**Mathematics for Economics – I**
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**Lecture No. 26**
**Definite integral**

Hello and welcome to another lecture of this course, called Mathematics for Economics-Part I. And we are discussing the topic called Integration presently. We started with a lecture in the previous class and there we introduced the idea that integration can be thought of as the reverse of differentiation and let us see how far we can go from there. We also talked about the fact that integration can be thought of as an area under a curve.

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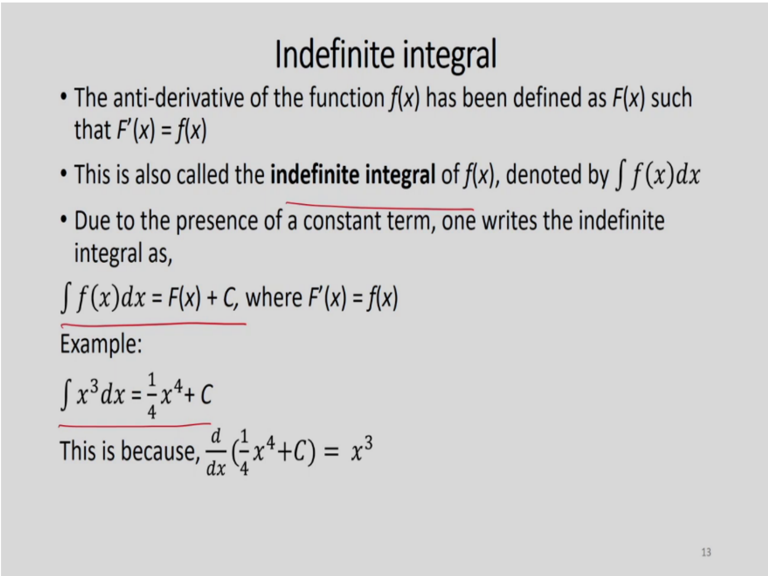


And we introduced the idea of indefinite integral so this is the idea. The anti-derivative of the functions f(x) as we defined it as capital F(x) such that capital $F'$ (x) is equal to f(x). So, this capital F(x) is the indefinite integral but we also have to notice that there is this constant term, capital C which has to be added because if you take the derivative of capital C, it will become 0. So, this is what we write. $\int$f(x) dx = Fx + C where C is the constant of integration.

So in this case, capital F(x) is the anti-derivative and it is capital F(x)+ C is the indefinite integral of small fx. And then we took one example where let us suppose $X^3$, we have to find the integral of that and we found that to be $\frac{1}{4}X^4$+ C.

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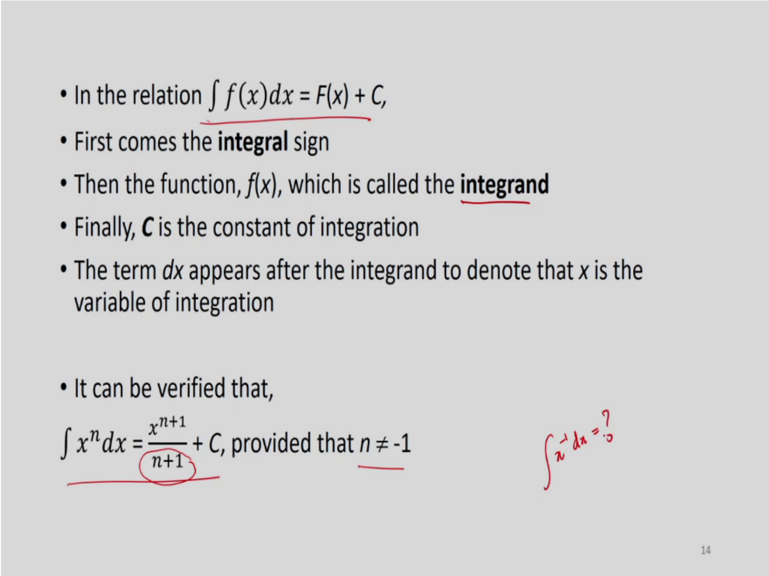


And here is the conventional way we write it. First comes the integral sign, then the function that we are trying to integrate dx, dx because x is the variable with respect to which the integration has been done. Here f(x) is called the integrand and on the right hand side we have capital F(x) + C. C is the constant of integration.

Now we are talking about certain rules of integration. We said that if you integrate $x^n$ with respect to the dx then what we get is $\frac{x^{n+1}}{n+1}$+C . C is the constant of integration. However, this rule is valid if n is not equal to minus 1. The reason being if n is equal to minus 1 then the denominator becomes 0 and so this quantity on the right hand side becomes undefined. So, the question that naturally arises is if you have to integrate $x^n$ and n is equal to - 1, then how do you do it?

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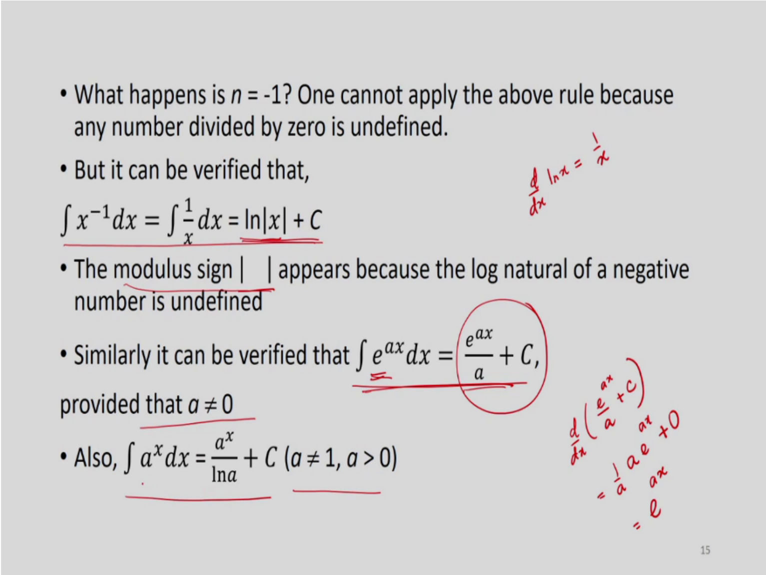


This above rule cannot be used but we have another formula, which is $x^{-1}$ as we know is $\frac{1}{x}$ and this can be written as $ln|x| + C$. Remember why we are writing it because we know if I take the derivative of log (x) then it becomes $\frac{1}{x}$. So, if I integrate $\frac{1}{x}$, I should get back $ln|x| + C$ and there is this plus C which is the constant of integration.

Now how is the modulus sign making an appearance here? The modulus sign is appearing because the log natural of a negative number is undefined. So, that is why you have to take the positive if x happens to be negative, you cannot just say log of x because x is negative. So, you take the modulus of that number and then you have $ln|x| + C$ .

