if you look at the table, you will see that we are getting these two equations one is $\sigma = \frac{1}{\beta+\delta}$, this is the case, and here is the other case, where $\sigma = \frac{2}{\beta+\delta}$.

Now, if I am representing $\beta + \delta$ along the horizontal axis and σ along the vertical axis, these two relations can be shown as two rectangular hyperbola. So this is the first one, and this is the second one. $\sigma = \frac{2}{\beta+\delta}$ and $\sigma = \frac{1}{\beta+\delta}$. Both of them are of the form $xy = k$, that k here is 2 and here is 1.

And we know that is the typical equation of a rectangular hyperbola. So these two rectangular hyperbola will help us demarcate these cases. These five cases, as we can see that the unstable cases are these, either $\sigma = \frac{2}{\beta+\delta}$ or $\sigma > \frac{2}{\beta+\delta}$. Now, which are these regions that we are talking about, these regions are actually these cases above the higher rectangular hyperbola or even on the rectangular hyperbola, these are the cases where there is no stability.

So that is what I have written here, below the curve $\sigma = \frac{2}{\beta+\delta}$, the price converges to the intertemporal equilibrium value. What was the intertemporal equilibrium value? It was that P^*, the particular integral. So, if we are picking up points from here that is below the upper rectangular hyperbola, then we are actually in that region of parameters where there is stability.

Either you are always in the equilibrium or even if you are not on the equilibrium, you are going to converge to the equilibrium, and in other cases that is if you are on the northeast of this upper rectangular hyperbola or if you are on that this rectangular hyperbola there is no convergence.

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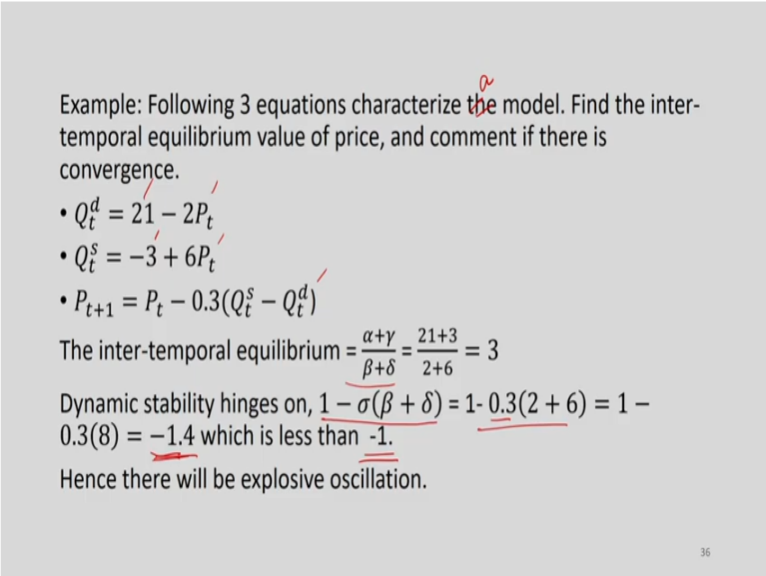

Here is an example of an exercise; the following three equations characterize a model that is a model of inventory market model. Find the intertemporal equilibrium value of price and comment if there is convergence, so these three equations are given the demand function, the supply function and the price adjustment function. We have to answer two questions; we have to first find the intertemporal equilibrium value of price.

Now that is given by this formula right, $P^* = \frac{\alpha+\gamma}{\beta+\delta}$. So, I just pick up the values from here and put it here, and if I do so, I will get 3, $\alpha = 21, \beta = 2, \gamma = 3, \delta = 6$. I put all of them here and I get 3, 21+ 3 is 24 divided by 2+6. That is 8, 24/8=3. Now then we have to answer the second question, comment if there is a convergence.

Now we know that convergence depends on this particular expression: $1 - \sigma(\beta + \delta)$ and again I put the values here. σ is 0.3, so I put that value here and I know the rest of the values and I get -1.4 and -1.4, as we know, it is less than minus 1, so there is going to be instability. And it is instability, plus it is going to be oscillatory instability, because this is negative. The b term is negative and it is less than minus 1. So there is going to be an explosive Oscillation.

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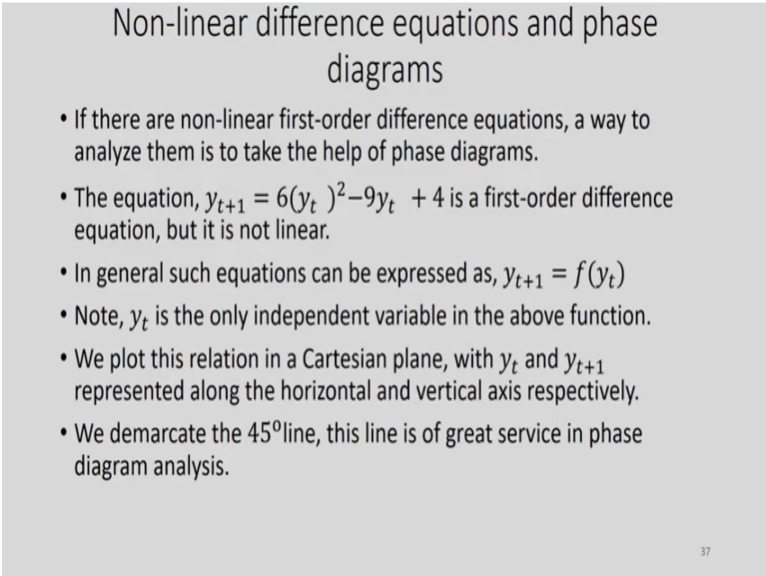

Now we take up another topic of difference equations, which is non-linear difference equations and phase diagrams. So, if you do not have linear difference equations then how do you deal with them. If there are non-linear first order difference equations, a way to analyze them is to take the help of phase diagrams, so we are going to use phase diagrams.

So here is an example: $y_{t+1} = 6(y_t)^2 - 9y_t + 4$ is a first order difference equation, but it is not linear. First order because the lag as you can see is of one. But there is a power of y_t here, which is 2. So it is not a linear difference equation, it is non-linear. Now the first thing to note is that in general, such equations can be expressed in this form $y_{t+1} = f(y_t)$, that is a function of y_t.

