Mathematics for Economics - I
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Module: Integration 2
Lecture 27
Oil extraction, income distribution, PDV

Hello, and welcome to another lecture of this course Mathematics for Economics Part I. Now, the module that we have been covering is of integration. So, we have talked about how integration can be interpreted as an area under curve and we talked about indefinite integrals, we also talked about definite integrals and right now we are talking about certain exercises where integration and its tools can be used.

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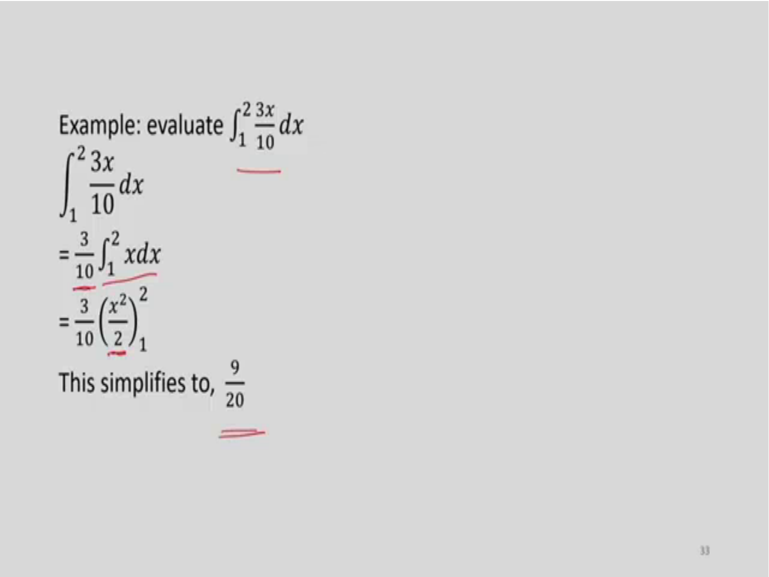

And here on your screen you can see one particular exercise. So, this is an exercise of definite integral. The question is evaluate integration 1 to 2 and the integrant that is the function we want to integrate is 3x/10. Now, how to do that? As we have seen earlier, if there is a constant in the integrant that can be taken out. So, here the constant is 3 /10 we take that out. So, inside you just have x. And integration of xdx as we know its $x^2/2$ and limits are 2 and 1. So, essentially what we are doing is 3/10 multiplied by half of 2 square minus 1 square and that essentially boils down to 9/20. This was a simple problem of definite integral.

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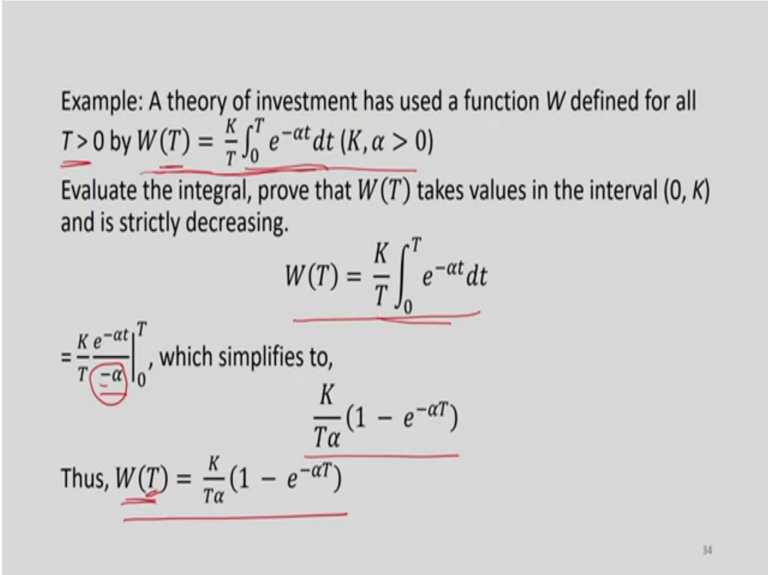

Here is a problem from finance and economics. A theory of investment has used a function ca W defined for all capital T by WT .That is, W(T) is equal to K/T and $\int_0^T e^{-\alpha t}dt$. Here K is a given constant, positive α is also a given constant positive. What we need to do, evaluate the integral and prove that W(T) takes values in the interval 0 to K and is strictly decreasing. So, there are three parts of this problem.

Now, let us try to understand what is this about. So, W(T) in some sense measures the value of investment and how this investment is measured, it is, remember it is a function of capital T. Capital T is the total time over which the investment will give some return. So, that is the total time span. That is the interpretation of capital T. It is so that W(T) is a function of capital T, but there are some complications here. There is an integral here, definite integral. And you have $e^{-\alpha t}$ which has to be integrated over the interval 0 to T.

We shall later on see that this particular thing $e^{-\alpha t}$.integration of that has an interpretation of present discounted value, but that will come later on. The first thing that we have to do is that we have to evaluate this. W (T) is $\frac{K}{T}\int_0^T e^{-\alpha t}dt$. Now, this is not very hard to integrate. Basically you have an exponential function whose power is -αt . α is a constant. So, we know how to do that.

It is $e^{-\alpha t}$ divided by the coefficient of t and the coefficient of t here is -α. So, that is coming in the denominator. And in this context K and T they are sitting outside, because they are not functions of small t. And we have to take the limits value, at the limits and $-e^{-\alpha T}$ minus of $e^{-\alpha}$ multiplied by 0. And you have to be careful that there is a negative sign here. So, this basically boils down to this, $\frac{K}{T}$ α (1-$e^{-\alpha T}$).

As you can see, this expression is a function of capital T. That is why you have W as a function of capital T on the left hand side. So, the first part is done. Now, what we need to show is that W(T) takes the values in the interval 0 to capital K. What is the range of capital T? Depending on the range of capital T, W will take different values.

Now, we have been given this information that capital T is not negative. It always takes a positive value. So, it basically varies between 0 and plus infinity. So, we can take these two limits and try to see how the value of the function changes, functions means W(T), how W(T) changes as capital T takes these two extreme values.

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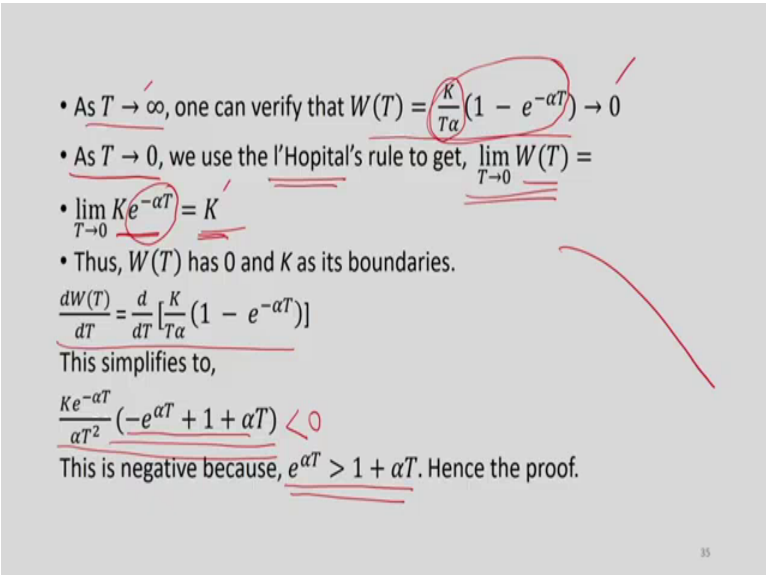

