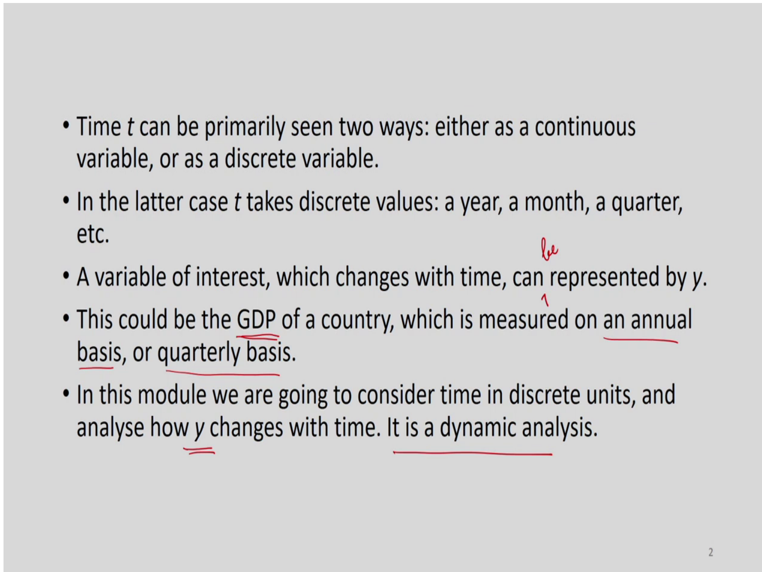
Mathematics for Economics – 1
Different Equations
Professor Debarshi Das
Humanities and Social Sciences Department
Indian Institute of Technology, Guwahati
Lecture – 29
First order difference equations, solution

Hello and welcome to another lecture of this course Mathematics for Economics part one. So, today we are going to start with a new module and this module is the last module of this course, the name of the module is difference equations. The difference equations, this particular module is a little bit different from what we have been discussing so far. The last topic that we have covered was integration and before that we were taking the optimization exercise as the subject of the module.

Now, the difference equation is a little bit different from all these topics in the sense that now, we are talking about time and how the variables change with respect to time. So, our focus will be solely on time. Time is the independent variable and we are trying to see how the variable of interest for us, for example, the price level or the GDP of a country or the output produced by a firm, they change with respect to time.

So, that is sort of dynamic analysis that we are going to introduce from this module and we are talking about difference equations. So, we are going to discuss what it means when we say difference equations.

(Refer Slide Time: 02:03)



So, this is the title slide as you can see. Time since, since we are talking about time t, now, this t can be primarily seen in two ways either as a continuous variable or as a discrete variable. So, if you are taking time as a continuous variable, then it means t can take any possible real value within any interval. Suppose 0 to 1 is the interval then t can take any possible value within 0 and 1. But, we are talking about discrete time.

So, in the latter case, that is the discrete time t takes only discrete values, it could be a year, a month, a quarter, etc. The variable of interest which changes with time can be represented. So, this can be represented by y. Now, what is y, this variable of interest? It could be the GDP of a country, the national income of a country, let us suppose. So, that is suppose y and this y changes with respect to time.

Or it could be GDP of a country but this GDP could be measured in different ways, it could be on an annual basis. Generally, that is the idea of GDP one has that this is the textbook definition of GDP that we are measuring the output produced within a country the final value of goods and services produced within a country in a year. So, when we are saying in a year that means, we are taking the GDP on an annual basis maybe a calendar year, let us say 2020 that is a calendar year or it could be a financial year.

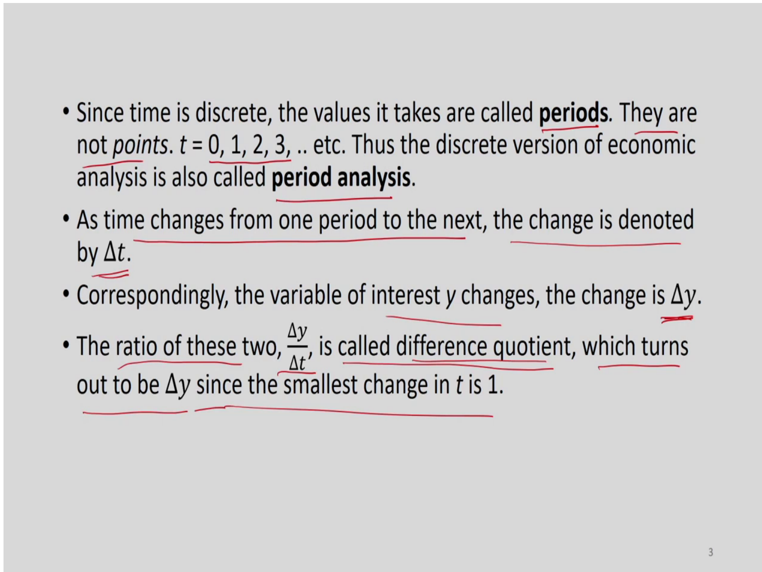
So, for example, in India, the calendar year and the financial years are not the same. In India, the financial year starts from April and it ends in March. So, that is the financial year. And as you can see, it does not actually coincide with the calendar year, which is from, January to December. But GDP can also be measured not on an annual basis, but on a quarterly basis. Quarterly basis means every three months.

So, you divide the twelve months into four parts. That is why it is called quarterly. So, if you divide the entire year into four parts then each part will consist of three months. And so, what we do in this case is that we look at the total value of goods and services final goods and services produced in a country in a three months period. So, that will be like the quarterly GDP. So, it could be like in Indian case, the first quarter will be April to June, April, May June, so that is a quarter the first quarter, then there will be second quarter or q2.

Likewise, there will be four quarters, but whatever it is, we are talking about a period of time in this case. A time is not a continuous variable, it can be divided into small small, irreducible periods, it could be a quarter, if we are talking about quarterly GDP or it could be annual GDP. So, one does not talk about, for example GDP of a second that is absolute. So, that basically means that here the GDP that we are talking about is measured with respect to time, but that time is discrete time.

In this module, we are going to consider time in discrete units and analyze how y not necessarily equal to GDP. It could be some other variable for example prices, how y changes with time and it is a dynamic analysis, because we are looking at the behavior of a variable with respect to time.