Note y_t is the only independent variable in the above function. We plot this relation in a cartesian plane with y_t and y_{t+1} represented along the horizontal and vertical axis respectively, and we demarcate the 45 degree line. This line is of great service in phase diagram analysis.

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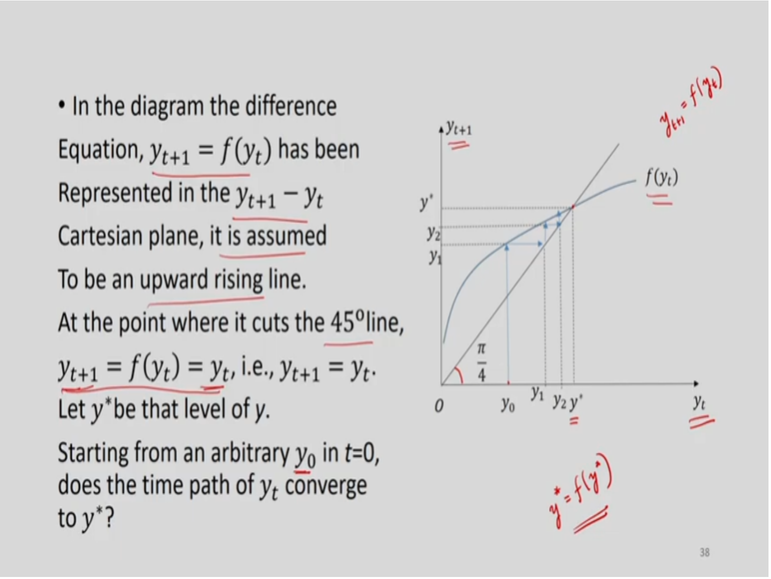

So let us see if we can find a diagram. Here it is. Here is a cartesian plane that is a two-dimensional plane on the horizontal axis I have taken y_t and on the vertical axis I have taken y_{t+1}. So it is not x y, but the value of the variable that we are dealing with in two successive periods, the previous period is on the horizontal axis and the subsequent period is on the vertical axis.

Now, additionally, what we have done is that we have plotted this function. This function is what is $y_{t+1} = f(y_t)$. This line, as you can see, it is not a straight line, and that is precisely the reason why it is a non-linear relation that we are dealing with and plus we have demarcated the 45 degree line.

This 45 degree line will be very critical in the analysis of phase diagrams. Now, why is it going to be critical? Because, the 45 degree line, along the 45 degree line, as we know the value of the x and the y, the coordinates are equal. So that is the property that we are going to be using in this analysis. In the diagram the difference equation $y_{t+1} = f(y_t)$ has been represented in this plane, Cartesian plane, y_{t+1} y_t.

It is in this case, I have assumed it to be an upward rising line. It is an upward rising line with a slope greater than 0. It is not necessary that all $f(y_t)s$ will be like this. There could be downward sloping lines as well. We shall deal with them later on, but right now to start with I am assuming that this $f(y_t)$ is an increasing function.

At the point where it cuts the 45 degree line what we have is this: that $y_{t+1} = f(y_t) = y_t$, the y and the x, they are equal. Because you are talking about this point, in this point you are on the 45 degree line as well. So y_{t+1} and y_t are same at that point and let us suppose that the particular value of y_t, it is denoted by y^*. So, what we essentially have is $y^* = f(y^*)$. That is how y^* is defined.

Now start from an arbitrary y_0 in $t = 0$, so what we are saying is suppose we are starting in the initial period where $t = 0$. Now, what is the value of y in that period? Let us suppose that is y_0. That could be any value, an arbitrary value, so we are taking an y_0 like here. It could be somewhere else also. Now, the question that we are asking is: does the time path of y_t converge to y^*? That is the question of dynamic stability or the lack of it.

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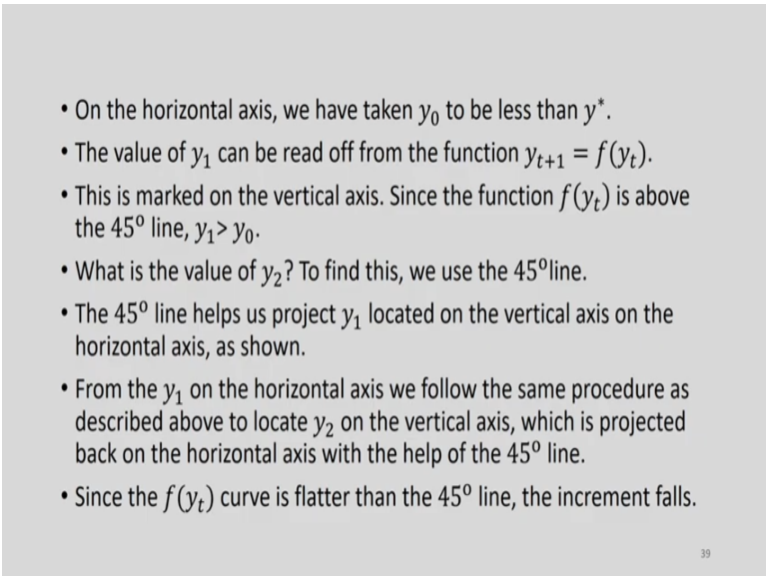

On the horizontal axis, we have taken y_0. Now y_0 is here, then the question is: where is y_1? Well, that can be easily found out by using this function. So from y_0 we draw a vertical line and go to the point of at this curve, and that will give me y_1, because that is the next period y, which is determined by the function. So the value of y_1 can be read off from the function $y_{t+1} = f(y_t)$.

This is marked on the vertical axis, since the function $f(y_t)$ is above the 45 degree line. Therefore, $y_1 > y_0$. So at this point the function is above the 45 degree line. It is above this line, 45 degree line, so that obviously means that y_1 will be more than y_0. So this is y_1. Now the question is: where is y_2?