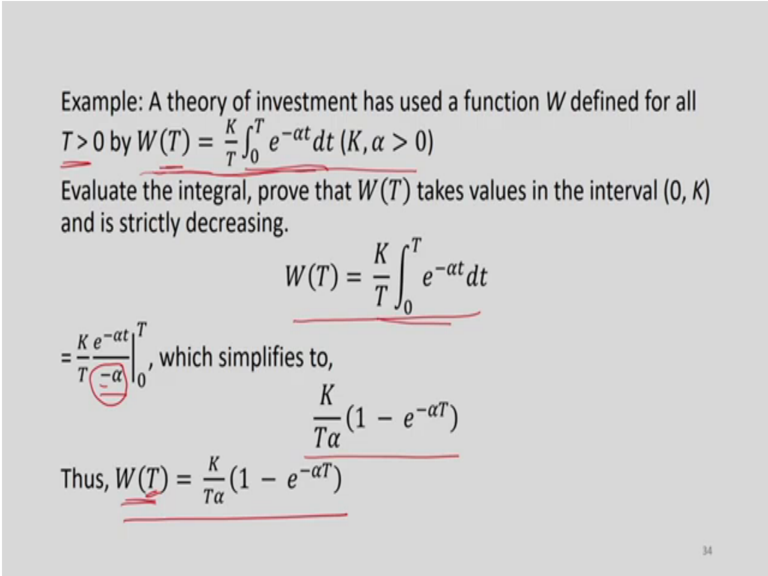

That is what we have done here. As capital T goes to infinity, that is positive infinity, this is the form of the function. You can straightaway see that K/T this term goes to 0. So, the whole expression will go to 0. So, capital W(T) goes to 0 as capital T goes to infinity. Now, that is one extreme. What about the other extreme? The other extreme is suppose capital T approaches 0, the minimum value, we know that capital T is positive. So, it can go as close to 0 as possible from the right hand side.

Now, so we have to evaluate this to be more precise T tends to 0 plus. Now, this we can evaluate by applying the L'Hopital's rule. So, we differentiate both the numerator and the denominator in this expression with respect to capital T. And if we do so, basically, W(T) becomes this expression, which is K$e^{-\alpha T}$.

Now, we can take the limit. And if we do that, as T goes to 0, then this term becomes 1. It goes to 1 because e^{0} is 1. So, the value of the function approaches capital K as T goes to 0. And that was our second task. Because what was the second task saying that this function takes value from this range 0 to K. That is what we have shown here. If T goes to 0, the value of the function approaches K. And as T goes to infinity, the value of the function approaches 0.

Now, we are also given the information that K is positive and as we can see that if K is positive then as T increases, it seems that the function is declining in value. It started from K which is positive and it goes to 0 as T goes to infinity. Now, we are not sure whether it is a monotonically decreasing function or not. For that, to show that, and that is the third task actually, that it is strictly decreasing. So, for that I have to take the derivative of the function that is derivative of W(T) function with respect to capital T.

So, that is what we have done here and we simplify this expression and simplify to this $\frac{Ke^{-\alpha T}}{\alpha T^2}(-e^{\alpha T}+1+\alpha T)$. Now, this expression turns out to be negative. The reason being, $e^{\alpha T}>1+\alpha T$. So, this is true, $e^{\alpha T}>1+\alpha T$, which basically means that this expression is negative and that essentially proves that the function that is W(T) is declining in T. So, that was our third task.

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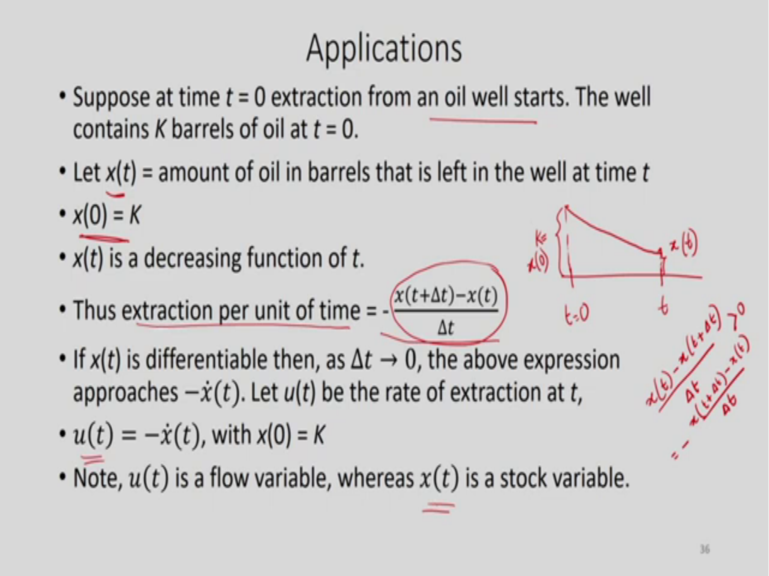

Now, we come to some applications of integration, in particular, definite integral. We will be concentrating more on the applications of definite integrals. Suppose at time 0, t is equal to 0, extraction from an oil well, so this is a problem of extraction of oil from an oil well, it starts extracting at time 0. The well contains capital K barrels of oil at that point of time. That is, at t is equal to 0 the oil well contains capital K barrels of oil.

Let us suppose x(t), so it is a function of small t is the amount of oil in barrels that is left in the well at the time small t. So, think about this in terms of a diagram. Here you have t is equal to 0. Now, here how much oil is there? Suppose this is capital K. Now, and take another time let us suppose a generically small t is the time, now here what is the amount of oil that is left let us denote that by x(t). So, this is the value of the oil or the amount of oil that is left in that oil well as let us suppose this is denoted x(t).

x(t) is a decreasing function of t. How much oil will be left in the oil well that goes on declining, because extraction is taking place at each point of time. To begin with when time was 0, the total amount was K. So, I can write K = x(0). So, you can think of this total amount of oil left in the well to be a declining function of time. What is the extraction per unit of time? So, how much oil is being extracted per unit of time that can be written as - $\frac{x(t+\Delta t)-x\,(t)}{\Delta t}$.

Why is that? The reason is how much oil is extracted per unit of time. If I assume this extraction rate of extraction to be positive, then notice $\frac{x\,(t)-x(t+\Delta t)}{\Delta t}$, because with time the value of the x(t) is declining, so this term will be positive. So, this is going to be a positive expression. And so the rate of extraction can be given as minus of…

Now, if x as a function of t is differentiable, and as t goes to 0, the above expression approaches to minus of x dot t, because this is nothing but the Newton quotient. Now, let us denote u(t) as the rate of extraction at point t. Therefore, we can write this as u(t) is equal to minus of x dot t where x(0) = K. Note, u(t) is a flow variable, whereas x(t) is a stock variable. u(t) is the change in the stock of oil and that stock of oil is given by x(t). u(t) is a flow and x(t) is a stock.