(Refer Slide Time: 06:54)



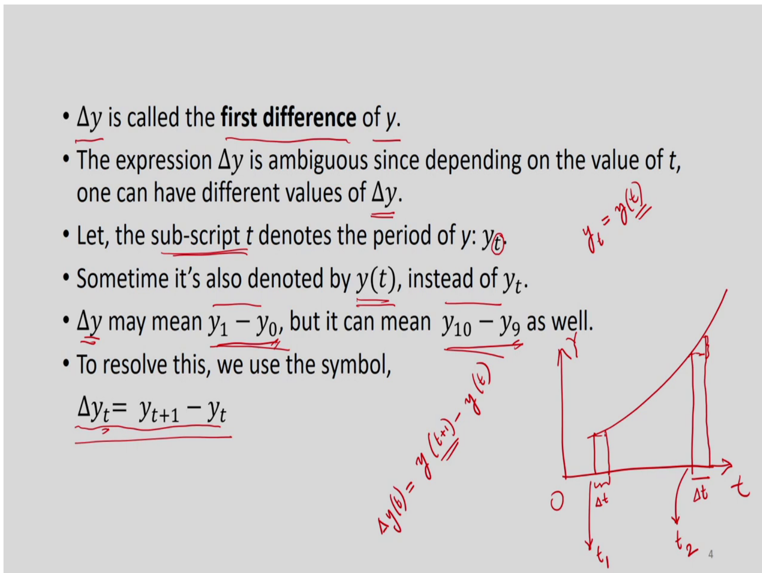
Since time is discrete, the values it takes are called periods, they are not points of time, they are not points, but there are periods of time. So, t, that is time can take these values, whole numbers 0 1 2 3 like that, and each of these is called a period. Thus, the discrete version of economic analysis is also called period analysis. Since we are talking about periods not points and therefore, the analysis that we are going to carry out the economic analysis will be called period analysis.

Now, as time changes from one period to the next, the change is denoted by this notation Δt, Δ means change. So, Δt is the change in time in particular, what is important here is that we are implying that the time is changing in a discrete manner. Correspondingly the variable of interest y also changes and that change let us suppose that is given by Δy to maintain consistency, since change in time is denoted by Δt, the change in let us suppose the GDP is also denoted by Δy.

Now, we can look at the ratio of these two. The ratio of these two means $\frac{\Delta y}{\Delta t}$. So, what does it mean? It means the change in the variable of interest that is let us suppose GDP with respect to 1 unit change in the time. And this is called the difference quotient $\frac{\Delta y}{\Delta t}$, which turns out to be only Δy since the smallest change in t is 1. So, if Δt is by that we mean the smallest possible change in time and we know time is discrete. And so, the minimum possible change that can take place in time is 1.

So, $\Delta t = 1$. So, $\frac{\Delta y}{\Delta t}$ boils down to Δy, because here by Δt I mean the smallest change that is possible because time changes from one period to the next is equal to 1. So, $\frac{\Delta y}{\Delta t}$ is same as Δy.

(Refer Slide Time: 10:04)



Δy is called the first difference of y. The expression Δy is ambiguous since depending on the value of t one can have different values of Δy. Think about this so, suppose this is how y is changing, y and here you have time. Now, if you take this change in t so, this is Δt which is equal to 1 then here you can see the change in y is very small whereas, if you take the same change in time here, here the change in the variable is quite large compared to here.

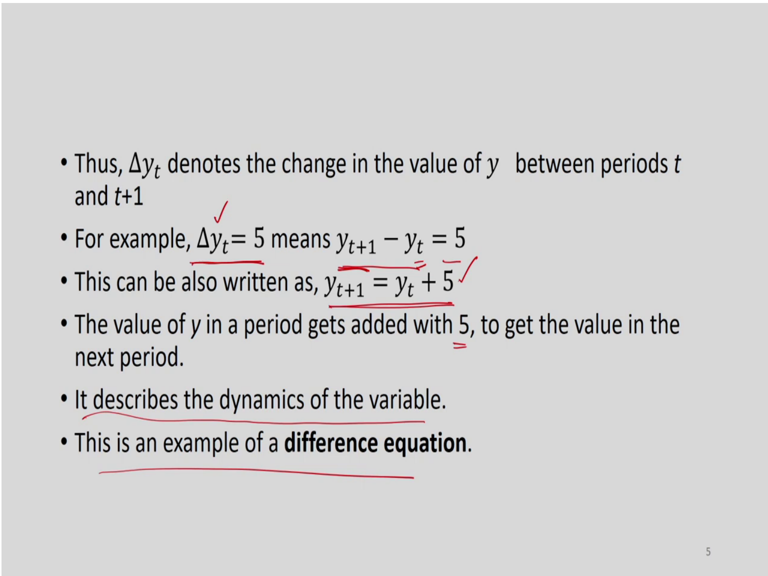
So, Δy is an ambiguous expression. It can change depending on what t we are considering. What is our reference t? So, to get around this problem of what is the value of Δy in a very concrete manner we use the subscript t. And that denotes the period of t. So, what is the reference point, are we talking about this t, this could be t_1 or are we talking about this t which is t_2. So, to denote the t that we have in mind we use this small subscript here.

So, y_t so, that will give us the reference point of time. Sometimes it is also denoted by $y(t)$. So, instead of writing this we can also write y which is dependent on t. Both of these things mean the same. Now, Δy may mean $y_1 - y_0$ or it can mean $y_{10} - y_9$ as well. In both cases time is changing by just 1 unit and we are looking at how much the y variable is changing. But as you can see how much y variable is changing might be quite different depending on what original t or the reference t we are considering here the reference t was 0, here the reference t is 9.

So, as you can see in the picture that I have drawn depending on where the t is 9 or t is 0, these values might be quite different. So, this is the thing that we use that Δy_t, what is meant by Δy_t? $\Delta y_t = y_{t+1} - y_t$. So, the change in time is taking place by 1 unit with respect to that the change in y is taking place. But from what t time is changing here that is denoted by this small subscript.

So, in case if we take this bracket expression it will be like this, $y(t + 1) - y(t)$. So, that would be $\Delta y(t)$ had we used the bracket expression. But in this module at least we are not going to use the brackets, we are going to use the subscript. So, this is a matter of convention and we are going to stick to this convention that we are going to use $\Delta y_t = y_{t+1} - y_t$.