Similarly, it can be verified that $\int e^{ax}$dx = $\frac{e^{ax}}{a} + C$, provided of course small a is not equal to 0 because small a if it is equal to 0, then again the right hand side becomes undefined. So, we have to make sure that small a is not equal to 0. And this relationship can be verified again you take the derivative of this $\frac{e^{ax}}{a} + C$, , what do you get, $\frac{d}{dx}e^{ax}$ = a$e^{ax+0}$ and this is simply equal to $e^{ax}$, which is on the left hand side as the integrand. So, this rule is all right.

So, instead of the exponential function like this if I take $a^{x}$. a is a constant then what happens, then it should be equal to $\frac{a^{x}}{lna} + C$ and of course I have to make sure that a is not equal to 1. If a is equal to 1, then log of 1 as we know is 0. Again there is a problem on the right hand side, so a cannot be equal to 1 and a cannot be equal to 0 or negative either because in either case, you have either minus infinity or it is undefined. So, these numbers – these conditions have to be satisfied. And I can again check whether this formula is all right or not

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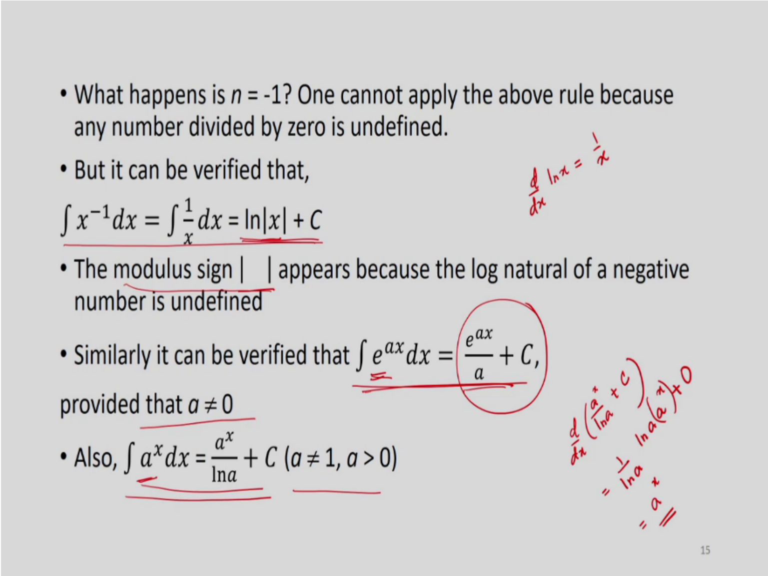


What I do is I take the derivative of the right hand side, which is $a^x$, this and this we know is, this is a constant term, it will come out. $\frac{d}{dx}ax$ What is that? It is log of a, $a^x$and plus 0. And so log a, log a will get cancelled, so you are going to get simply $a^x$ and which is an integrand on the left hand side. So, this formula is in order.

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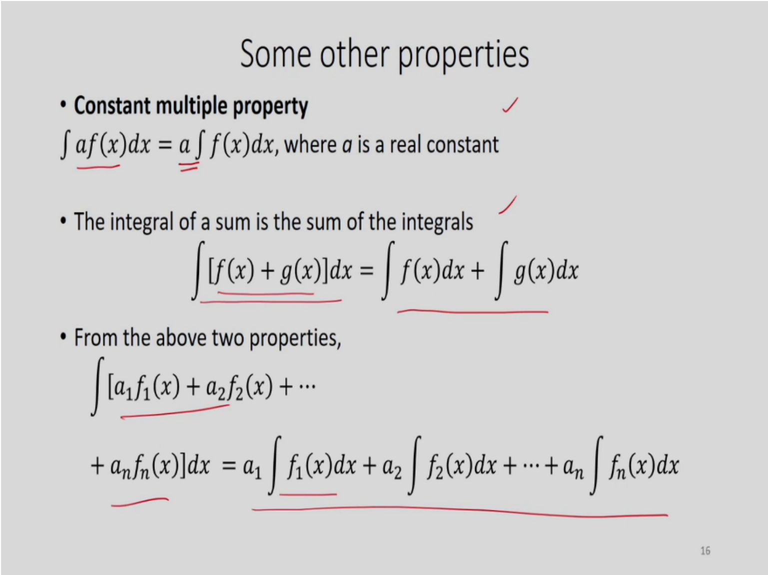

Some other properties, what is known as the constant multiple property. So, here what is there in the integrand you have a constant which is small a multiplied by a function of x which is f(x). So, if that is the integrand then I can write this simply as a $\int$f(x)dx where a is a real constant.

So, essentially what is happening if in the integrand there is a constant term, real constant term then I can take the real constant term out and whatever is left of the integrand remains inside. Now this makes the estimation of the integral simple because now you do not have to deal with the constant.

And what happens if you have some of two functions? So, here on the left hand side that is exactly what you have integrand is the summation of 2 separate functions, f(x) and g(x). Now this can be written as summation of 2 integrals. So, $\int$ f(x) dx + $\int$ g(x) dx. So, essentially what is being said is that the integral of a sum is the sum of the integrals. And if we combine this property with this property, we get a third property which is integration of $a_1 f_1(x)$+ $a_2 f_2$(x) +.......... so on.

Let us say the last term is $a_n f_n(x)$ and the whole thing dx. This becomes $a_1 \int f_1(x)dx$+ $a_2$ $\int f_2$(x) dx plus so many terms and the last term is $a_n \int f_n(x)dx$ . So essentially, I am first breaking this term down into n different parts and then taking the constant terms outside the integral in each of this n terms.

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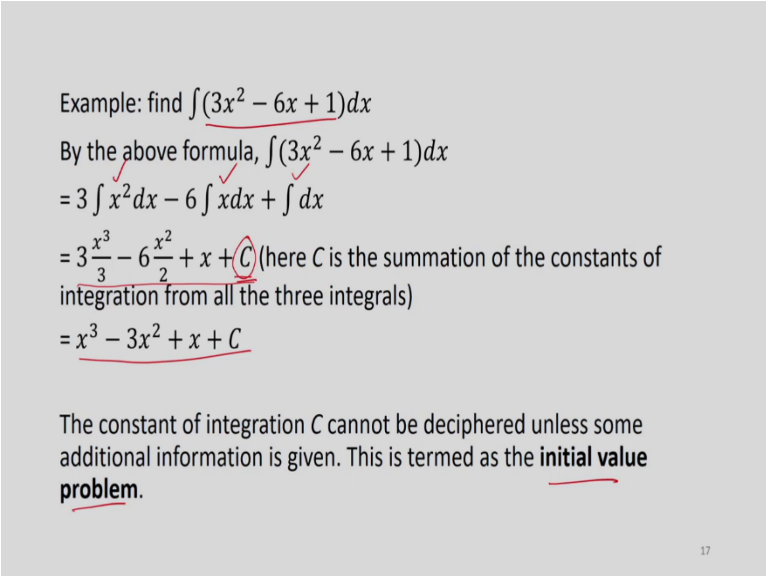


Here is an application of what we have just seen. Suppose you have an expression like this a polynomial, $3x^2 - 6x + 1$ is what we have to integrate and what we do is that we apply the formula before where I write this as separate integrals. So, 3 $\int x^2$ dx because 3 will come out minus 6 $\int$xdx because minus 6 is coming out plus $\int$ dx.