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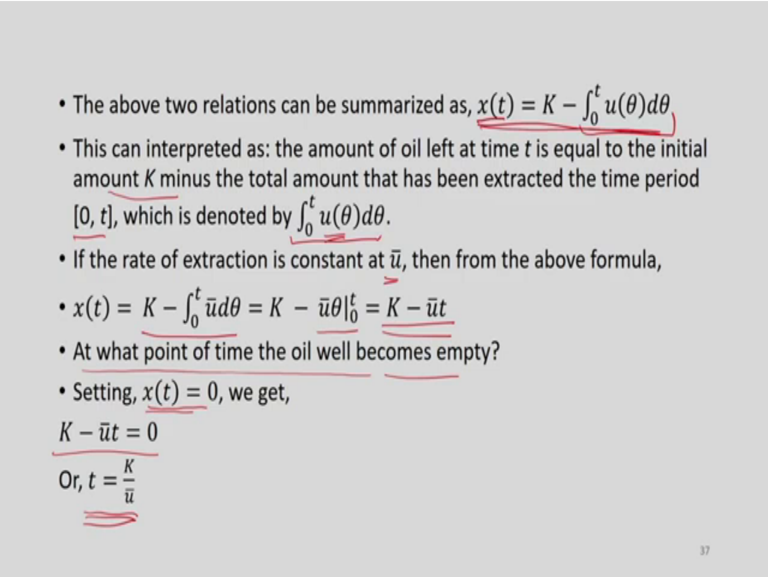

The above two relations can be summarized as follows. The x(t) or the amount of oil that is left in the well at point t is equal to K- $\int_0^t u(\theta)\, d\theta$. What is the interpretation of this? The amount of oil left at time, t that is x(t) , is equal to the initial amount which is K minus the total amount that has been extracted in the period 0 to t. So, that is given by this the total amount of oil that has been extracted from the well in the period 0 to t.

Remember, u(θ) is the rate of extraction at point θ. So, if I take the integration of 0 to t that will give me the total amount of extraction from the point 0 to the point t. So, we can take a very simple example where suppose the rate of extraction is constant and it is given by let us suppose $\bar{u}$, then we can play around with the above expression, this expression. So, in this case u(θ) is equal to constant. It is $\bar{u}$. So, I can replace that in this formula.

And if I do so, $\bar{u}$ will, it can be taken out. It is a constant. So, if I integrate 1, it becomes only θ. And so this becomes K- $\bar{u}t$. So, this is the total amount of oil left in the well after t amount of time. And this is quite intuitive, because at each time, at each point of time the rate of extraction is $\bar{u}$. So, after t point of time the total extraction will be $\bar{u}$t.

Now, one question that can be asked is that at what point of time the oil well becomes empty? Now, I can solve that by taking x (t) = 0 and solve for this t. We want to find the time in which the stock will deplete to 0. And the stock we know is x(t). So, that is why we are setting x(t) = 0. And I use this formula here. So, K- $\bar{u}t$ = 0 and that is equivalent to saying that t = $\frac{K}{\bar{u}}$.

This expression that we are getting here it is giving me the time at which the oil well will run dry. Now, this could have been done directly also, because the total stock of oil is K and per unit of time $\bar{u}$ is being extracted, which is constant. So, the time that will be required to extract all the oil from the oil well will be $\frac{K}{\bar{u}}$.

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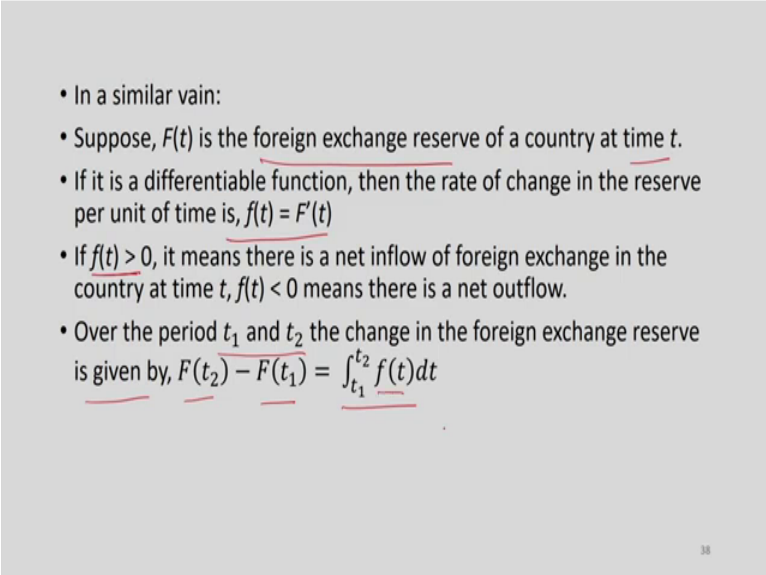

Now, this problem of oil extraction can be extended in a different context. So, suppose capital F as a function of small t is the foreign exchange reserve of a country at time t. So, what is foreign exchange reserve? Every country has a stock of currencies which are issued by other countries. So, for example, India will be having a stock of, let us say, US dollars. US dollars are the most accepted currency all over the world.

So, India whatever the foreign exchange reserve it has will be mostly denominated by US dollars. But there could be other currencies also like euro or other currencies like the Chinese currency or the Japanese currency. So, all those country's currencies will be held as stock by India and that stock is called the foreign exchange reserve. Likewise, other countries also have their own foreign exchange reserve.

Now, this F which is a function of t, so it basically means that with time the foreign exchange reserve can go on changing. If it is a differentiable function, then the rate of change in the reserve per unit of time is, let us suppose, given by f(t) which is equal to F'(t). Now, if small f(t) > 0, it means there is a net inflow of foreign exchange in the country at the time t. And if f(t)< 0 it means there is net outflow.

So, for example, if f(t) < 0 that means that foreign exchange reserve, which is F(t) is depleting. On the other hand, if f(t) > 0, then the foreign exchange reserve is accumulating in value. Now, notice, I said that this example is similar to the oil well example. It is similar, but not exactly the same.

In the case of oil well, the extraction goes only in one direction. So, as long as the rate of extraction is positive, the stock of oil that is there, which was x(t) will always be declining. But here, F(t) which is the stock variable here can increase if f(t) is positive. It can also decrease if there is a depletion of the foreign exchange which occurs if f(t) is negative.

Now, over the period t_1 and t_2 the change in the foreign exchange reserve is given by Ft_2-Ft_1 and that we have seen is it can be expressed in the form of an integration. It is a definite integral. So, the integration of t_1 to $t_2$and the integrant is f(t)dt.

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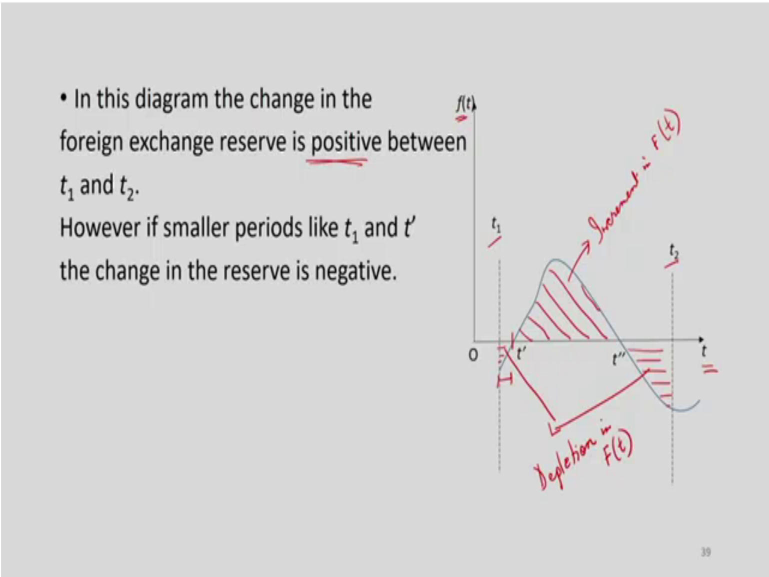

In this diagram, the change in the foreign exchange reserve is positive between $t_1$and t_2. So, in this diagram notice what we have done here along the horizontal axis I have time and along the vertical axis, mind you, I do not have F(t), I have f(t). And the area under the curve, remember that is the idea, the area under the curve, this curve that is f(t), if I take two limits, here suppose the limits are t_1 and t_2, then the area under this curve f(t) will denote how much the foreign exchange reserve is changing over this particular time frame.