We are interested to know the locations of y_1, y_2, y_3, etc, because we want to find out whether we are approaching y^* or not because y^* is the point where if you go there, then you will stay there. If you are at y^*, then in the next period also, you are going to be y^* in the third period. Also, you are going to be at y^*, etc etc.

So the question of dynamic stability is when you are starting from an arbitrary y_0, then the sequence of y's, do they go towards y^* or not to converge to y^*? So that is why we are interested to find out, where is y_2? If we know about y_1 to find this, we use the 45 degree line. The 45 degree line helps us project y_1, which is located on the vertical axis on the horizontal axis, as shown.

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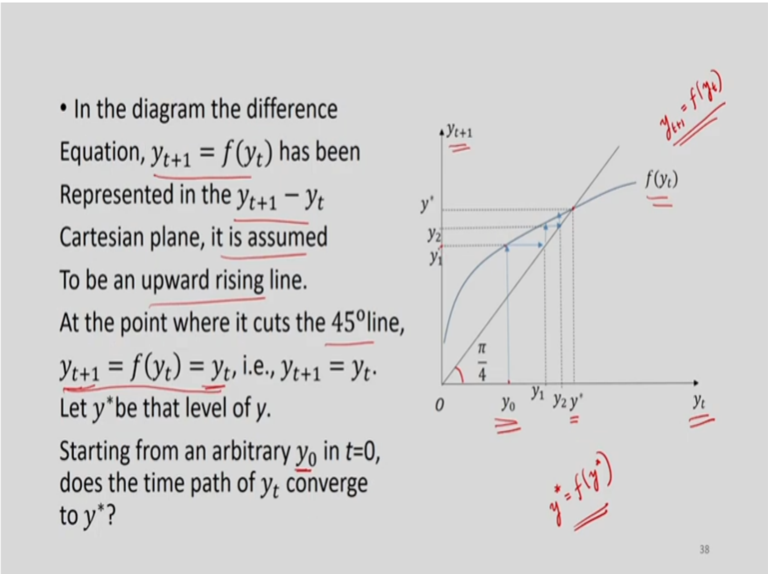

So here is y_1. I am projecting this on the horizontal axis here. How am I doing that? I am taking this y_1 and going to the 45 degree line and drawing a vertical straight line, and I am reading y_1 from the horizontal axis. So this is y_0, therefore, and on the same line on the same axis, I am able to locate. Where is y_1, and if I know y_1, then I will straight away know y_2, because from here I draw a vertical line straight going to the f function.

$f(y_t)$ and $f(y_t)$ will give me y_2, which is here from the function. And y_2 is here, but we wanted to find out what is y_2 on the x axis. So again, I project that on the x axis, taking the help of the 45 degree line, so here is y_2. So this is the method we are deploying here, we are using the 45 degree line to project the values on the y axis on the x axis.

And simultaneously we are using the f function as well, so that is what we have written here from y_1 on the horizontal axis, we follow the same procedure as described above to locate y_2 on the vertical axis, which is projected back on the horizontal axis, with the help of the 45 degree line. Since $f(y_t)$ curve is flatter than the 45 degree line, the increment falls.

So this is very critical, the last statement, the f function. I have purposefully drawn it in a manner that it is flatter than the 45 degree line, at least in the neighbourhood of y^*, at least where I have located y_0. Now, if it is flatter, then it basically means that subsequently, the gaps between y's will go on declining.

So what I mean by that is that this gap, the gap between $y_2 - y_1$, is less than this gap, which is $y_1 - y_0$. You can see that with your naked eyes in the diagram, but that is coming because this y, the f function, is a flat function.

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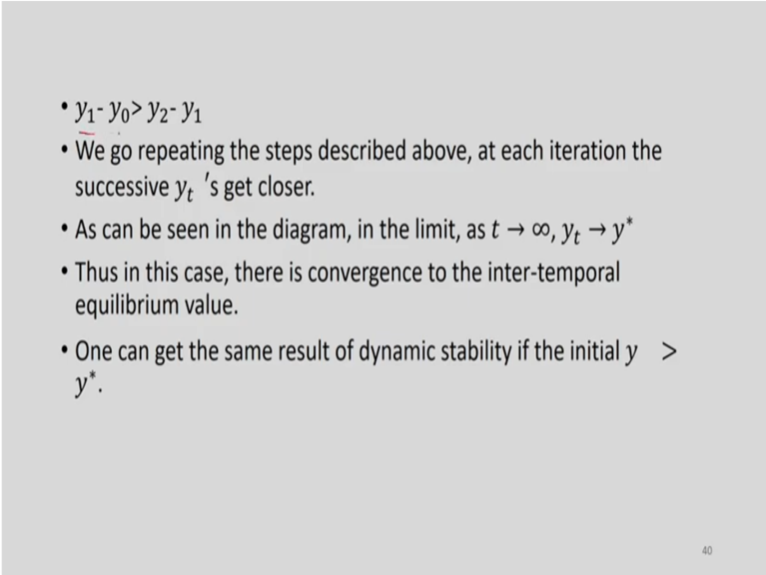

So that is what I have written here, $y_1 - y_0$, which was the first increment, is strictly greater than $y_2 - y_1$. This was the second increment, so likewise you can go one step further. The next stage will be $y_3 - y_2$ and that $y_3 - y_2$ will be further lower than $y_2 - y_1$.

We go on repeating the steps described above at each iteration, the successive $y_t s$ get closer. That means the gap between the two successive y's they keep on declining, as can be seen in the diagram in the limit, as t goes to infinity y_t goes to y^* all, so this is how it is happening.