And from here I get this expression $3\frac{x^3}{3} - 6\frac{x^2}{2} + x + C$. Now at this stage you might be wondering that since you have three integrals, so there will be 3 constants. From each of this integrals you get one constant. There are three operations of integration, so 3 constants, so how come there is just a single constant. What we have done is that I have assumed that the single constant a is the summation of all these 3 constants.

Since these are arbitrary constants, so I can do that because c is undefined. It is just a constant. All right in the next step I am simplifying this 3 and 3 will get canceled, so $x^3 - 3x + x + C$. So, the constant term integration C cannot be deciphered unless some additional information is given. This is termed as the initial value problem.

Right now C is just an arbitrary constant. We cannot say anything about the value of C so to understand what will be the value of C we need a certain condition, some additional information and this is termed as the initial value problem.

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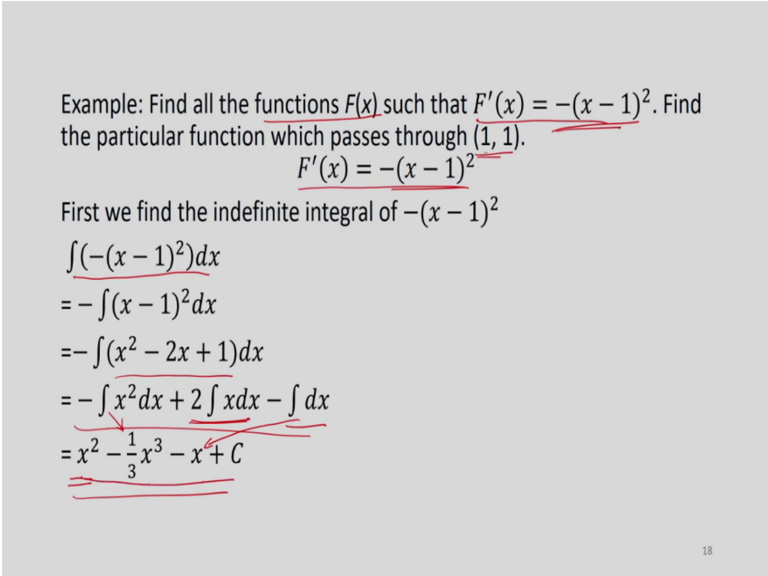


Here is an application of that. Find all the functions fx – capital F(x) such that $F^{/}(x) = -(x-1)^2$. Find the particular function, which passes through this point (1, 1). So, there are two parts to this question. We have to find F(x) such that $F^{/}$ is a particular function of x. And secondly there will be many such functions which will satisfy this condition. We have to find that particular function which passes through 1 and 1.

So, first we have to basically integrate this -$(x-1)^2$ because this is the antiderivative. Now here we are trying to find the indefinite integral of -$(x-1)^2$dx because x is the variable of integration. One can take the minus 1 outside so it becomes - $\int(x-1)^2dx$. Now this is $(a-b)^2$ formula. So, this will be $x^2 - 2x + 1$ and then I can break it up. It will become - $\int x^2dx + 2\int xdx - \int dx$

And if you simplify this, so this term if you integrate this, it will become $x^2$ because $x^2$ divided by 2, 2 and 2 will get cancelled. So, I am writing this as the first term and then I am writing the integration of this as the second term. This will be $\frac{x^3}{3}$ and integration of 1 is just x plus the constant of integration. So, this is the indefinite integral that we have found.

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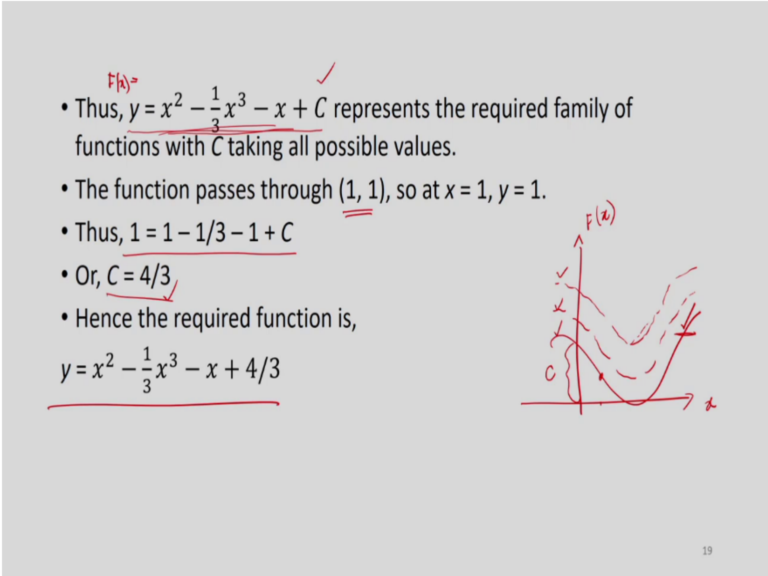


So, y $= x^2 - \frac{1}{3}x^3 - x + C$ represents the required family of functions with C taking all possible values. So, instead of y, I could have written it as F(x) as well because that is the form that is written in the question.

Now we have to find that particular function, which passes through the point 1, 1. It is written there, find the particular function which passes through 1, 1. So, F(x) is fine, F(x) is this but here capital C is undefined. C is just a constant. We do not know the value of C. So, I have to give a particular value to C only then the function becomes a definite function. Otherwise, the function might have different positions. So, in terms of a diagram you can imagine this so suppose this is the x and this is F(x) that is y and suppose the function I do not know looks like this, it is just an arbitrary figure.

Now since C is undefined, this function could as well have this position depending on what the value of C is because as you can see if I put x = 0 here then the value of the function becomes C. So, this, this is C. But C is unknown so depending on the value of C you have different positions of the function. But our problem is that we want to find that particular function, that specific function which passes through a given point which is 1, 1.