Now, in this diagram I have drawn the diagram in such a way that this area which is above the horizontal axis, I have shaded that, is more than in value than these areas which are below the horizontal axis. So, the area above the axis, above the t axis these are the increments in the exchange that is foreign exchange reserves. And these are what, these are depletion.

Now, the picture has been drawn in such a manner that the total amount of increment is more than the total amount of depletion. This area, the slanted shades that region, is more than the horizontal shades region. So, that is why I am saying that the foreign exchange reserves change over t_1 and t_2 is positive. However, if smaller periods like t_1 and t' are taken then the change in the reserve is negative. So, that is quite obvious.

So, I take the period from t_1 to t' so this small period. Here you can see that there is this area which is below the t axis. And if it is below the t axis that means f is negative. And therefore, there is depletion in foreign exchange in this particular timeframe.

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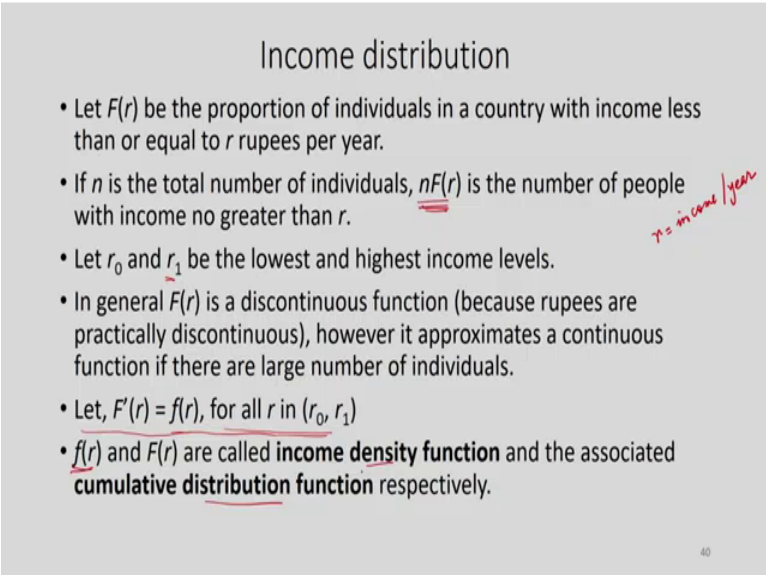

Now, we come to a third application of this idea of definite integrals. Let capital F(r) be the proportion of individuals in a country with income less than or equal to r rupees per year. So, r here is income per year and F(r) is the proportion of individuals in a country with income less than or equal to r rupees per year. So, to give you an example, suppose someone says that in India 60 percent of India's population have an income which is less than 1 lakh rupees per year. Suppose that is the statement. So, in that case here F(r) will be like 0.6, 60 percent and r is 1 lakh.

The second point is this. If n is the total number of individuals then nF(r) is the number of people with income no greater than r. So, this is the second statement. Let us try to understand this statement in terms of that example I have just given. I said 60 percent of India's population have income less than 1 lakh rupees. Now, I did not say how many people in India earn less than 1 lakh rupees. To get that number, how many people earn income less than 1 lakh rupees, suppose that number I have to find out. Now, how do I find that out?

I have to multiply 60 percent with the total population of India and that is what I have done here. Let us suppose India's population is, let us take an arbitrary figure, let us suppose this is 100 crore people is India's population. So, 100 crore multiplied by 60 percent that means 60 crore people in India earn income less than 1 lakh rupees. So, by this manner we can actually find out how many people are earning income less than a particular level of income.

Thirdly, let us suppose $r_0$and r_1 be the lowest and the highest income levels. So, you can imagine there is a minimum income level below which people do not earn so that is r_0. And in the other extreme there could be a maximum or the highest income among all the income levels in a country and that highest income level let us denote that by r_1. So, all the incomes in the country will lie between these two extreme values r_0 and r_1.

Now, in general, this F(r) is a discontinuous function, because rupees are practically discontinuous. However, it approximates a continuous function if there are a large number of individuals. So, this is quite practical that r is what, it is income level. And income levels are measured in terms of the currency of the country. In India's case it is Indian rupee. But income levels are not continuous.

It could be, let us suppose, 1000 rupees or 1001 rupees, 1002 rupees. So, there are gaps between 1000 and 1001. So, it seems that this F(r) function is likely to be a discontinuous function. But one can imagine that there are a large number of people. In India's case it is close to 140 crore people. That is a huge number. So, one can actually say that this function approximates to a continuous function.

Let us assume a particular function which is f(r), which is the derivative of F(r). And suppose this is valid for all levels of income in this range r_0 to r_1.. Then actually there are standard names for these two functions: f(r) and F(r). They are called income density function that is f(r). And F(r) is called the associated cumulative distribution function. So, one is density, the other is distribution.

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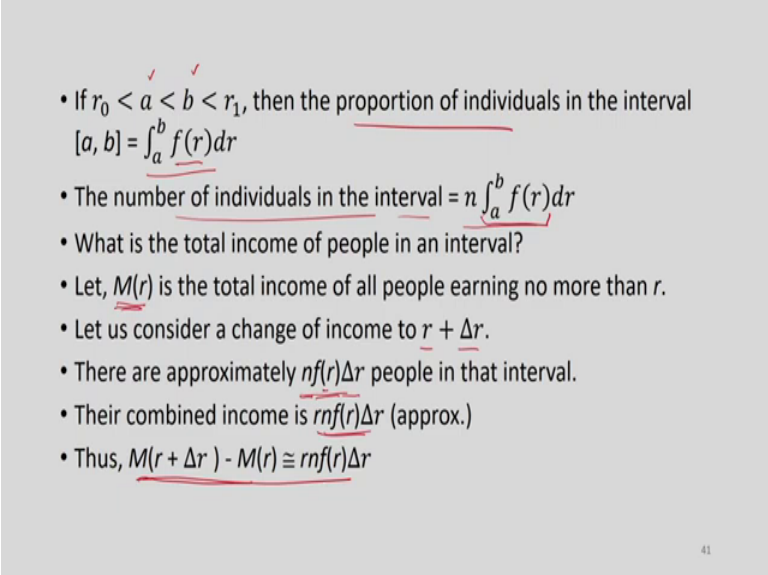

Now, let us pick up two income levels a and b from this range that is r_0 to r_1.. then the proportion of individuals in this range will be, this is very easy, it is definite integral $\int_a^b$f(r)dr. f(r) is the income density function, and this was obtained by differentiating the distribution function. So, if I take the definite integral from a to b of the density function then I get the proportion of individuals who earn between these two income levels a to b. And how many people are there in that interval. So, like the previous case I multiply this term with n. n is the total number of people in the country. So, this is the total number of people earning income from a to b.