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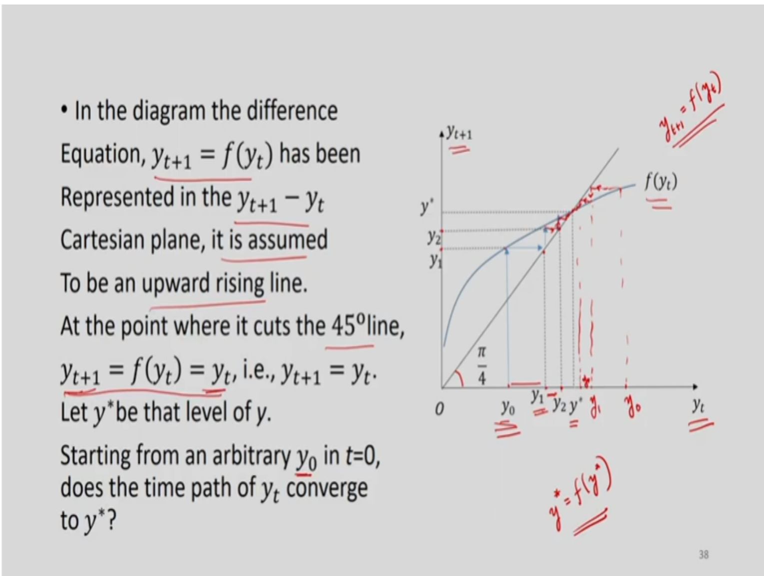

First, you go here, then you go here and like this, you ultimately reach y^*. So these are the arrows one can draw to show how the path is demarcated. So in this case there is actually a dynamic stability. Thus, in this case, there is convergence to the intertemporal equilibrium value as t goes to infinity y_t actually tends towards y^*.

One can get the same result of dynamic stability if the initial y is greater than y^*. So here we took y_0, that is the initial value to be less than y^*, but if we had taken it somewhere here then also the same result would have been applicable. So here this will be y_1, and from here you can find y_1 here.

This will be y_1, and from here you can find y_1 here, this is y_1, so you can see that actually you go and converge to the point of intersection which is y^*, so it does not really matter where we are starting from y_0 could be above y^*, it could be below y^* in either case we are able to converge to y^*.

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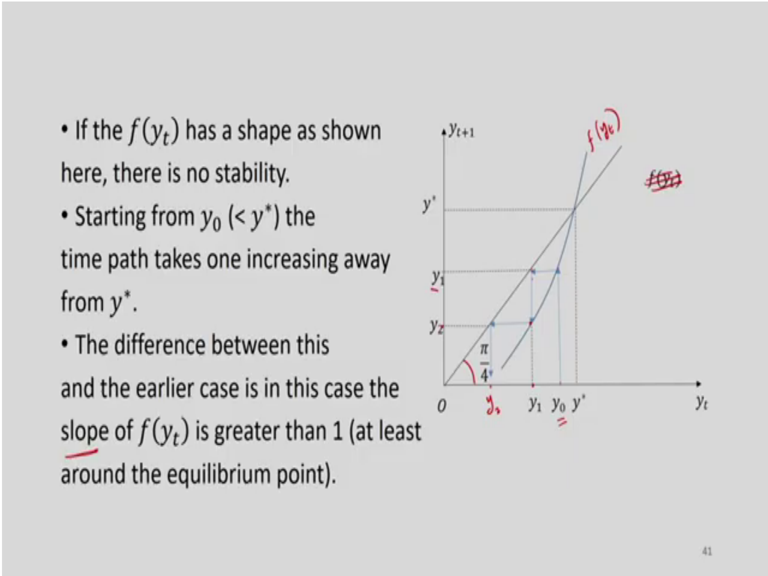

If the $f(y_t)$ has a shape as shown here in this diagram. Then however, there is no stability. So let us try to trace out one path here we are taking y_0 here below y^*, so y_1 will be in this value and again projecting back we get y_1 here, so y_t will be here and as you can see, y_0 was here, y_1 is further away from y^* and y_2 will be even further away, and so we are actually going away from the intertemporal equilibrium value of y.

Starting from y_0, the time path takes one increasingly away from y^*. The difference between this and the earlier case is in this case, the shape of $f(y_t)$ is the slope. The slope of $f(y_t) > 1$, at least around the equilibrium point that you can see here. This line $f(y_t)$ should not be here, $f(y_t)$ is a steep line. In particular it is intersecting the 45 degree line from below, which means that it has a slope which is greater than 1, which is the slope of the 45 degree line, and in this case, we are getting the case of dynamic instability.

So in both these cases one was stable and the other was unstable. There was one common point that the f function was an increasing function, but what happens if the function is not increasing?

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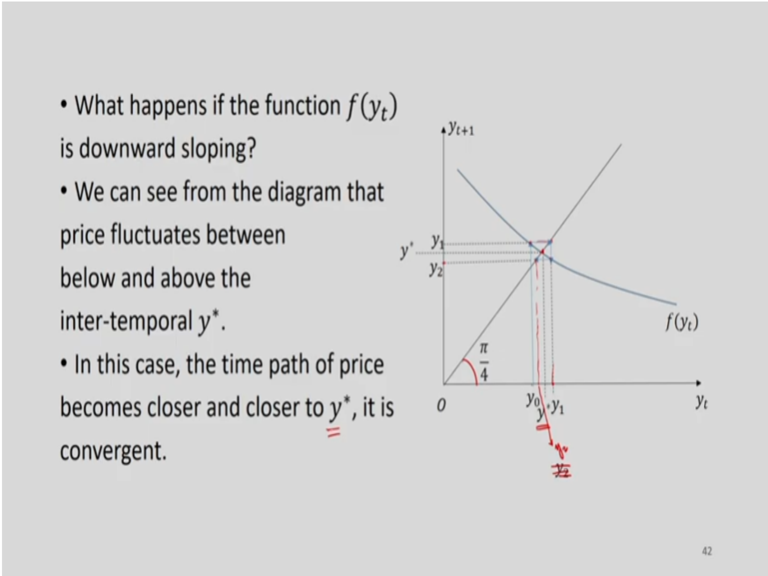

If it is downward sloping, then you have something like this right. I have drawn this $f(y_t)$ function. We can see from the diagram that price here fluctuates between below and above the inter-temporal y^*. So here y star is at the intersection point like before, which is the long run intertemporal equilibrium value; but let us see how the time paths look like, so we are starting from y_0 here. What is y_1?