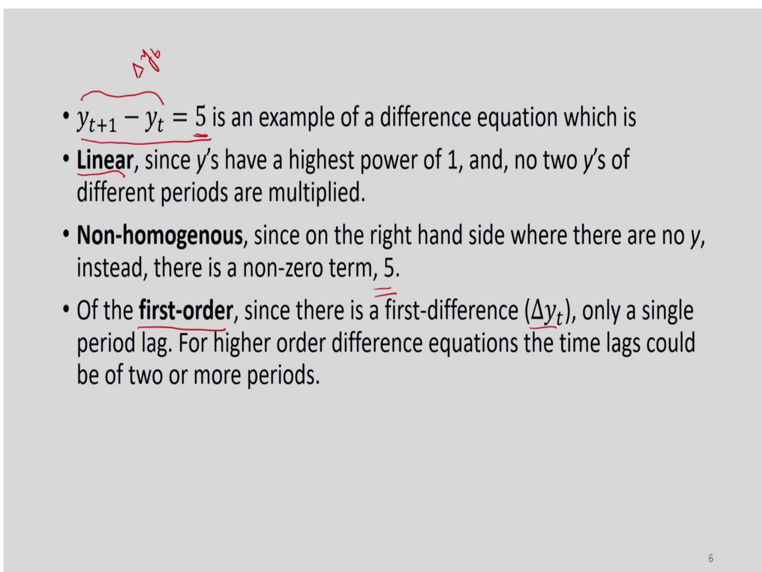
(Refer Slide Time: 14:34)



So, Δy_t denotes the change in the value of y between periods t and t plus 1. So, let us take an example. So, suppose $\Delta y_t = 5$ that means, if we are starting from point t and we are taking t plus 1, then this gap $y_{t+1} - y_t = 5$. And notice this can also be written as $y_{t+1} = y_t + 5$. The value of y in a period gets added with this number 5 to get the value in the next period. So, that is what it basically means.

In that same way, when we say delta $\Delta y_t = 5$ that means the same thing that if we want to get the value of y in the next period, then we take the value of y in this period and add 5, and that will give me the value of y in the next period. This describes the dynamics of the variable, this variable y. And this is an example of a difference equation. So, this is a difference equation, $\Delta y_t = 5$. But this is also a difference equation because both these two things mean the same $y_{t+1} = y_t + 5$.

(Refer Slide Time: 16:09)



$y_{t+1} - y_t = 5$ is an example of a difference equation, which is linear. Why is it linear because y's have a highest power of 1 and no two y's of different periods are being multiplied here. So, look at the power of the y's, there is no y^2 or y^3 or even $\sqrt{y}$. So, the only power of y's that is there is 1 and one can see that the 0 is also there because $y^0 = 1$. So, this is the reason why we are saying it is a linear equation and also it is the case that if we take the y's of two different periods, then they are not multiplied together.

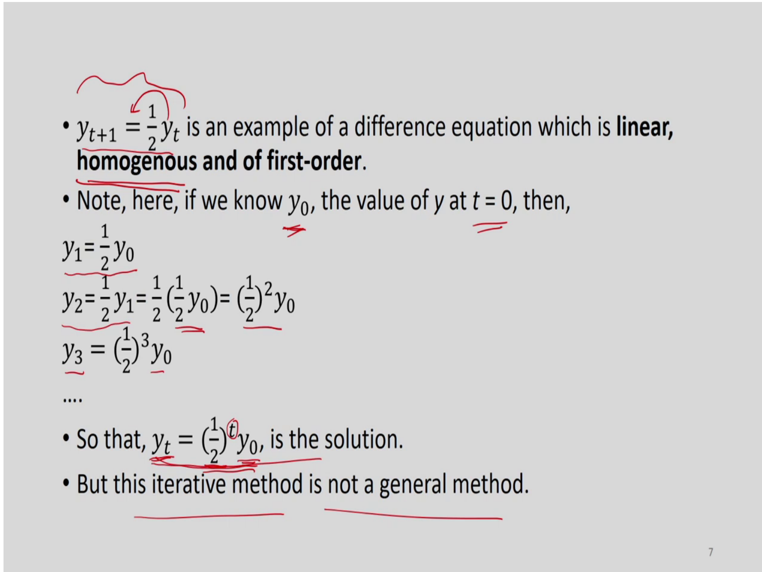
So, these conditions are being satisfied here and so we are calling it a linear difference equation. The second comment about the typology of this difference equation is that it is a non-homogeneous difference equation. And the reason for that is that on the right hand side that is here where there are no y, what we have is a nonzero term 5. So, had we have a 0 here instead of 5 then we would have called this a homogeneous difference equation.

But since we have a nonzero constant on the right hand side. So, what is special about the right hand side here? The special thing about the right hand side here is that on the right hand side there are no y's. And if there are no y's we have a nonzero constant which makes this difference equation a non-homogeneous difference equation. Now, the third point and this is very important, this difference equation is of first order, what is meant by first order?

Since there is a first difference that is Δy_t only a single period lag is there. So, look at this, this left hand side expression, it is $y_{t+1} - y_t$ and this is nothing but Δy_t. So, there is just one period lag and that means that this difference equation is a first order difference equation. For higher order difference equations, the time lags could be of two or more periods. So, we are going to talk about higher order difference equations as well after we have discussed the first order difference equations.

But for the time being, let us try to summarize this difference equation that we have here is a linear difference equation it is a non-homogeneous difference equation and it is a difference equation of the first order.

(Refer Slide Time: 19:34)



Here is an example of a difference equation which is linear and a first order but homogeneous. So, this is the new thing that is there in this difference equation which was not there in the earlier one. And look at the form it is saying that $y_{t+1} = \frac{1}{2}y_t$. So, you can see that there is no constant term here. So, if I take y_t to this side then it will just become $y_{t+1} - \frac{1}{2}y_t = 0$.

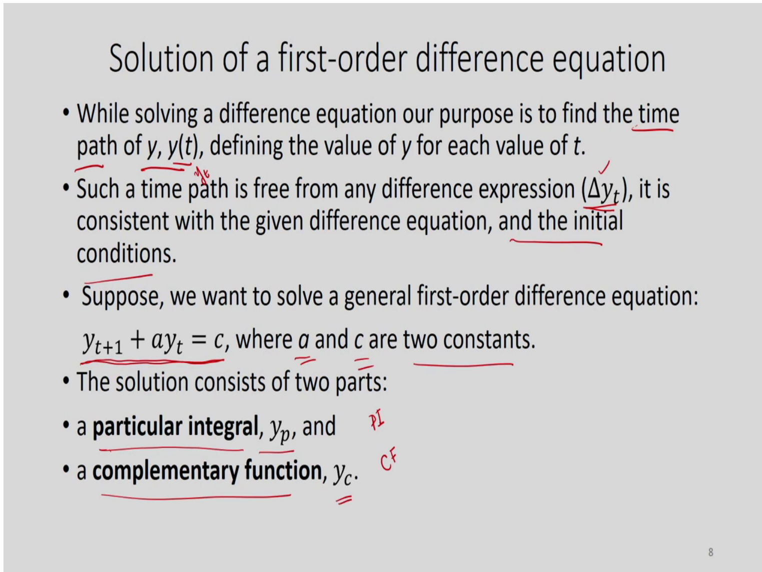
So, there is no constant term on the right hand side and that allows us to conclude that this is a homogeneous difference equation. The other two criteria, that is linear and first order, those two criteria are being maintained like in the previous example. Now, actually this particular equation can be easily solved. Here suppose we know y_0 what is y_0, y_0 is the value of this variable when time is 0. What is the speciality about that?