But the next question is how much income are these people earning who belong to this interval a to b. So, that is an interesting question. What is the total income of these people? Let us assume the function which is M(r). M(r) is the total income of all people earning no more than r. r is a particular level of income. So, there will be many people who will be earning less than r. I want to measure their total income that is being denoted by M(r).

Let us now consider a change in the income from r to r +Δ r. So, the change in the income is Δ r. Now, there are approximately nf(r)Δr people in that interval. How am I saying that? f(r) Δr is the proportion of people, approximately, this is the proportion of people in that Δr interval. And I am multiplying that by n. So, nf(r)Δr is the number of people who belong to that interval.

And what is their combined income? So, these number of people are there, but what is their total income? I can multiply that with r, because I am taking the value at r. So, I can multiply that with the level of income which is r and that will give me approximately the total income earned by these people in that interval, Δr interval. In other words, I can write this M(r+ Δr) $-$ M(r).

What is the left hand side? It is nothing but the same thing that we are talking about. It is the total income of the people in that interval, Δr interval. It is M(r+Δr) $-$ M(r). And that is what we have just discussed. It is roughly equal to rnF(r)Δr. Now, we can divide both sides by Δr then the left hand side becomes the Newton quotient.

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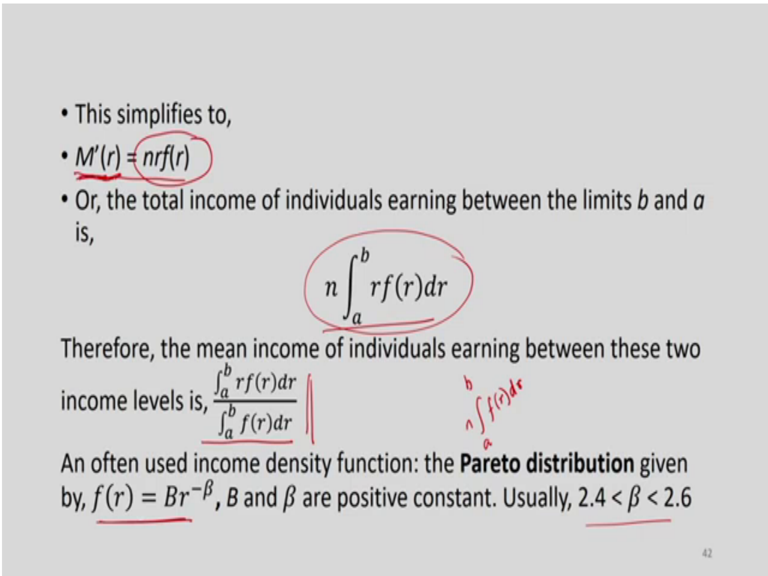

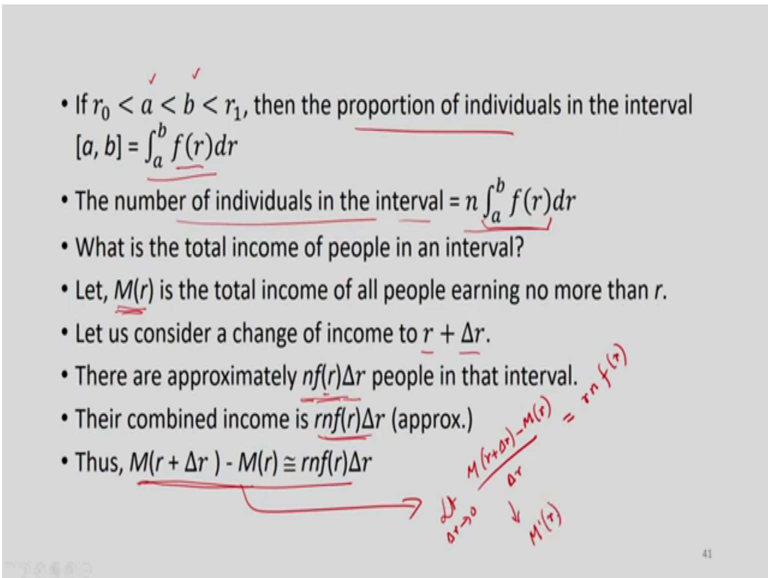

And if I take the Newton quotient and take the delta r approaching 0, then I get $M'(r)$ on the left hand side. So, from here I can get M(r+Δr) $-$ M(r) / Δ r and limit Δ r going to 0 on the right hand side it is simply rnf(r) and this is nothing but M'(r). So, that is what I have written here, M'(r)= nrf(r). Or the total income of individuals earning between the limits b and a is this, because this is the derivative.

So, remember, if I take the integration, definite integral of the f'function that is the differentiated function which is M'(r) here, then I get the total income of the individuals between that interval. So, that is what I have done here. I have taken the integral of a to b of this function, n will come out because n is a constant and what is the variable of integration it is r, because r is changing. That is the income level is changing.

So, we have found out the total income of individuals earning income between two limits a to b. It is given by n$\int_a^b$ rf(r)dr. Therefore, the mean income of individuals earning between these two income levels is given by this. So, essentially what I have done is that I have taken this, which is the total income of the individuals earning between a to b and divided it by the total number of people in that interval.

And that number of people this denominator is $\int_a^b$f(r)dr that we have just seen before. So, n and n will get cancelled from numerator and denominator and we are going to get this expression. So, this is the mean income of individuals earning between two limits a to b.

And often used income density function is called the Pareto distribution given by f(r). So, this is Pareto. In this case its density function f(r). It is given by $Br^{-\beta}$. Here β and B are positive constants. Usually, β has a value between 2.4 and 2.6. So, this is an often applied density function in the study of income distribution, the Pareto distribution.

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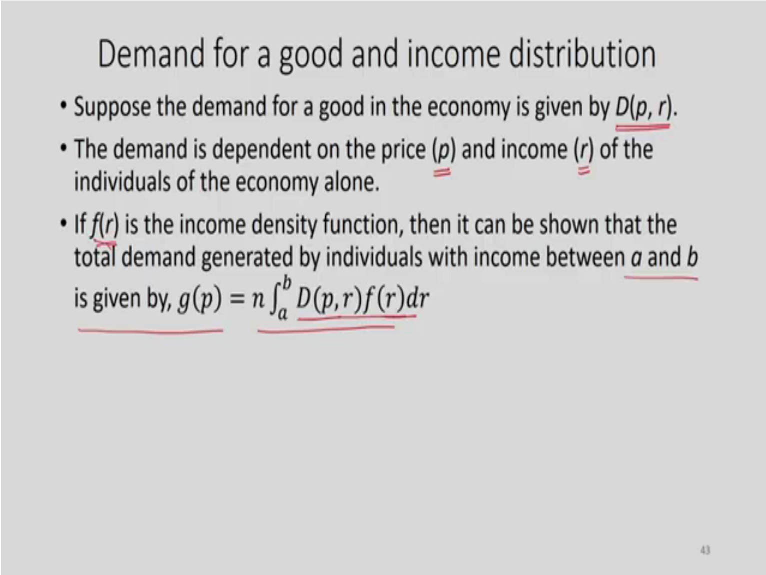