So suppose 1 is here and 1 is here. So, this is the 1, 1 point. So, we want to find this function which is represented by this continuous line, not the dotted lines. So, what will be the equation of this particular line? So, how do I do that? This particular function that we are after passes through this point 1, 1.

So, therefore on the left hand side, you are going to get 1, if you put x = 1, so this is the equation you will get and therefore I get C= 4/3 after simplifying and I put this 4 / 3 here and I get this particular function. y = $x^2 - \frac{1}{3}x^3 - x$+ 4 /3. So, this is the required form.

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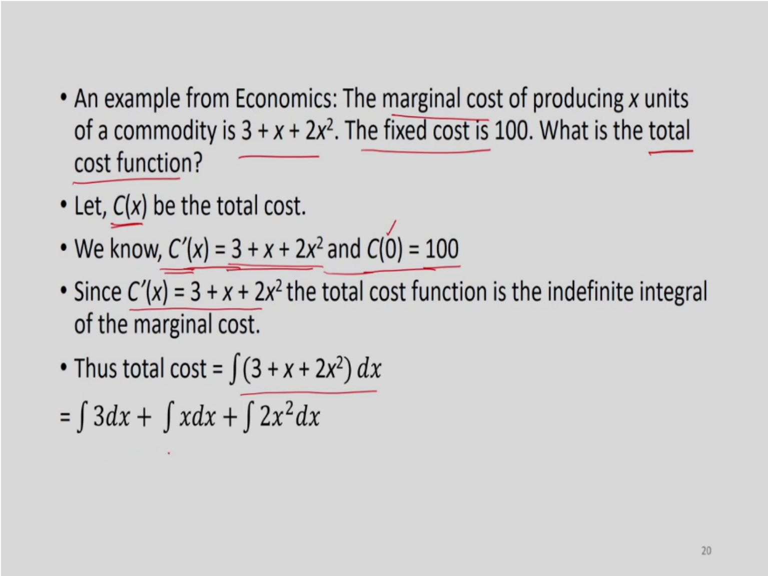


Here is an example from economics. The marginal cost of producing x units of a commodity is 3 + x + $2x^2$. The fixed cost is 100. What is the total cost function? So this is what we want to find, the total cost function. Now let us assume that C which is a function of x is the total cost function. What we know is that if I take the derivative of C(x), then I get $C'$( x). This is $C'$( x) = 3 + x + $2x^2$. This is the first information we know.

And why am I writing this? Because marginal cost. If you have noticed my previous lectures when we introduced the idea of derivatives then we met this point that the marginal cost is nothing but the derivative of the total cost function and marginal cost is given here in the problem, so $C'$( x) = 3 + x + $2x^2$. Remember the marginal cost is a function of x.

And further we also know that fixed cost is equal to 100. What does it mean that if the producer does not produce anything then also he has to bear certain costs, the value of that cost is 100. In terms of our mathematical language the cost function at 0 that is x is equal to 0 that means C is equal to C of 0 is equal to 100. So, these two information we know and from these two information I have to find C(x). So, that is the problem.

Now $C^{/}(x)$. Let us try to find the indefinite integral of this and that will give me the total cost function but there will be a constant term and then we shall deal with the constant term. So, the total cost = $\int(3 + x + 2x^2)\,dx$, so this becomes $\int 3dx + \int xdx + \int 2x^2\,dx$ .

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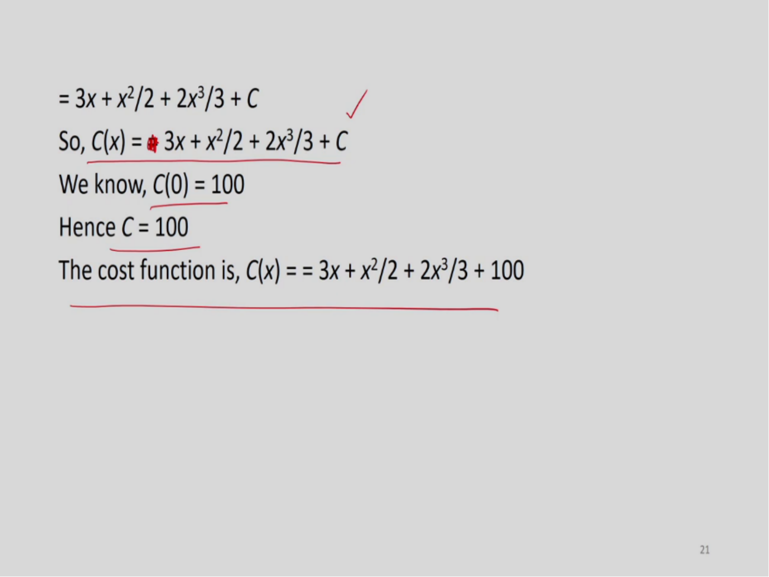


And this turns out to be 3x + $x^2/2 + 2x^3/3$ + C , C the constant of integration. So, this is the cost function. But here obviously there is this plus C terms and I have to get rid of that and here I am using this information that is C of 0 is known to be 100. So, if I put there are 2 equal signs here. One is redundant, so if I put x is equal to 0, then the right hand side becomes C and the left hand side is known to be 100. So, therefore C is equal to 100. And this is substituted back in this function so the total cost function is found to be 3x + $x^2/2 + 2x^3/3 + 100$ . Done.

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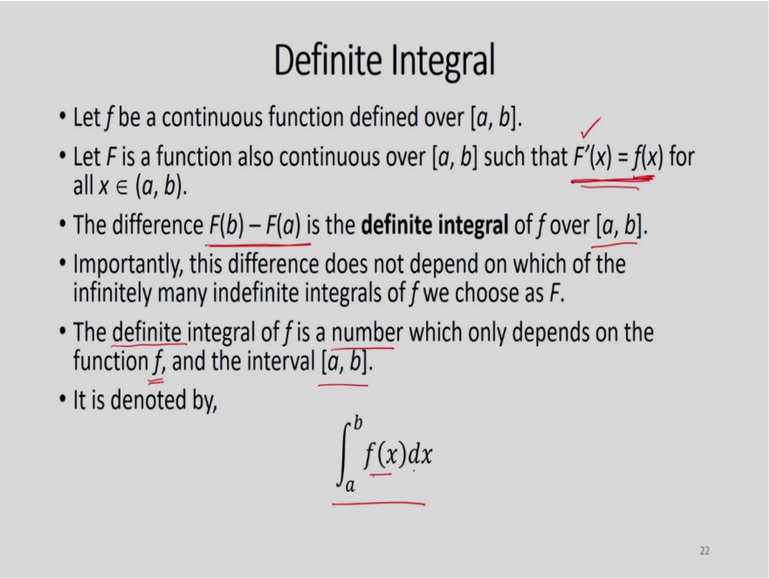


Now we come to the idea of definite integrals and let us see what is the definition saying. Let small f be a continuous function defined over the closed interval a to b. Let capital F is a function also continuous over the closed interval a to b such that $F^{/}(x) = f(x)$. For all x belonging to the open interval a to b. Then the difference F(b) - F(a) is the definite integral of small f over the closed interval a to b. This is the definition of a definite integral.