We go to the f function; this is y_1 and we are projecting it back right. So this is y_1 on the horizontal axis. Now, what is y_2? We go up right. This is y_2, and so this should be y_2. So this is y_2, but as you can see that price first it was below y^*. Then it goes y_1, which is above y^*, and then it again comes back to y_2, which is below y^*, which means price is fluctuating in this case from below to above the intertemporal equilibrium y^*.

That is the first thing to note, but the second thing to note is that the price over time is becoming closer and closer to y^*. So thus, in this case, the time path is actually convergent.

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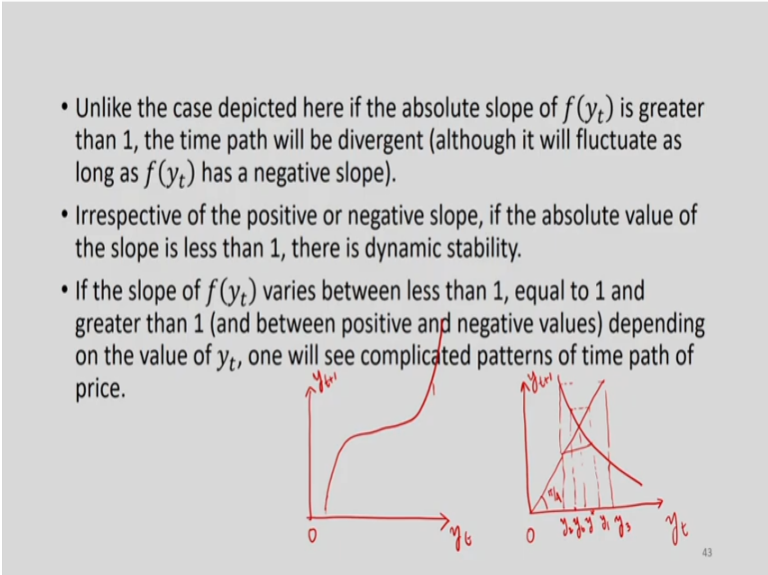

Unlike the case depicted here, if the absolute slope of $f(y_t) > 1$, then the time path will be divergent. Although it will fluctuate as long as $f(y_t)$ has a negative **slope**, so the reason why we are going to have fluctuation is because $f(y_t)$ is downward sloping that is giving me fluctuation, but whether you are going to have stability or not that depends on the absolute value of the price.

So what is being written here is this suppose, this is the 45 degree line and you have a downward sloping line, but which is very steep like this. So in this case you are going to have instability. So let us start from suppose this is P^* or y^*, and you are starting from here y_0 going up to y_1, so this is y_1. This is y_1, what is y_2? y_2 is this, so this is y_2.

What is y_3? This is y_3, So, as you can see, we are moving away from y^*. First we were here, quite close to y^*. Then we are going a little bit away from y^* y_1 and then y_2. Again, the gap is rising from y_2 to y^* and y_3. The gap is rising even more so actually, you are going away from the equilibrium. There is divergence, irrespective of positive or negative Slope. If the absolute value of the slope is less than 1, there is dynamic stability.

So this is the conclusion, if we want to have dynamic stability, then the absolute value of the slope of $f(y_t)$ that has to be less than 1. If it is positively sloped, you are going to have a monotonic convergence. If there is a negative Slope, then there can be dynamic stability, but it will be a kind of fluctuation of price and if the slope of $f(y_t)$ varies between less than 1, equal to 1 and greater than 1 and between positive and negative values, depending on the value of y_t, one will see complicated patterns of time path of price.

So this is a general case that it is possible that if you have $f(y_t)$, but it is not always less than one and not always greater than one, but the slope goes on changing it can be equal to 1 less than 1 greater than 1 and it can be positive and negative as well. Then we can have a really very complex pattern of time path of price, just think about it in terms of a diagram.

Suppose you have something like, so you can see first, we are starting from a very steep line, but then it is flattening out and again it is getting quite steep right at this range, so here, the slope is varying between less than 1 equal to 1, greater than 1 etc, etc. In this case, like S kind of function, you can have a very complicated pattern of price over time.

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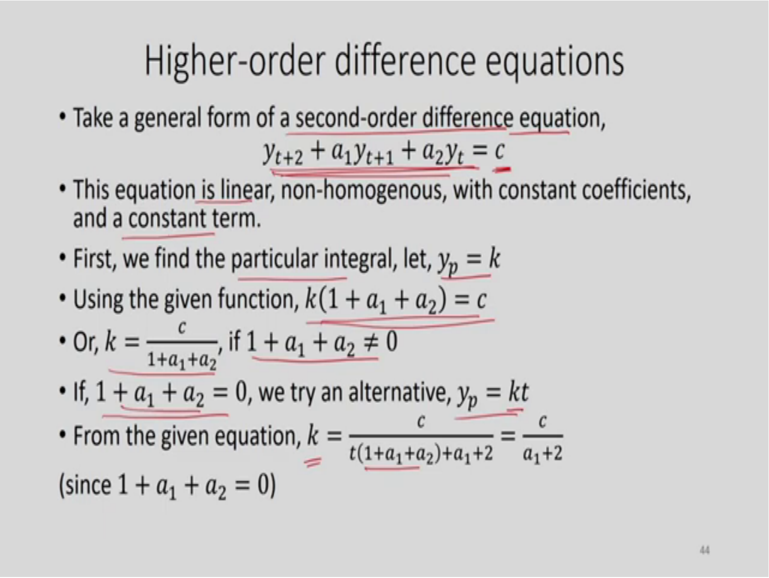

I think we shall start this discussion of higher order difference equations, maybe I will not be able to finish it today itself, but at least we can introduce this new topic. Now, so far we have talked about only first order difference equations, there was only one lag. But if there are higher order difference equations, then how we can deal with them. In particular, we are going to talk about second-order difference equations.

So here is the general form of a second order difference equation with some additional properties that this equation is a linear equation. It is non homogeneous because you have a non-zero term on the right hand side, and we have constant coefficients, a_1, a_2 are all constants and on the right hand side I have a constant term c. So this is that equation, which satisfies all these properties, and we want to find out how to solve it.