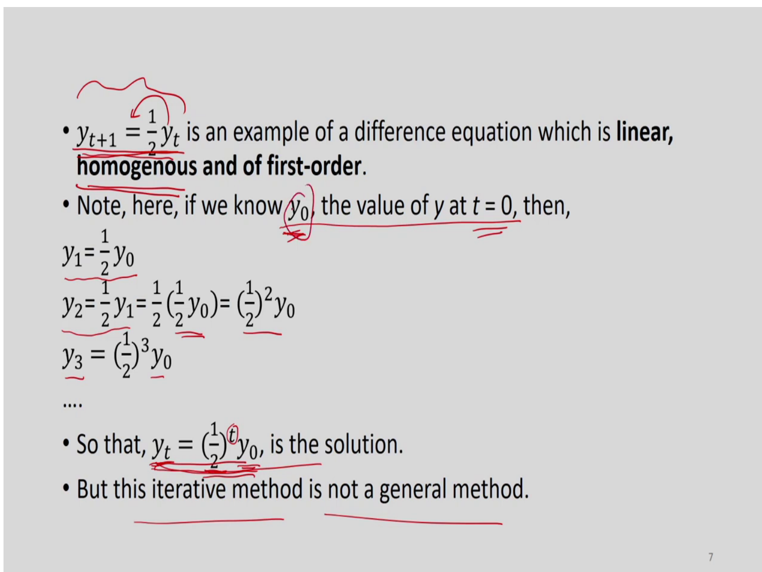
Time is 0 is the point of origin from which time everything is starting t_0 then it could be t_1 t_2 like that, time cannot be negative. So, t_0 is the point of origin if I know the value of y at that point of time that is suppose it is y_0 then actually I can do the following from this equation I can immediately see that y_{t+1} that is y_1 will be $\frac{1}{2}y_t$ that is in this case y_0. So, I know $y_1 = \frac{1}{2}y_0$ similarly, $y_2 = \frac{1}{2}y_1$.

But I already know y_1 so, I substitute that here and that gives me y_2 as a function of y_0 it is $y_2 = (\frac{1}{2})^2 y_0$. Likewise, $y_3 = (\frac{1}{2})^3 y_0$. So, you can see there is a pattern that is coming out from this series and in general one can write y_t that is the value of y at period t is $y_t = (\frac{1}{2})^t y_0$ and this is the solution of this difference equation. So, this method is the iterative method of finding y_t.

Finding y_t as a function of the constants and time. So, you can see here that on the right hand side, there is no term involving y_t or y_{t+1} or y_{t-1} the only thing we have is a constant y_0 and here is also another constant, $\frac{1}{2}$ and time. So, this pattern that we are getting here will be called a solution. It will basically give me the value of y at any arbitrary point of time. So, if I am given the value of t, I can immediately tell you what is the value of y_t I will just put that t here and I know what is y_0. So, I will get the value of y_t. But this iterative method cannot be applied everywhere. It is not a general method.

(Refer Slide Time: 23:28)





Here is the general method, Solution of a first order difference equation. While solving a difference equation, our purpose is to find the time path of y. So, this is a new term I am using: time path and this is denoted by $y(t)$. And we have seen that $y(t)$ is also written as y_t. It defines the value of y for each value of t. That is what I have just said, that if I am given the value of t from the solution, I can give the value of the y that is the variable. Such a time path is free from any difference expression.

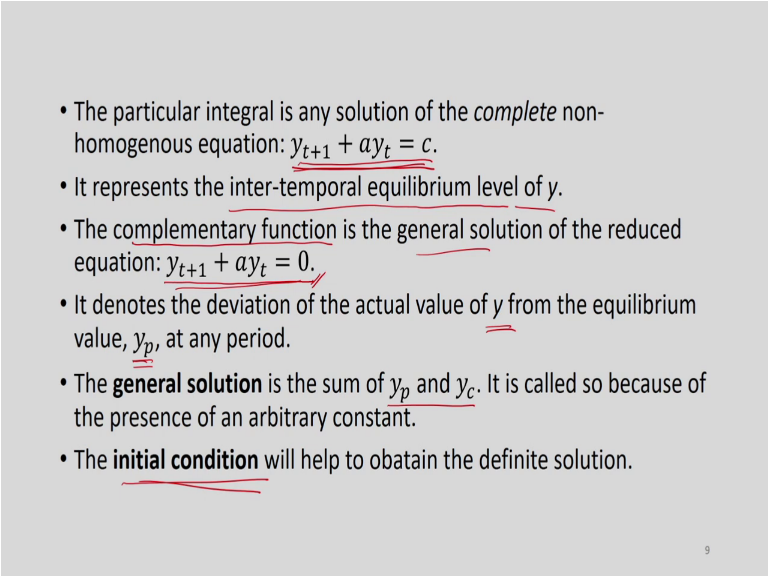
So, it does not involve this kind of expression Δy_t it is consistent with the given difference equation. So, let us take this example. This solution that we got here is consistent with this given difference equation. You can just check that so if suppose I have to find out what is y_1, $y_1 = \frac{1}{2}y_0$. And that is consistent with this general form. Here also $y_1 = \frac{1}{2}y_0$ and the initial conditions what is meant by the initial conditions the initial conditions is this is an example of an initial condition that at $t = 0$, I know what is the value of y, it is given by y_0.

So, that is an initial condition. So, the time path or the solution of the difference equation has to satisfy these criteria. Firstly, it should not include an expression like this Δy_t it is consistent with the difference equation that is given to us and thirdly it should satisfy the initial condition. Suppose we want to solve a general first order difference equation of this form $y_{t+1} + ay_t = c$ where a and c are two constants.

So, these are given to us we know the value of a we know the value of c and the difference equation is this $y_{t+1} + ay_t = c$. We have to solve this. And as you can see, it is quite general this expression because we are having a non-homogeneous difference equation if we put $c = 0$, it becomes a homogeneous difference equation. But that will be a special case and lag is just 1. So, that means it is a first order difference equation and it is linear.

Now, the solution of this difference equation will consist of two parts, the first one is called the y_p or the particular integral, sometimes it is also written as PI and the second part is y_c and it is called the complementary function and it is sometimes denoted as CF. So, the solution will be a combination, a summation of these two things y_p and y_c.