Importantly, this difference does not depend on which of the infinitely many indefinite integrals of small f we choose as capital F. The point that is being made is this since we know $F^{/}(x) = f(x)$. one can take the indefinite integral of f(x) and that will give me F(x) + C where C is the constant of integration.

So, depending on the value of C you are going to get a host of functions which will satisfy this condition but the interesting point of definite integrals is that it is immaterial which particular function that satisfies this you use the value of the definite integral will remain the same, this value remains the same.

The definite integral of small f is a number. Definite integral is a number, which only depends on the function f and the interval a to b. And it is denoted by this notation so first you have the usual integral sign and you note that here I have written a and b. On top I have written b and on bottom I have written a then comes to integrand which is small fx and then comes dx.

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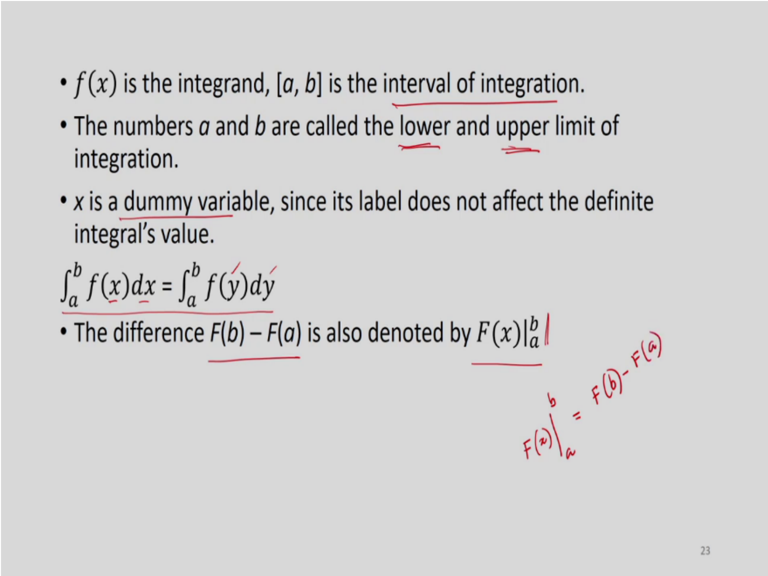


Small f(x) is the integrand we have noted this a to b the close interval is the interval of integration. So, this interval of integration is a new concept. It was not there when we talked about indefinite integrals. The numbers a and b are called the lower and the upper limit of integration. So, from where to where the integration is being done. The definite integral is defined. Those 2 things a and b are called the limits. One is called the lower limit, the lower value and the other higher value is called the upper limit.

Small x is a dummy variable here. Since its label does not affect the definite integral's value. So, it is just a place holder. It doesn't really matter what variable you use, the value of the definite integral remains the same. So, to make it clear I have written this $\int_a^b$ f(x)dx is same as $\int_a^b$ f(y)dy. So, as you can see on the left hand side I have used that variable x.

On the right hand side, everything remains the same, only the variable has been changed from x to y and the claim is that the left hand side is equal to the right hand side. The reason being x is a dummy variable. It doesn't really matter what variable I take, the value of definite integral remains the same.

What is the value of the definite integral? It is F(b) - F(a). It is also denoted by in short fx a to b, so here b and a they denote the lower limit and upper limit. So, in other words I can write f(x) this is equal to – and this is what we are going to see later on repeatedly.

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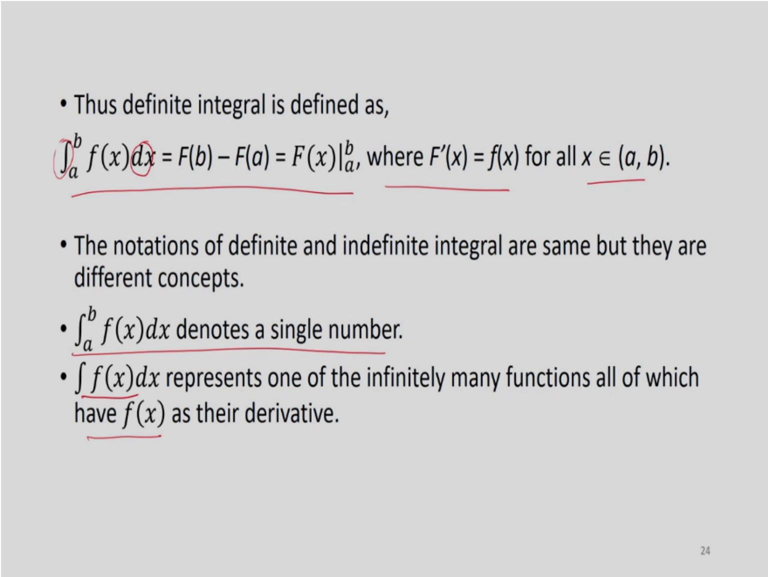


So, this is what is being written here, $\int_a^b$ f(x)dx = F(b)- F(a) = F(x)$|_a^b$ where $F^{/}(x)$ = f(x) for all x belonging to the open interval a to b.

Thus, the notations of definite and indefinite integrals are the same. The notations are same integral sign is there, dx is there. There is a small difference where in the definite integral you have the lower and the upper limits. In the case of indefinite integrals the limits are not there, but they are different concepts. Conceptually these two things are very much different. The definite integral is a single number, $\int_a^b$ f(x)dx, denotes a single number.

On the other hand if I talk about the indefinite integral, $\int$ f(x)dx, it represents one of the infinitely many functions all of which have fx as their derivative. We have talked about this before so that is how indefinite integral is defined. It is not a single number, it is one of the many functions and all these functions satisfy a common property.