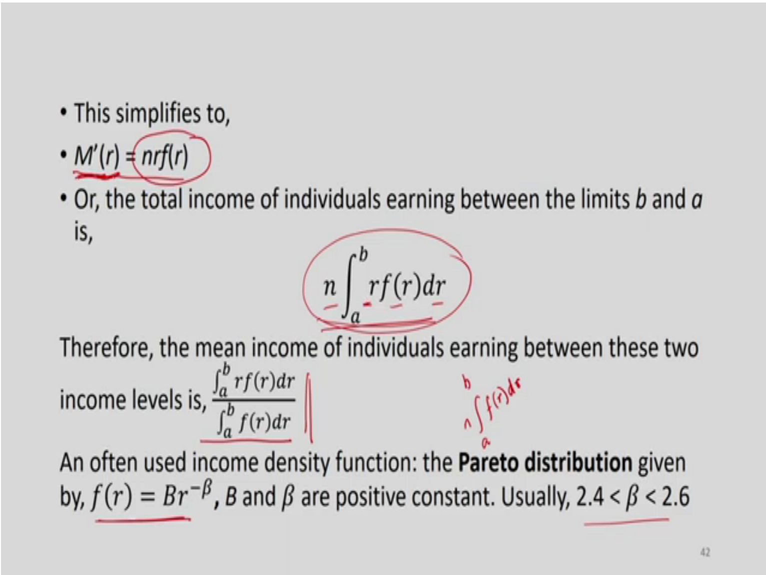

Here is a related application of the same idea. Demand for a good and income distribution. Suppose the demand for a good in the economy is given by D(p, r). So, demand that is D, it is a function of p and r. It is p is price and r is income like before, r is income, of the individuals of the economy. So, demand is only dependent on these two variables.

The price of the commodity that we are talking about that price is given by p and the income of the individuals. And let us suppose f(r) is the income density function. Then it can be shown that the total demand generated by individuals with income between a and b is given by g which is a function of p and it is equal to $n\int_a^b D(p, r)f(r)\,dr$.

So, look at the general pattern here f(r) is occurring D(r) is occurring, but in front there is this the demand function which is a function of r. You can see the same pattern here also. Here we are talking about not demand, but total income of people who are earning between a and b. So, here also you have nf(r)dr, but here r is occurring just before f(r). In this case, it is the demand because we are not talking about the total income but we are talking about demand. So, f(r) is preceded by the demand function which is D (p , r).

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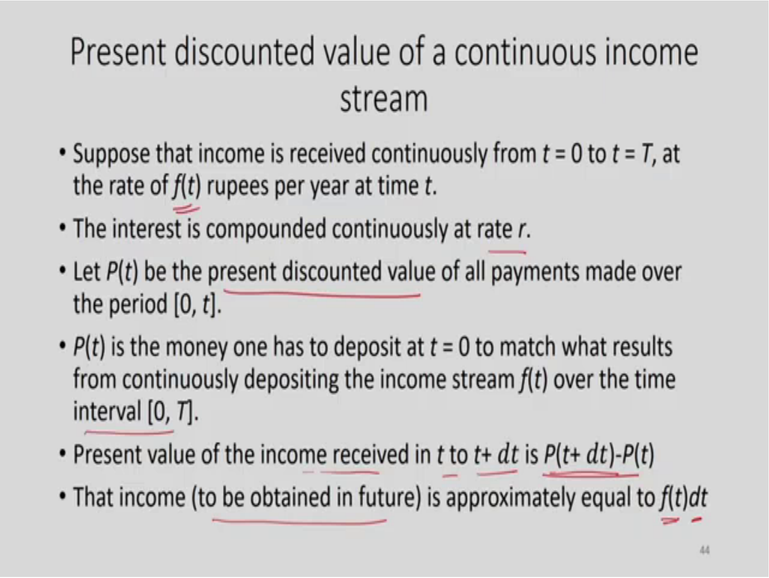

Now, we come to another, the fourth probably, application of integration. The present discounted value of a continuous income stream. Suppose that income is received continuously from t to 0 to t to T at the rate of f(t) rupees per unit of time at time T. So, per unit of time you are getting f(t) and the total time in which you are going to receive is from 0 to T. The interest is compounded continuously at the rate of r. Let capital P(t) be the present discounted value of all payments made over the period 0 to t.

Now, P(t) is the money one has to deposit at point 0 to match what results from continuously depositing the income stream f(t) over the time interval 0 to t. We talked about the idea of present value. So, here the present value is denoted by T. So, it will give me some return over a period of time and that return must match with the income stream of f(t) that I am talking about.

Now, the present value of the income received in t to t+dt. So, I have taken a change of time from t to t+dt is given by this, P (t + dt) - P(t). This is the change in the present value if I take two different points of time. That income to be obtained in future, remember, this is a present value of some income to be obtained in future.

And what is that income that is going to be obtained in future. It is approximately equal to f(t), f(t) is the stream at point t and you are changing the time a little bit, so this amount of income that you are going to earn in future in that dt small period is approximated by f(t)dt.

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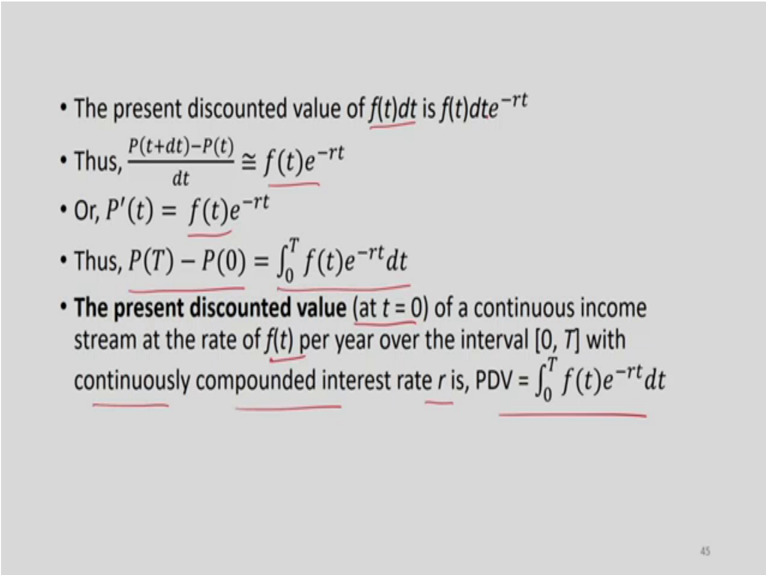

And the present discounted value of f(t)dt is what, it is f(t)dte^{-rt}. Thus, $\frac{P(t+dt)-P(t)}{dt}$, I have taken dt to the left hand side, is approximately equal to f(t)e^{-rt}. And I can take dt going to 0 and so the left hand side becomes P'(t), P'(t) is equal to f(t)e^{-rt}.

And so I can now talk about the total amount of money, the present value of that, that will be obtained in future, it is given by P- P(0) and this will be nothing but the definite integral of this integration $\int_0^T$ f(t) e^{-rt}dt.

The present discounted value evaluated at t is equal to 0 of a continuous income stream at the rate of f(t) per year over the period 0 to T with a continuously compounded interest rate r given by this expression. So, this is the expression for present discounted value. Remember, we saw something similar in that exercise of investment that we discussed just now.