(Refer Slide Time: 27:33)



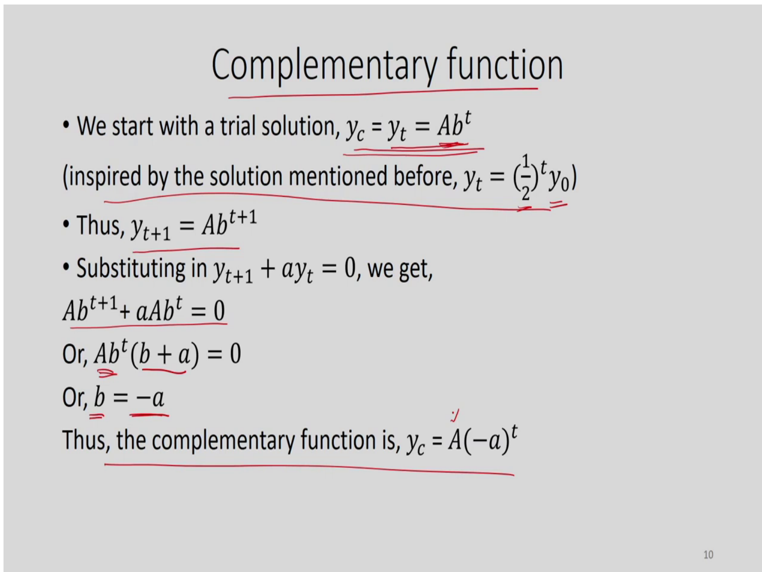
The particular integral is what? The particular integral is any solution of the complete non-homogeneous equation. So, this is given to us, the particular integral is any solution to this equation. It represents the intertemporal equilibrium value of y. Intertemporal equilibrium value of y, since y that this particular integral is going to satisfy this. So, we can say that this solution that we are talking about is an equilibrium value.

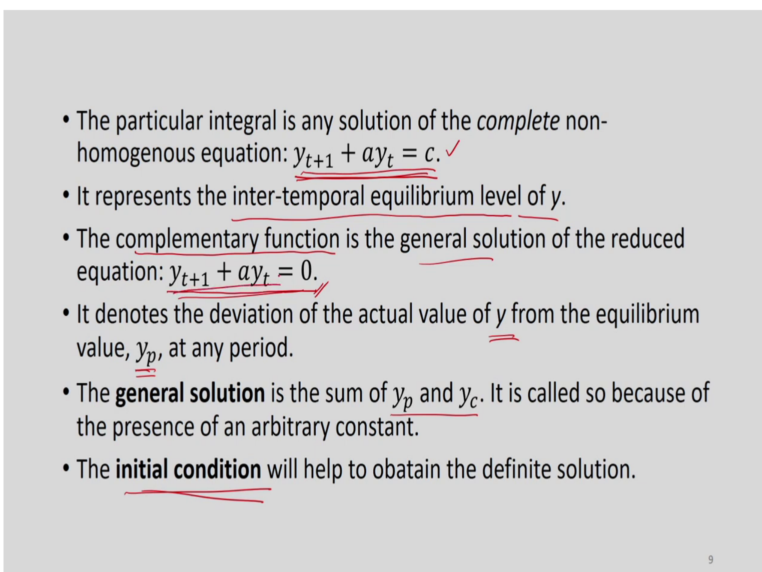
What is the intertemporal ness of this that we are going to find out shortly. The second part that is the CF, the complementary function is the general solution of the reduced equation, this $y_{t+1} + ay_t = 0$. So, notice the difference between these two here, the constant term has been gotten rid of, so it is a homogeneous equation now. So, the CF or the complementary function is the general solution to this homogeneous part of the difference equation.

Now, what does the CF denote? It denotes the deviation of the actual value of y from the equilibrium value and what is the equilibrium value that is y_p at any period. Equilibrium is not only equilibrium, it is the intertemporal equilibrium value. y_p is the intertemporal equilibrium value of y. The general solution is the sum of y_p and y_c. This we have just mentioned, the solution is the summation of these two parts y_p and y_c.

It is called so, because of the presence of the arbitrary constant, we are going to talk about that, that in the general solution, an arbitrary constant will be there and the initial condition we have talked about initial conditions before that will help us obtain the definite solution. At this point of time, it might seem a little bit abstract a little bit vague what we are talking about. So, let us try to solve a particular equation and try to see concretely what we are trying to say.

(Refer Slide Time: 30:14)





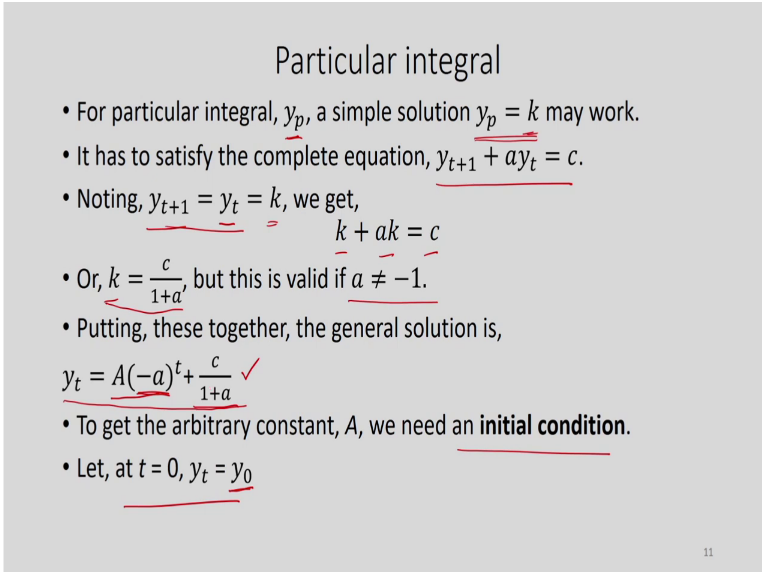
For this difference equation, which is the general form this is the general form, let us try to find out the complementary function and the particular integral and try to see what the solution look like. We start with a trial solution. So, we are trying to find a complementary function of this equation. We try the solution of this form Ab^t, where A and b are constants. Now, why did we take this particular form, this is basically coming from this solution that we have just done.

So, here you can see there is a constant term here, and then it has been raised to the power t and there is a constant term multiplied with that. So, that is the form that we are using here to get the complementary function. Now, if $y_t = Ab^t$, then $y_{t+1} = Ab^{t+1}$. So, I know y_t, I know y_{t+1} and then I will substitute these in this homogeneous equation. Now, if I do that, then I get this and I can take Ab^t common on the left hand side.

So, within the brackets I will get $Ab^t(b + a) = 0$. Now, $Ab^t \neq 0$, I am assuming b and t they are not 0. So, this will not be equal to 0. So, the other possibility is that this part is 0, $(b + a) = 0$ and that gives me the solution for b, remember b was something which we do not know, I just assumed that it is a constant nonzero constant. So, I know $b = \ - a$.