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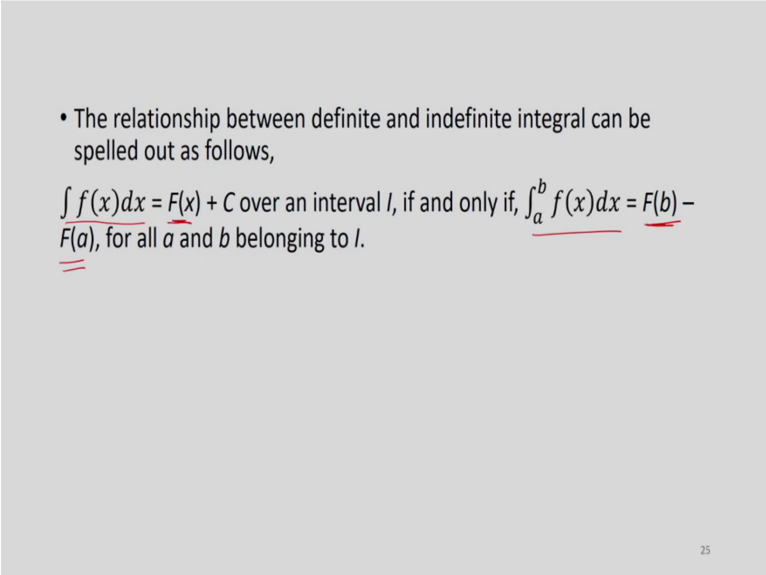


The relationship between definite and indefinite integral can be spelled out as follows. So, I take $\int f(x)dx = Fx + C$ over an interval I if and only if $\int_a^b f(x)\, dx = F(b) - F(a)$ where a and b belong to this interval I. The thing to note here whatever the statement that is written here is that here F(x) this function is used and that same function is making an appearance here also. And when I am talking about the definite integral then this constant term is not there.

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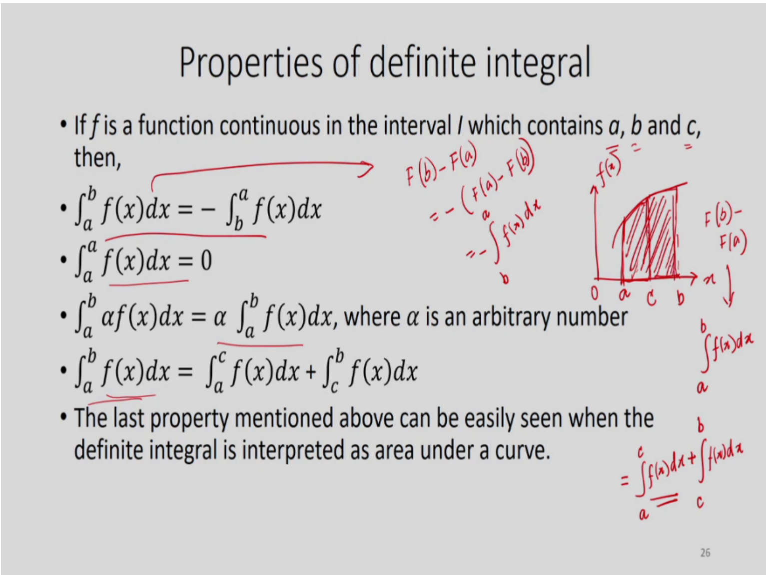


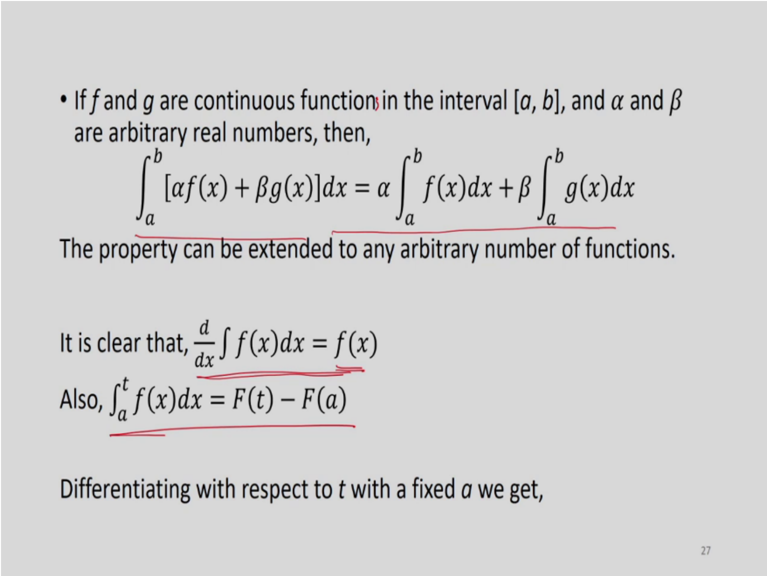
We now talk about properties of definite integral. If f is a function continuous in the interval I which contains 3 points let us say a, b, c, then $\int_a^b f(x)\, dx = -\int_a^b f(x)\, dx$. This is not very difficult to see because what we are going to get from here is F(b) - F(a) assuming that $F'(x)$ = f(x). And you can write this as F(a) - F(b) and what is this, this is b to a fx dx. So, that is why you are going to get this particular property.

Secondly, $\int$ f(x) dx defined over the interval a to b, a = 0. And this is easy to see if you imagine integration as an area under a curve, so suppose you have x here on this axis and you have f(x) on the y-axis and the curve looks like this. You have a here and let us say b here. Now this entire area is F(b)- F(a). This entire shaded area and that is the same as saying this is equal to $\int_a^b f(x)\, dx$. So, these three things are equivalent. This shaded area F(b)- F(a) and thirdly $\int_a^b f(x)\, dx$ . These three things are same

Now if I take the limits to be a and a so the same point so there is no area that we are enclosing therefore this left hand side is equal to 0. Thirdly, $\int_a^b \alpha f(x)dx = \alpha$ multiplied by $\int_a^b f(x)dx$ where α is an arbitrary number. Now this is equivalent to what we have seen in the case of indefinite integral. There also if you have a constant term in the integrand then it can be brought out. Here the same thing is occurring, the difference is being that we are talking about definite integrals here.

The fourth property is $\int_a^b f(x)dx = \int_a^c f(x) + \int_c^b f(x)dx$. The last property mentioned above can be easily seen when the definite integral is interpreted as area under a curve. This can be seen from this diagram itself so a and b are 2 numbers a is less than b and I take suppose c somewhere in between here is c. Now I know the left hand side that $\int_a^b f(x)dx$ is the entire shaded region.

Now this entire shaded region can be subdivided into two separate regions. One is from a to c and the other is from c to b. So, this a to b can be seen as the area between a and c and that will be denoted by this plus area between c to b which is denoted by this. So, essentially I am talking about this particular area on the left plus this particular area on the right. So, their summation is the total area.



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If f and g are continuous functions in the interval, closed interval a to b and α and β are arbitrary real numbers then we have this property that $\int_a^b [\alpha f(x) + \beta g(x)]\, dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$. Again this is similar to the property of indefinite integral that we have seen before. The difference being here there are limits to the integral because we are talking about definite integrals here.