Now the general pattern remains the same. That first, we find out what is the particular integral and the complementary function. And given the initial conditions, we find out the value of the arbitrary constants and we write the general solution. So that is the strategy of solving it. Let us first start with the particular integral and we assume a simple solution, $y_p = k$, k is a constant.

And I put it here $y_p = k$ and so that means y is invariant with time. So I get this equation and that boils down to $k = \frac{c}{1+a_1+a_2}$. But I can write this if the denominator is non-zero, that is $1 + a_1 + a_2 \neq 0$. But what happens if $1 + a_1 + a_2 \neq 0$. Then we have to try some alternate solution, I take y_p that is the particular integral to be a function of t, kt.

So again, I put this in this equation and if I do so and solve it, then I will get $k = \frac{c}{t(1+a_1+a_2)+a_1+2}$. However, this part drops out because it is equal to 0, so I am getting $k = \frac{c}{a_1+2}$. So therefore, the solution will be this multiplied by t.

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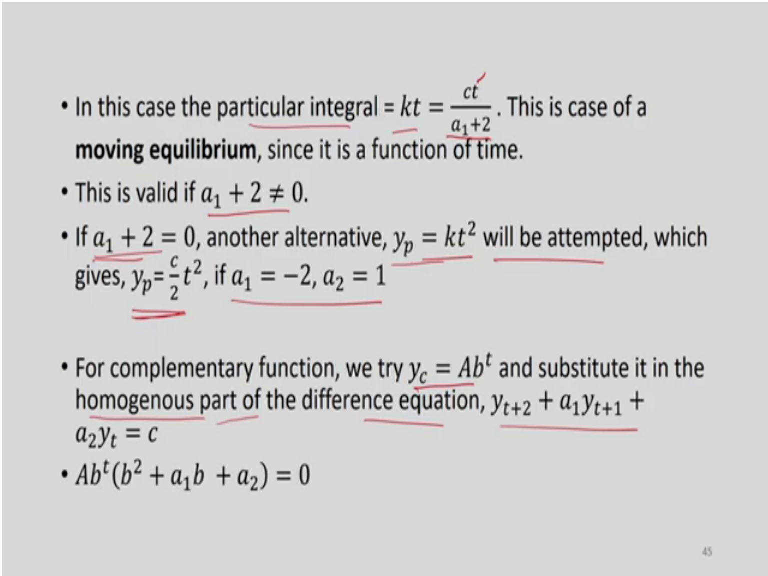

In this case, the particular integral will be kt that is $kt = \frac{ct}{a_1+2}$, and this is called a moving equilibrium because it is a function of time, t is there, but this again is valid. If the denominator is not 0, that is $a_1 + 2 \neq 0$. So I have to cover the case where $a_1 + 2 = 0$.

This is the third case, now, if $a_1 + 2 = 0$, then we try another solution, which is a little bit more complicated, $y_p = kt^2$. So earlier we took $y_p = kt$. In the first case, we took only k, and now we are taking $y_p = kt^*$ and again this can be substituted in the difference equation and solved for k, and we are going to get $k = \frac{c}{2}$.

Therefore, $y_p = \frac{c}{2}t^2$, and this is the case where this condition is satisfied and the earlier condition, this is also satisfied. So that these two conditions actually will give me, these two values $a_1 =- 2$ and $a_2 = 1$. So this is the case of particular integral, as you can see, there could be three possible cases. Now we come to the complementary function and $y_c = Ab^t$.

This is the complementary function that we use and we substitute it to the homogeneous part of the difference equation, and this was the difference equation. $y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$, and this will give me this thing: $Ab^t(b^2 + a_1 b + a_2) = 0$.

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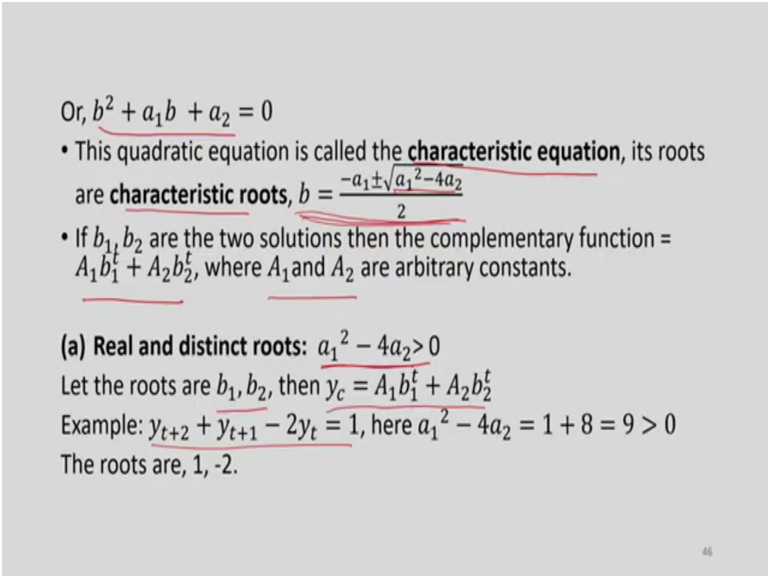

So the first part that is $Ab^t \neq 0$. Therefore, only this part is equal to zero, $(b^2 + a_1 b + a_2) = 0$. This is a quadratic equation in b, and this is in particular. This is called the characteristic equation and its roots are called characteristic roots, and it is given this form $b = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$.

These are two characteristic roots, one if we take the plus sign minus, can be taken, then that will give me the second root. If b_1 *and* b_2 are the two solutions, then the complementary function is written as this $A_1 b_1^t + A_2 b_2^t$ where A_1 and A_2 are arbitrary constants. So this is the general case, but there could be three possible sub cases from here.

One is where the roots are real and distinct right, where within the square root. This part, that is $a_1^2 - 4a_2 > 0$ that will give me real and distinct roots. So let us explore this possibility, $a_1^2 - 4a_2 > 0$. Now, in this case, let us suppose the roots that we are getting from here right are called b_1 *and* b_2.