So, that the complementary function is $y_c = A(- a)^t$. Now, remind you still we do not know the value of capital A, this was an arbitrary constant. We have to get hold of that at a later point of time.

(Refer Slide Time: 32:38)



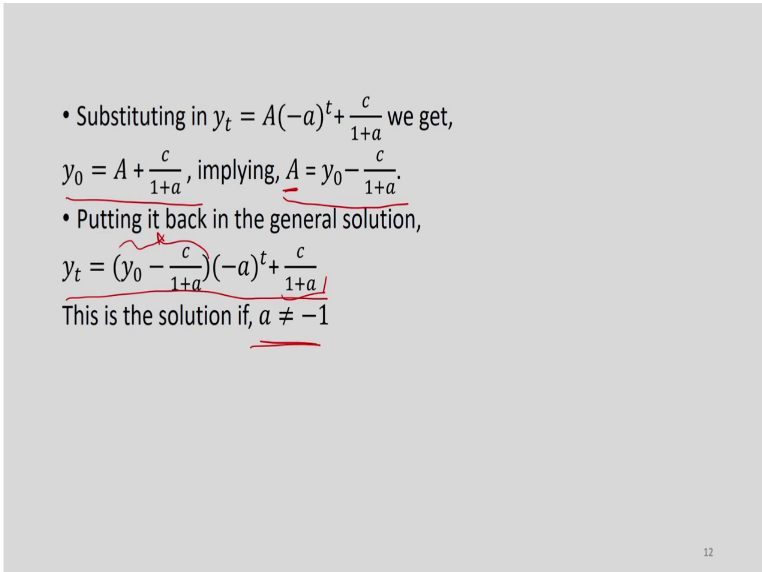
Now, PI is still left out, that is particular integral. Let us suppose it is given by y_p and we try a simple solution, remember a particular integral is any solution which solves the given difference equation. So, I take the simplest possible solution that can happen, that, I take $y_p = k$, k is a constant. Now, if k is constant then y_{t+1} and y_t both of them will be equal to k, $y_{t+1} = y_t = k$. Because k does not change with respect to time and I substitute these things in this equation.

So, what do I get? $k + ak = c$. So, that means that $k = \frac{c}{1+a}$. So, this is the solution that I am getting here of the particular integral. But mind you this can be a solution, this can be a particular integral if the denominator is not 0, $\frac{c}{1+a}$, $1 + a$ cannot be 0 that means, a cannot be equal to minus 1. So, this is something which we have to keep in mind. Now, putting all this together I get the general solution, this is the general solution y_t is a summation of the particular integral and the complementary function.

This was the complementary function $y_t = A(-a)^t + \frac{c}{1+a}$, this is the general solution. Now, we do not know the capital A. To find that out, we need an initial condition. So, suppose the initial condition is this that at $t = 0$, $y_t = y_0$.

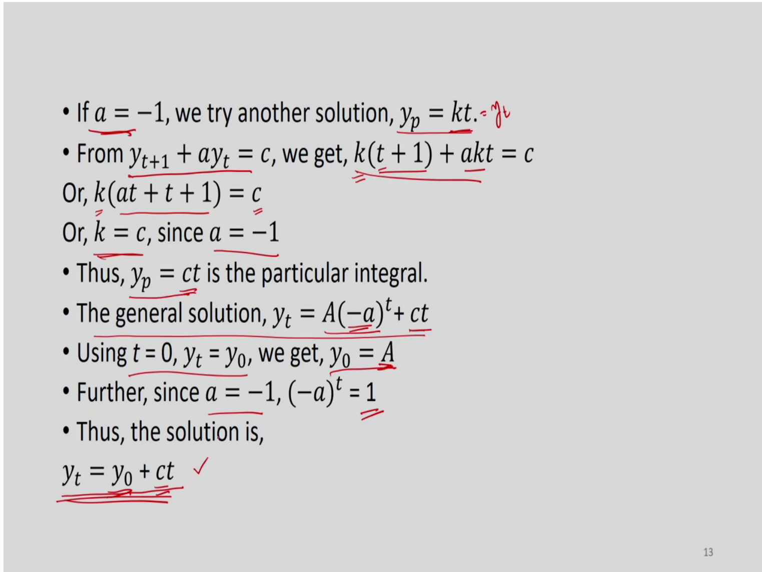
So, I use this information in this equation here. So, if I do that, think about what will be the outcome of that, on the left hand side, I am going to get y_0, I am putting $t = 0$. So, on the left hand side I am going to get y_0 and then equal to this will become 1, this becomes 1, because $(-a)^0 = 1$. So, this becomes $y_0 = A + \frac{c}{1+a}$.

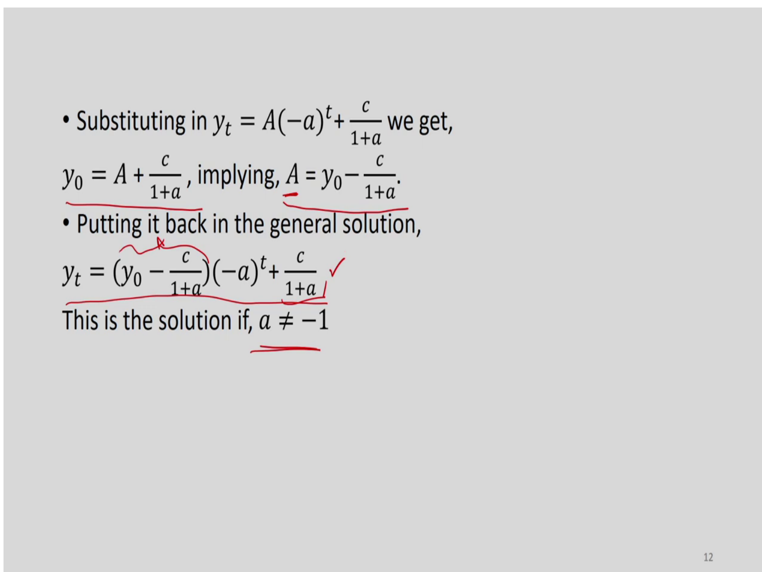
(Refer Slide Time: 35:19)



$y_0 = A + \frac{c}{1+a}$ and that gives me the value of A. So, the arbitrary constant has been gotten rid of. And so, I substitute this A back to the general solution and I get this expression $y_t = (y_0 - \frac{c}{1+a})$ this was the A multiplied $(-a)^t$ plus the particular integral which was $\frac{c}{1+a}$, $y_t = (y_0 - \frac{c}{1+a})(-a)^t + \frac{c}{1+a}$. The only thing here to note and which is a point of discomfort is that this solution is valid if $a \neq -1$. But obviously, we cannot a priori say that $a \neq -1$. So, we have to cover that case as well, what happens if $a = -1$.