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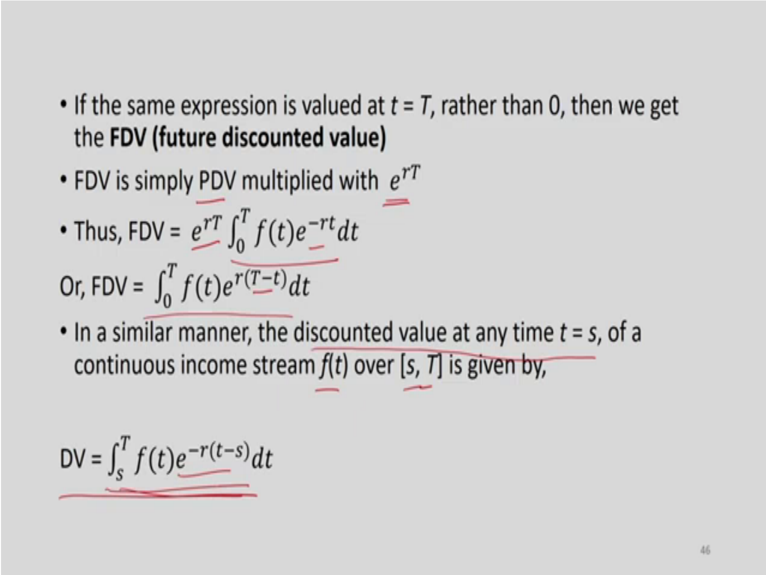

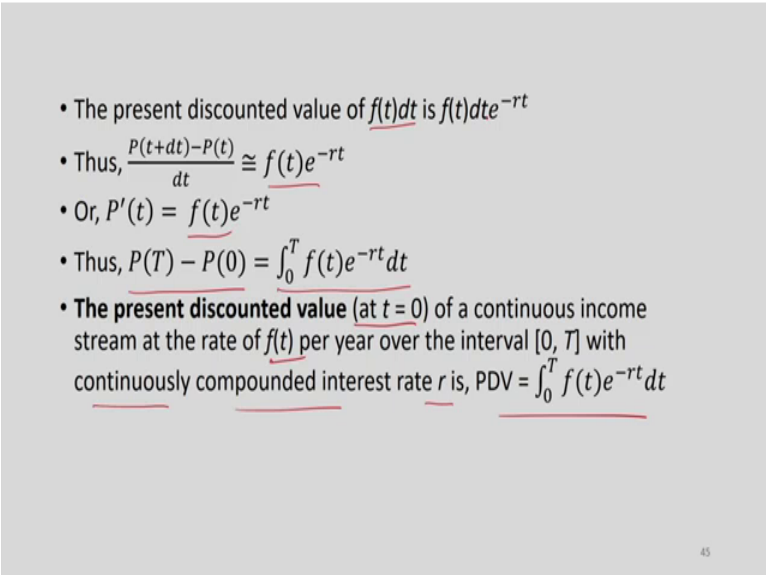

If the same expression is valued at t is equal to T rather at 0, then we get FDV or the future discounted value. So, here P(t) is the discounted value when we are standing at point 0 and we are evaluating the future income stream. But suppose that evaluation is done in future at T then what happens, then it is called the future discounted value. And how is it evaluated? I just have to multiply the PDV by e^{rT}.

So, in this case, FDV is equal to e^{rT} multiplied by the present discounted value and that simplifies to $\int_0^T$ f(t)$e^{(T-t)}$, because I am taking this inside, dt. In a similar manner, the discounted value at any time t = s of a continuous income stream f(t) over s to T is given by

DV, discounted value. So, here instead of standing either at 0 or at T, suppose a person is standing at s and s is between 0 and capital T.

Then how does she evaluate the future income stream, so that is given by this. It is integration of s to T, because it is starting from s and the income stream is going to go on till T, of f(t) $e^{-r(t-s)}$dt. You can verify that here if I change s to be equal to 0, I will get the PDV.

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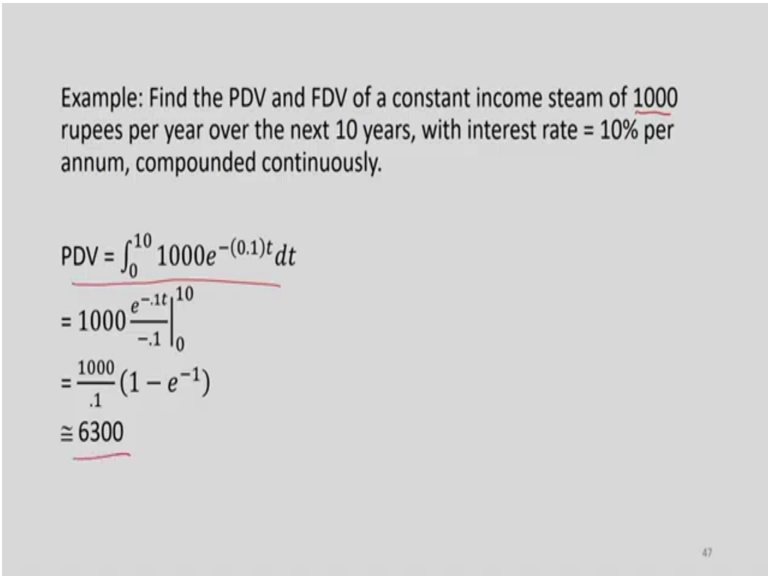

Here is an example. Find the PDV and FDV of a constant income stream of 1000 rupees per year over the next 10 years with interest rate 10 percent per annum compounded continuously. So, PDV, I just directly apply the formula, integration of 0 to 10, 10 years are there, multiplied by the stream of income, in this case it is constant, so f(t) is just 1000, e^{-rt}, r that is rate of interest is 10 percent, so it is minus 0.1 multiplied by tdt. And so things are now falling into place. So, I just have to integrate and take the definite integral. It comes out to be roughly 6300 rupees. This is the PDV.

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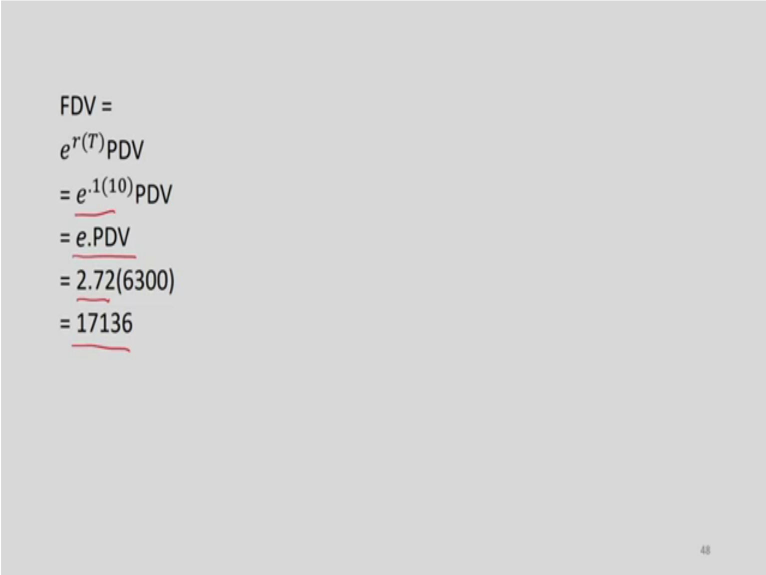

For FDV it is just converting the PDV into FDV. I have to multiply that with $e^{r(T)}$. So, e^{r} *is* 0.1, T is 10. So, 0.1 multiplied by 10 is 1. So, e multiplied by PDV and e is roughly that is, how much is it, 2.72 and that boils down to 17136. So, you can see that there is a huge change in the value depending on whether we are talking about PDV or FDV.