Then the solution, as we have just seen, is $A_1 b_1^t + A_2 b_2^t$. So here is an example: suppose this is the difference equation that is given to us $y_{t+2} + y_{t+1} - 2y_t = 1$. Here, what is $a_1^2 - 4a_2$

? It turns out to be 1 plus 8, which is equal to 9 and which is, as we know, is positive. So this is the thing that we are talking about here.

The roots will be real and distinct and, as it turns out, the roots are 1 and -2, so this 1 and -2 will be substituted here as b_1 *and* b_2. So this was the first case where the roots are real and distinct, and we have seen what is the complementary function in this case.

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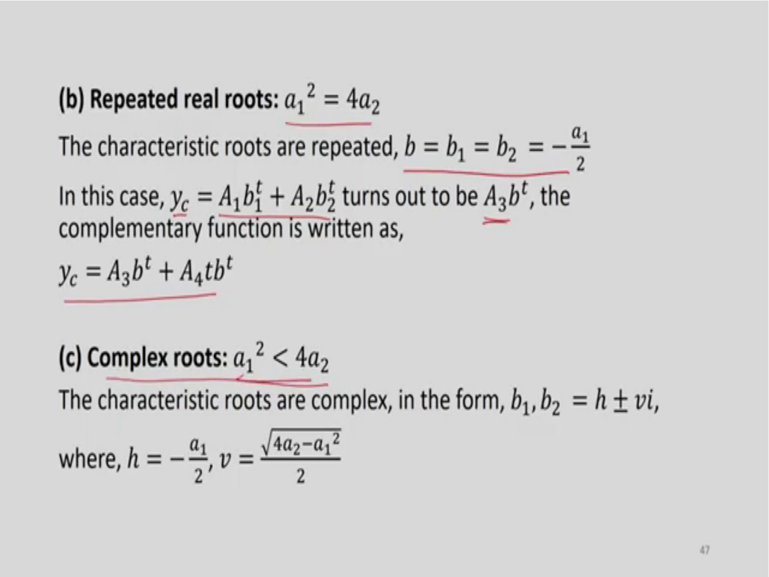

The second case is a case where there are repeated real roots in this case, $a_1^2 = 4a_2$. So basically, we are talking about this part becoming 0. In this case, there will be no distinct roots. There will be a single root and that is why we are calling it repeated real roots and what are these two roots? Actually, they are the same root. It is equal to $\frac{-a_1}{2}$, and this root is repeated twice in this case y_c.

That is the complementary function: $y_c = A_1b_1^t + A_2b_2^t$ and since b_1 *and* b_2 are same. This can be written simply as A_3b^t, and in this case the complementary function is actually written. As this $y_c = A_3b^t + A_4tb^t$, where like before A_3 and A_4, are arbitrary constants, which can be found out if we are given the initial conditions.

And finally, is the case of complex roots, so this is the case where the square root term, $a_1^2 - 4a_2$ turns out to be negative. So this is the condition, $a_1^2 < 4a_2$ where it is negative. In this case, the characteristic roots are complex. In the form, $b_1, b_2 = h \pm vi$, i is the imaginary number, where $h = \frac{-a_1}{2}$ and $v = \frac{\sqrt{4a_2 - a_1^2}}{2}$.

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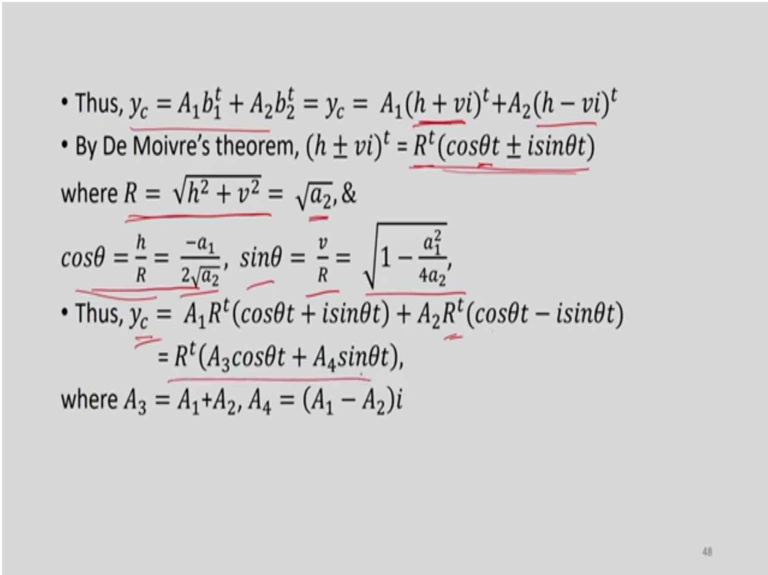

So therefore, the complementary function is $y_c = A_1 b_1^t + A_2 b_2^t$, and I know $b_1, b_2 = h \pm vi$, so I can write them and it turns out to be like this. Now by De Moivre's Theorem, $(h \pm vi)^t = R^t(cos\theta t \pm isin\theta t$, where $R = \sqrt{h^2 + v^2}$. In this case, what is $h^2 + v^2$? It is simply a_2 and what about theta?

We have to find out what theta is also. Well theta is given by this, this relation that $cos\theta = \frac{h}{R}$, and if we use the value of h and R, then it is coming out to be $cos\theta = \frac{h}{R} = \frac{-a_1}{2\sqrt{a_2}}$ and similarly $sin\theta = \frac{v}{R} = \sqrt{1 - \frac{a_1^2}{4a_2}}$.

So if we solve this, then we can find out θ. And if we solve this, I can find out R. So then we can put the value of R and θ here. Now, since this is known, then y_c I can just put in place of $h + vi$. I can put $(h + vi)^t$, I can put $R^t(cos\theta t + isin\theta t)$.

In place of $(h - vi)^t$.I can put $R^t(cos\theta t - isin\theta t)$ and this simplifies to be this $y_c = R^t(A_3 cos\theta t + A_4 sin\theta t)$ where $A_3 = A_1 + A_2$ and $A_4 = (A_1 - A_2)i$. So this is my complementary function in this case. This was the case of complex roots, so we have discussed all the four cases of real, distinct roots, repeated real roots and complex roots.