(Refer Slide Time: 36:25)





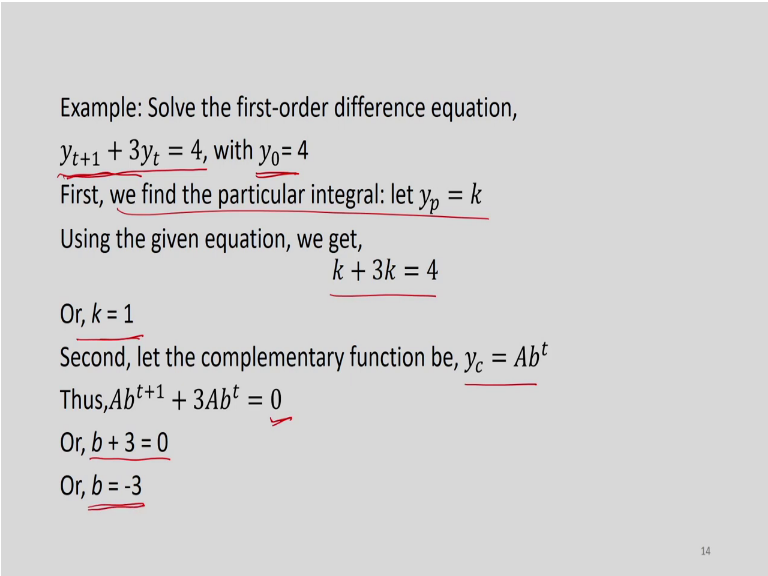
If $a = -1$, then our particular integral has to be different. So, instead of $y_p = k$ that we assumed earlier here we are taking this particular trial solution $y_p = kt$. So, it is a function of t now, a little bit more complicated than what we had before and let us try out whether this works out or not. So, this y_p, which is basically equal to y_t we substitute that here.

And if we do so, then what we are going to get instead of y_{t+1} I write $k(t + 1) + akt = c$ and from the left hand side I can take$k$ common and within the brackets I have $k(at + t + 1) = c$, from here I will get at then $t + 1$. On the right hand side I have only c. But if $a = - 1$, here, I can write $- 1$, then minus t and plus t will get cancelled. So, I am just left with $k = c$.

So, this is the solution of the particular integral, it is $y_p = ct$. So, the general solution in this case is $y_t = CF + PI$, CF is $A(- a)^t + ct$ the particular integral. But again, I have to get rid of A. So, I again use the initial condition if I do so. So, therefore, I will get, if I put $t = 0$ this part vanishes. So, $A = y_0$.

Further, since $a = - 1$, so this part just becomes 1. So, $(- a)^t$ becomes 1. So, the solution actually is a very simple looking solution it is $y_t = y_0$, y_0 is coming from this A part, complementary function plus the particular integral which is ct. So, this is the solution here in the case, where $a = - 1$. So, actually, you have two kinds of solution, this is the solution if you have $a \neq - 1$ and this is the solution where $a = - 1$.

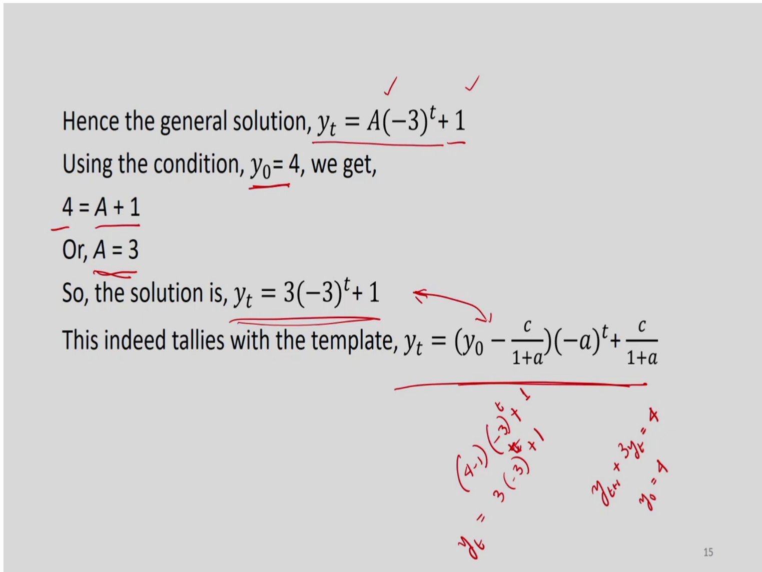
(Refer Slide Time: 39:25)

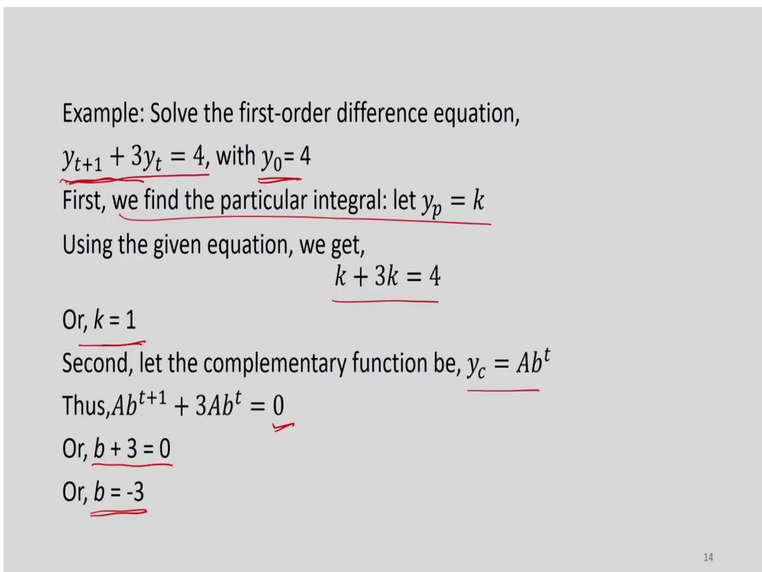


Here we are having an example. Solve the first order difference equation, this $y_{t+1} + 3y_t = 4$. The initial condition is given that the $y_0 = 4$. First let us try to find out the particular integral. So, y_p that is the particular integral let us suppose it is constant k. So, we substitute that here, k is not dependent on time. So, y_{t+1} and y_t both of them are equal to k. So, therefore, I have $k + 3k = 4$ or $k = 1$.