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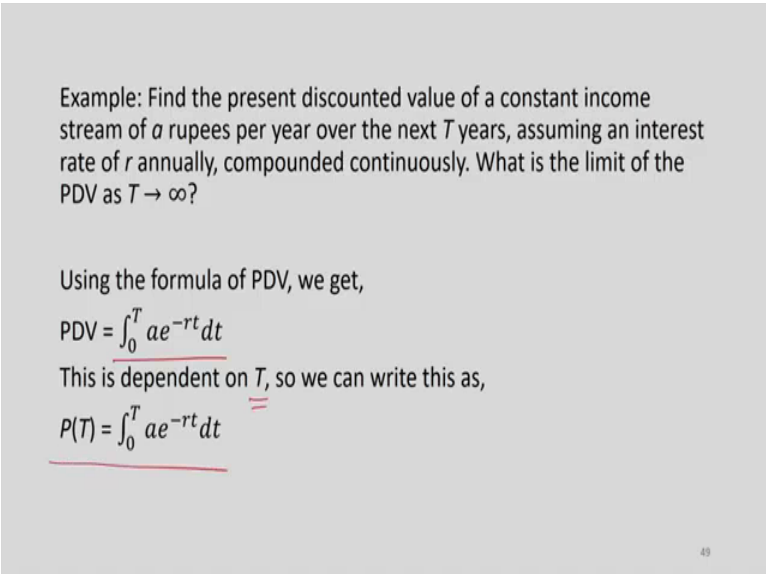

Let us do this one and then we shall call it a day. Find the present discounted value of a constant income stream of a rupee per year over the next T years, assuming an interest rate of r annually compounded continuously. What is the limit of the PDV as T goes to infinity?

So, there are two parts of this. First we have to find the PDV. Let us do that. It is simple. I just have to plug in these information into the formula. Here f(t) is equal to a, e^{-rt} and the time span is 0 to capital T. Now, this obviously is a, I have to evaluate that, but notice that this is a function of capital T. So, I can write this as P(T), denoting the present discounted value, which is a function of capital T.

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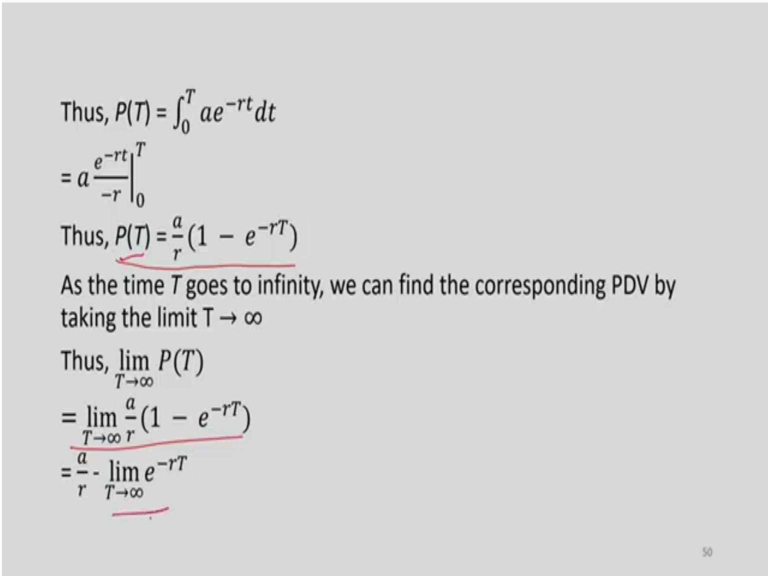

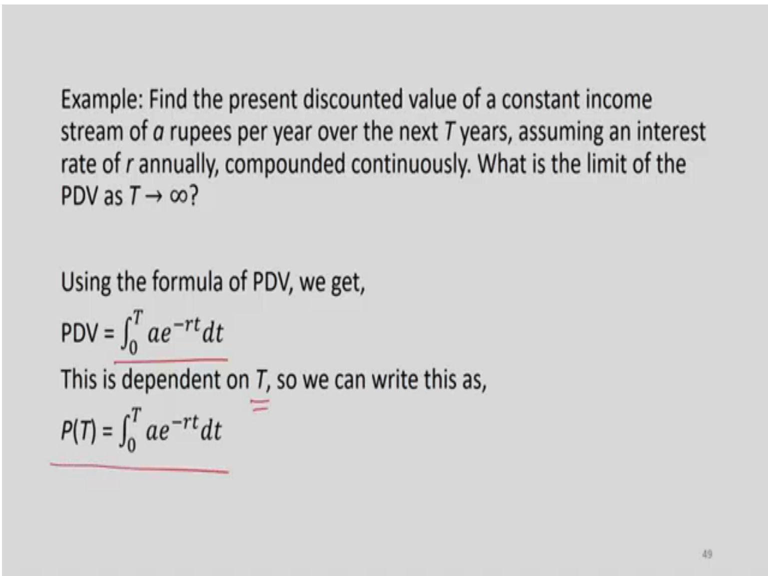

And I do this integration and I find this P(T) to be of this form, $\frac{a}{r}$ (1- e^{-rT}). Now, the second part was this that as T goes to infinity, then what is the limit of this PDV. Well, I have to take the limit of P(T) as T, T goes to infinity, so this is what we are getting into and so this will become just 0, because e^{-rT}, $when$ T goes to infinity it will become 0.

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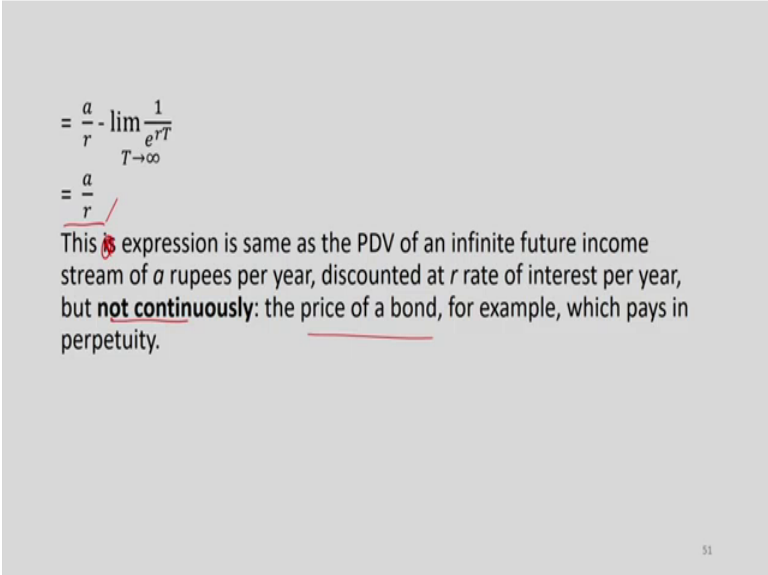

So, this becomes a / r. This expression is same as the PDV of an infinite future income stream of small a rupees per year discounted at r rate of interest per year, but not continuously. So, this we have talked about before when we were talking about geometric series. There I did not talk about continuous discounting. So, there was discounting, but the discounting was discrete. So, there also I got the same expression, a divided by r is the present value. So, here also we are seeing the same thing.

The exact example that I gave was that our bond which pays return in perpetuity there the value of the bond or the price of the bond was given by a divided by r. I think I will stop here and I think there will be just one more lecture that will be required to finish this topic of integration. So, I will do it in the next lecture. Thank you for joining us. Thank you.