Now it is clear that if I take the derivative of the integral basically these 2 things cancel each other because integration is anti-derivative. So, if you take the derivative of the anti-derivative what you get is there – these 2 things are nullifying each other and you are going to get the integrand itself. So, that is what is being written here.

Also if I take the limit from a to t, t is let us suppose a particular variable then we get F(t)-F(a) a is a constant. Now I take this relationship and I differentiate this with respect to t because as I told you t is a variable whereas a is a constant, then what do we get?

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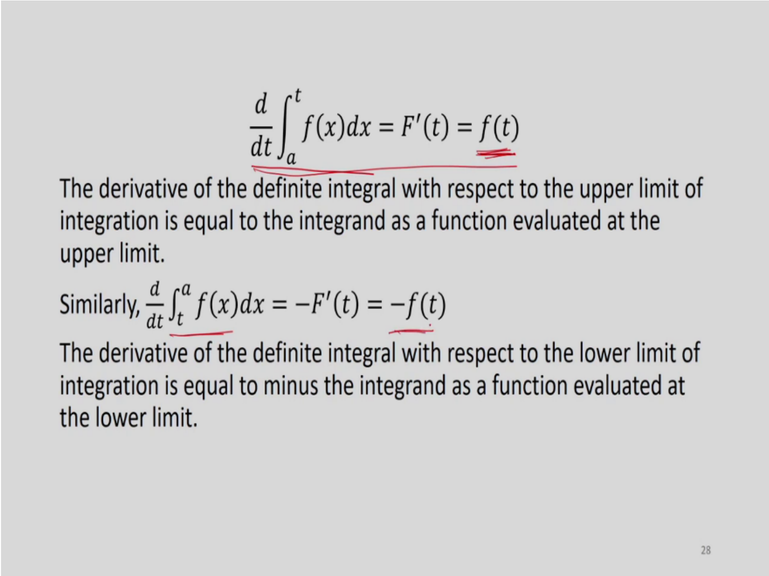


If I differentiate this, then the left hand side will be this, $\frac{d}{dt}\int_a^t$ f(x) dx. And on the right hand side this term will drop out because this is constant, a differentiation of F(t) is what we know, it is equal to t – f(t). So, what does it mean this relationship? The derivative of the definite integral with respect to the upper limit of integration is equal to the integrand as a function evaluated at the upper limit.

So, this is the integrand right, a f(x) that is the integrand. We are evaluating this at the upper limit, upper limit is t that is what we have got on the right hand side. So, that is the same thing that is written in words here. By a similar logic if I just reverse it, if I take the upper limit to be a and the lower limit to be t that is what we have done here. Then the derivative of this with respect to t will come out to be -f(t) and again what does it mean?

The derivative of the definite integral with respect to the lower limit of integration is equal to the minus the integrand as a function evaluated at the lower limit. These two properties might be used in some exercises.

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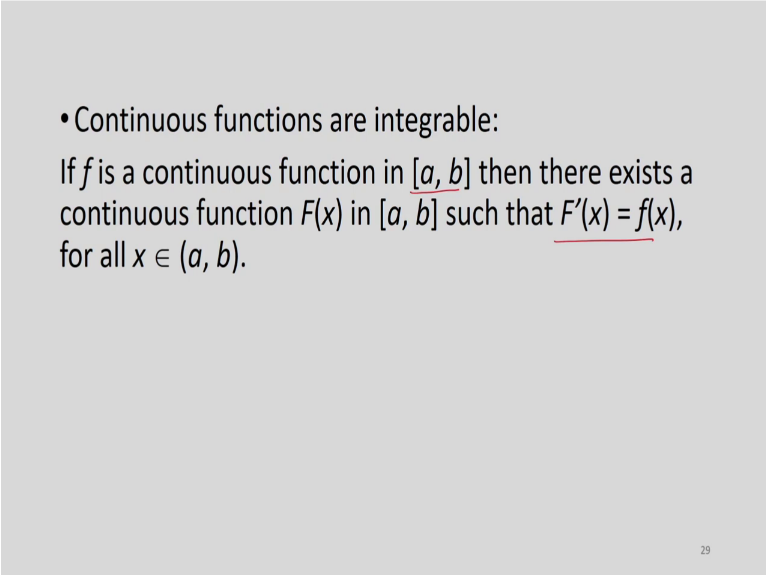


Continuous functions are integrable. So, If you have a continuous function, if small f is a continuous function in the closed interval a to b then there exists a continuous function capital F(x) in the same interval a to b such that $F^{/}(x) =$ f(x) for all x in the open interval a to b which basically means that f(x) is integrable, it is possible to integrate f(x).

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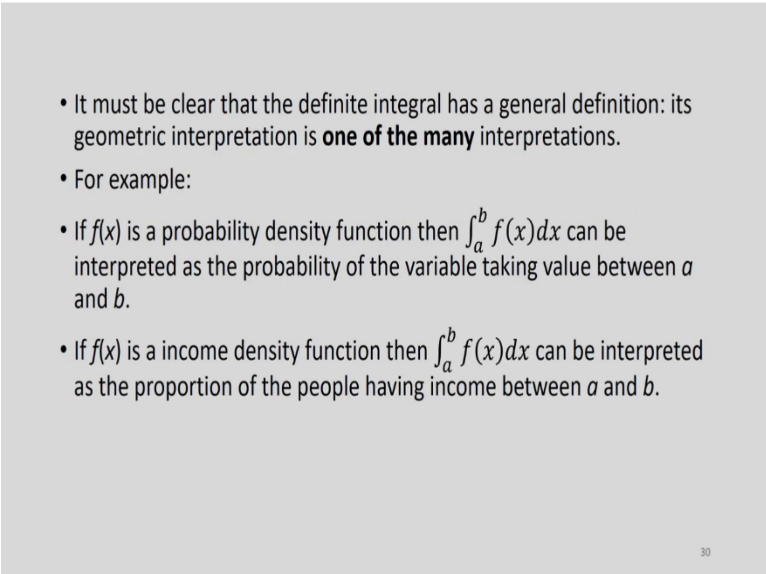


Now here is a clarification or sort of disclaimer, it must be clear that the definite integral has a general definition. Its geometric interpretation is one of the many interpretations. By geometric interpretation what we mean is the area under a curve. So, that is the interpretation we are often invoking but the definition, the general definition of definite integral as such, is more than just this interpretation of the geometric interpretation.

So, here are some other sort of interpretations that one can adduce to. If f(x) is a probability distribution. This is coming from the subject of statistics, so suppose f(x) is a probability distribution, then $\int_a^b f(x)\,dx$ can be interpreted as the probability of the variable taking value between small a and small b.