This is the particular integral as far as the CF is concerned I am assuming $y_c = Ab^t$ and that I substitute back to this equation on the right hand side I do not take 4, I take 0, the homogeneous equation and thus we get $Ab^{t+1} + 3Ab^t = 0$ and that simplifies to $b + 3 = 0$ or $b = -3$.

(Refer Slide Time: 40:46)



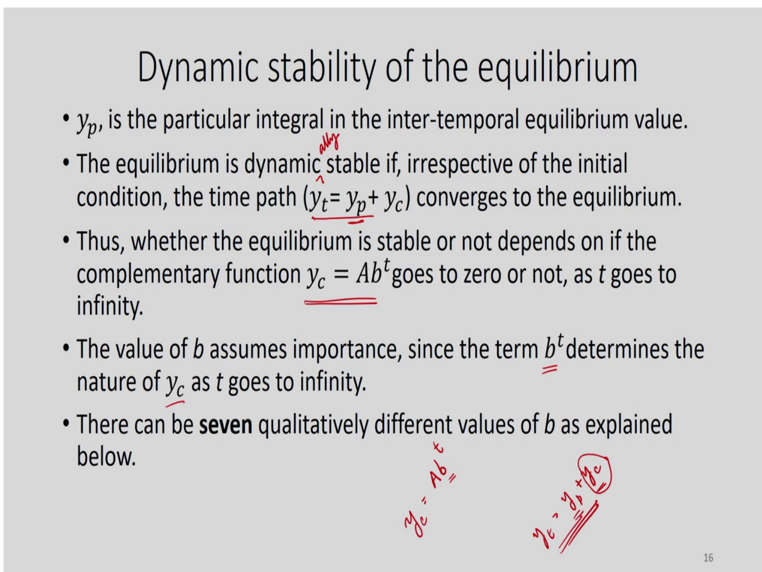


Therefore, the general solution is this $y_t = CF + PI$. $y_t = A(-3)^t + 1$ and I have to use the initial condition which was $y_0 = 4$, that is, if you take $t = 0$ then on the right hand side you are going to get A.1 because $(-3)^0 = 1$ plus 1 that is A + 1 and on the left hand side $y_0 = 4$ that is 4. So, from here I get $A = 3$.

So, I get the solution to be this $y_t = 3(-3)^t + 1$ and if you have noticed what was the general template of the solution of first order difference equation it was this, and this actually matches with this, this and these two are tallying with each other because in this case, this particular example that we have it is $y_{t+1} + 3y_t = 4$ that is the form that is given to us and $y_0 = 4$.

So, in this case $A = 3$ and $c = 4$. So, $A + 1 = 3 + 1$. So, that will be 4 and $c = 4$. So, $\frac{4}{4} = 1$ and therefore, you have after that $(4 - 1)(-3)^t + 1$. So, this is nothing but $3(-3)^t + 1$. So, this is the y_t predicted by this template and that is exactly what we have found out by our more elaborate method.

(Refer Slide Time: 43:30)

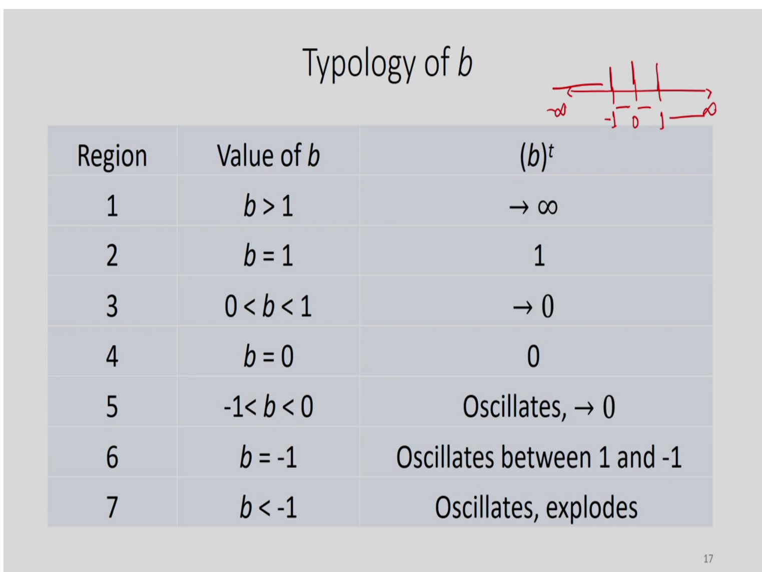


What about dynamic stability of the equilibrium? Now, when we say dynamic stability of equilibrium let us try to define what we mean by dynamic stability. Now, in this time path that is $y_t = y_p + y_c$ this is what we have seen before, y_p is the particular integral it is the intertemporal equilibrium value. So, irrespective of the time this gives me the equilibrium value, this part.

The equilibrium is dynamically stable if irrespective of the initial condition, the time path, this converges to the equilibrium. So, y_t if it converges to y_p irrespective of where we are starting from that is where the y_0 is. If y_t converges to y_p then we say that the equilibrium is dynamically stable. Thus, whether the equilibrium is stable or not depends on if the complementary function that is the other part y_c, which we know $y_c = Ab^t$ goes to 0 or not, as t goes to infinity.

So, if this part slowly gradually drops out it dies down then y_t converges to y_p. And in that case, we are saying that the equilibrium is stable. Now, the value of b assumes importance in this context, since the term b^t determines the nature of y_c as t goes to infinity. Now, that is obvious, you see, $y_c = Ab^t$, t is going to infinity and it is becoming larger and larger. So, what is the value of b that will determine whether y_c slowly drops out or not. So, the value of b, that constant is of critical importance. Now, there can be several qualitatively different values of b as explained below.

(Refer Slide Time: 46:09)


Typology of b

Region	Value of b	$(b)^t$
1	$b > 1$	$\to \infty$
2	$b = 1$	1
3	$0 < b < 1$	$\to 0$
4	$b = 0$	0
5	$-1 < b < 0$	Oscillates, $\to 0$
6	$b = -1$	Oscillates between 1 and -1
7	$b < -1$	Oscillates, explodes

So, different types of b that can occur. So, this is called the typology of b, there can be seven possible regions from which we can take its value. I am just going to introduce this and maybe we can discuss it in more detail in the next lecture. So, b can be greater than 1. So, think about a straight line, the real number line here you have +1, here you have -1, $+\infty$, $-\infty$,. So, value b can be here, more than 1, it can be exactly equal to 1, it can be between 0 and 1, it can be at exactly 0. It can be between 0 and -1. It can be equal to -1, or it could be less than -1.

So, these are the seven distinct possibilities of b. And each of them might give us different kinds of stability. And that is what I have commented on in the third column here. I shall take it up in the next lecture and there I can carry it forward further. And so, let me stop here for the time being. I shall see you in the next lecture. Thank you.