These are basically the cases where the characteristic equation has to be solved and the characteristic equations can have three separate cases of roots, so we have talked about that.

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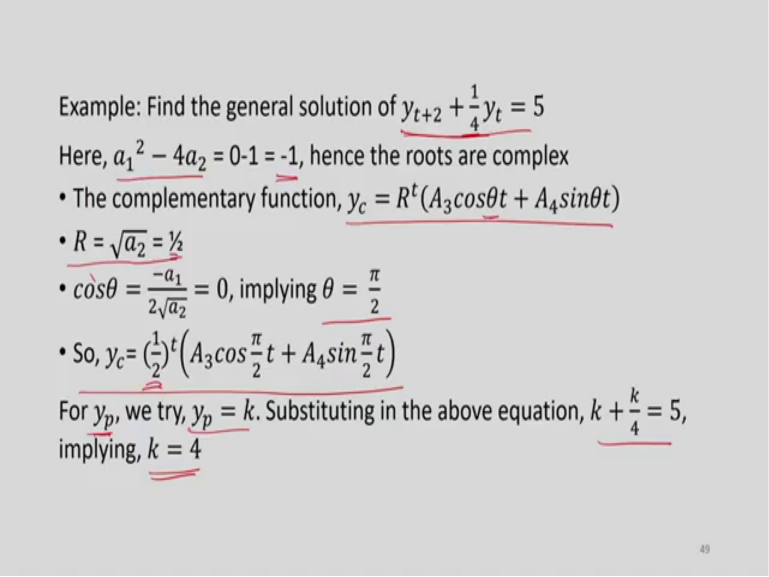

Now the next point is to take one particular equation, second order difference equation and try to solve it, where you can have complex roots. So here $y_{t+2} + \frac{1}{4}y_t = 5$. You can see that the lag is of two, so it is a second order difference equation, there is no non-linearity here.

So first we look at that term $a_1^2 - 4a_2$, that is crucial. So, in this case, it turns out to be minus 1, which basically means that the roots are going to be complex and if the roots are complex, we know that the complementary function we have just found out it is given by $y_c = R^t(A_3 cos\theta t + A_4 sin\theta t)$, where $R = \sqrt{a_2}$. What is a_2? $a_2 = 1/4$, so $R = \sqrt{a_2} = 1/2$.

So remember we are taking the positive root here and what is $cos\theta t$. We want to find out θ. Also, $cos\theta t$ and $sin\theta t$. So $cos\theta = \frac{-a_1}{2\sqrt{a_2}}$ and what is a_1? $a_1 = 0$ because there is no y_{t+1} term here. So the coefficient of y_{t+1} is zero.

So therefore $a_1 = 0$, so $cos\theta = 0$means that $\theta = \frac{\pi}{2}$, so that information is used to get the complementary function which is $y_c = (\frac{1}{2})^t (A_3 cos\frac{\pi}{2}t + A_4 sin\frac{\pi}{2}t)$. So this is the complementary function.

We still do not know what is A_3 and A_4. But if we know the initial conditions, then we can find that out, but we have not found out the intertemporal equilibrium, the particular integral again we take $y_p = k$and if we substitute that here then we get $k + \frac{k}{4} = 5$, that gives me $k = 4$, so both the y_p and y_c's are found out. Then the general solution will be the sum of these two, that is it.

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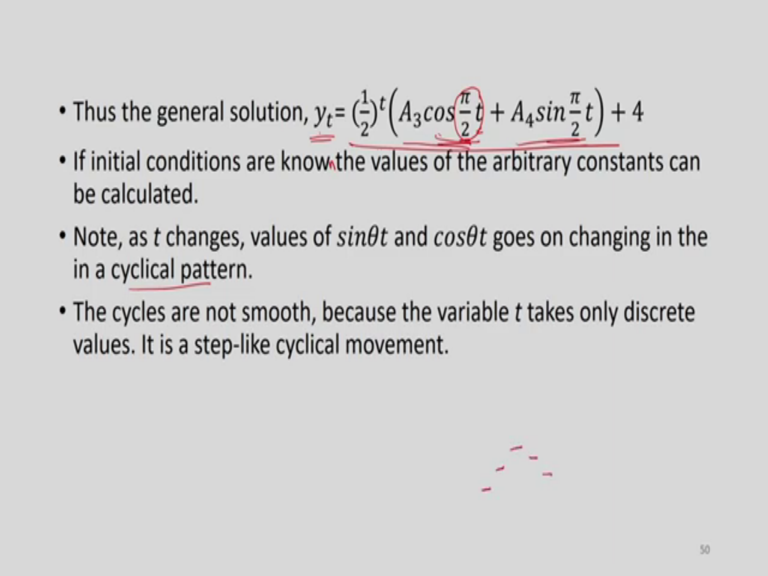

That is what we have written: if the initial conditions are known, the values of the arbitrary constants can be calculated, note as t changes, the values of $sin\theta t$ and $cos\theta t$ goes on changing in a cyclical pattern. So that we know from trigonometry that as this $\theta = \frac{\pi}{2}$, is given to us, as t changes so $cos\frac{\pi}{2}t$, t will take 0, then 1, then 2 etc, etc. So this angle is going to change accordingly and the cos of that angle will change in a cyclical manner and similarly, the sin of the angle will also change in a cyclical manner.

That is what I have written is going to change in a cyclical pattern. The cycles are not smooth because the variable t takes only discrete values. It is a step like cyclical movement. So t will take only values like 0, 1, 2, 3, 4 like that, discrete values. So corresponding to that the sins and the cos they will go on changing like that, according to that, so the value of y_t will change in a cyclical manner, but it is not going to be a smooth cycle.

It is going to be something like this. As you can see, there are jumps between two time periods. So that is what I mean that it is a step like cyclical movement, all right so I will stop here and in the next lecture, I am going to talk about the convergence. So we have talked about convergence and dynamic stability of first order difference equations, we have solved the second order difference equation, the general pattern, but what about the question of dynamic stability, under what conditions dynamic stability is going to be there that we can discuss in the next lecture. Thank you.