So, here one is interpreting the definite integral as probability of a particular event and then there is this other sort of interpretation from the subject of economics. If f(x) is an income this should be an, is an income density function then $\int_a^b f(x)\,dx$ can be interpreted as the proportion of the people having income between a to b. So, this interpretation is somewhat similar to the statistical interpretation but you can see what is going on that fx is the income density function and we are going to talk more about that later on when we deal with income distribution and the application of definite integrals there.

Now f(x) is income density function then if I take the definite integral of f(x) within in a particular range, particular limit a to b, then how do we interpret that number that definite integral number, it can be interpreted as the proportion of people having income between these 2 values that is a and b.

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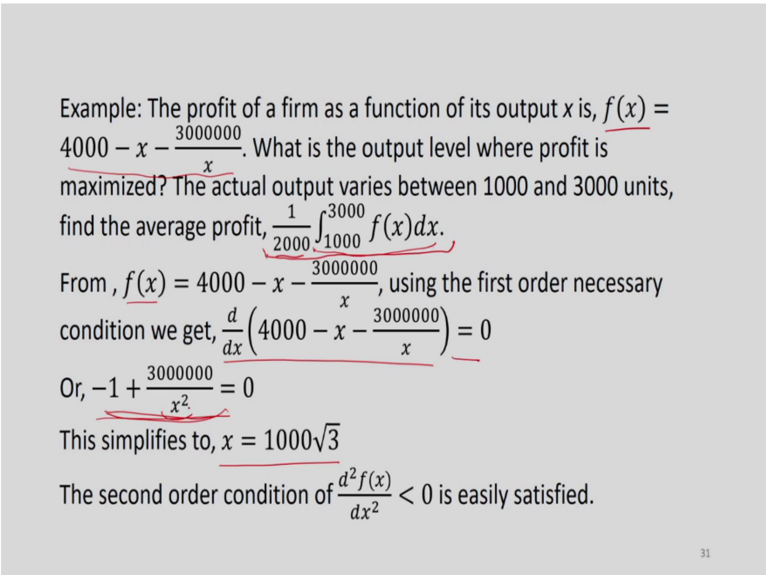

So, here are some examples, enough theory. We have talked about many theories so far. Now let us take some examples to fix our ideas. The profit of a firm as a function of its output is given by this f(x) and what is f(x), it is this 4000 - x - $\frac{3000000}{3}$. So, this function f(x) is the profit and what is x is the output. So, as you can see the function is not a kind of a very monotonic function as x rises it is not sure how f(x) will behave it might go up it might go down also.

What is the output level where profit is maximized that is the first part at what output level profit is maximized? Second; the actual output varies between 1000 and 3000 units, find the average profit which is given by this form. So, there are 2 parts to this question. First is at what output level the profit is maximized? Now this first part is simply a problem of optimization with a single variable. The second part is the part which is relevant to integration.

So, what is being said in the second part? Second part is saying that the actual output level which might not be maximizing the profit mind you. The actual output varies between 1000 and 3000, find the average profit and helpfully they have given us the form of the average profit. How does it define $\int_{1000}^{3000}$ f(x) dx. So, this is what?

This is the total profit, how because f(x) is the profit but profit can vary depending on the output level. So, if I integrate toward this range 1000 to 3000, I get the total profit and what is the average profit? It is the total profit divided by the midpoint between 1000 and 3000. So, this is the midpoint 2000 so that is how this is the average profit.

Now first thing first, we take this function and try to see where it is getting maximized. So, I use the first order necessary condition. So, I take the derivative as said that equal to 0. From this I get - 1 + $\frac{3000000}{x^2}$ because there is a what is the power of x it is minus 1. So, if I take the derivative of that minus 1 will come first and then minus, minus become plus and x to the power minus 1, minus 1 that is minus 2 and therefore I have divided by x square.

And then this gives me this solution x =$\sqrt{3000000}$. I take the positive root because output cannot be negative and $\sqrt{3000000}$ is 1000$\sqrt{3}$. Now this was coming from the first order condition, so I have to see whether the second order condition is satisfied or not. For that I take the derivative of this that is the second derivative of the profit function and this comes out to be clearly negative because here the power of x is -2, so - 2 will come first.

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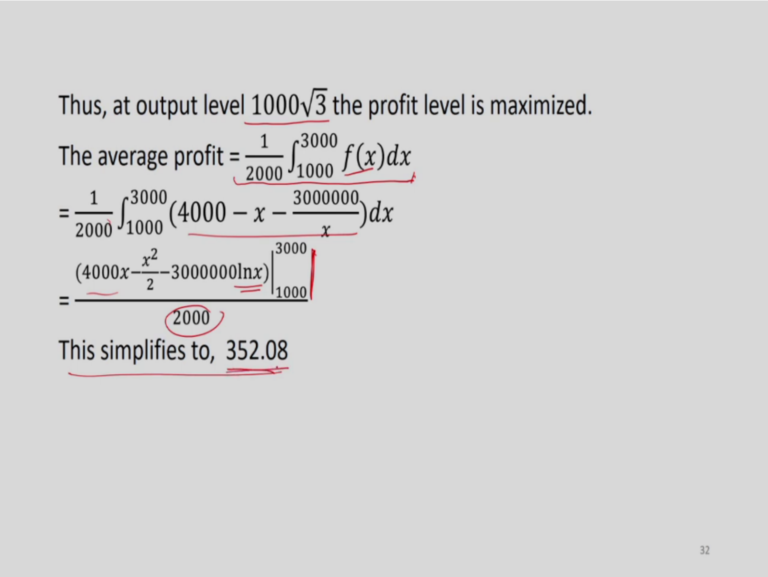


And so the second order condition is satisfied because the second derivative is negative. So, this indeed is the profit maximizing output level. So, the first part is solved. Now as we know the actual output does not remain fixed at 1000 $\sqrt{3}$. The actual output varies between 1000 and 3000 so the average profit is this much. So, we have to find the integration of this. I write the profit function here f(x) which is this, 4000 - x - $\frac{3000000}{x}$.

And I am dividing this whole thing by 2000 so 2000 is appearing here and integration of this is $\frac{4000x-x^2}{2-3000000}$, $\int \frac{1}{x}$ what is that it is log( x) . Now I have to take the value of this at 3000 and from that I have to subtract the value of this at x =1000. So, that is what is written here and if I do that actually I am going to get this figure 352.08. So, I have not shown this in the slide itself. It is a tedious mathematical procedure so at the end of the day I am going to get 352.08. So, this is the average profit of the firm in reality. So, this may not be equal to the maximized profit mind you.

I think I will stop here and take up the next topics in the next lecture. So, thank you for joining us and I will see you in the next lecture Thank you. Have a good day.