Mathematics for Economics – I
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Module: Single Variable Optimization 2
Lecture 24: Profit Maximizations

Welcome to another lecture of this course, Mathematics for Economics part one. So, we are discussing this particular topic in this course, single variable optimization and so, essentially what we are doing is that we have looked at the properties of optimization where there is a single variable, single decision variable.

And at what conditions the function will be maximized or minimized, conditions could be of different nature they could be necessary conditions, they could be sufficient conditions and then what we are doing right now, in the last lecture and today also shall be doing some application of this particular topic that is single variable optimization. I hope to finish this particular topic in today's lecture itself. So, let us go to where we left off in the last lecture.

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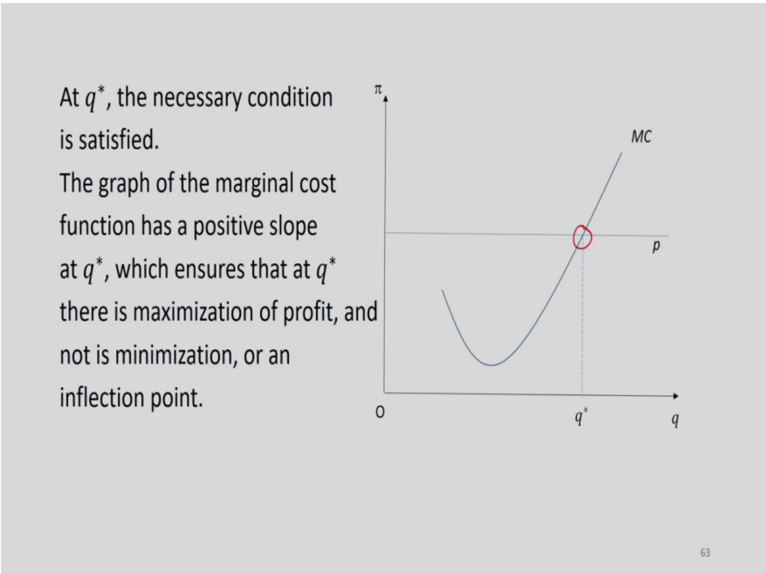

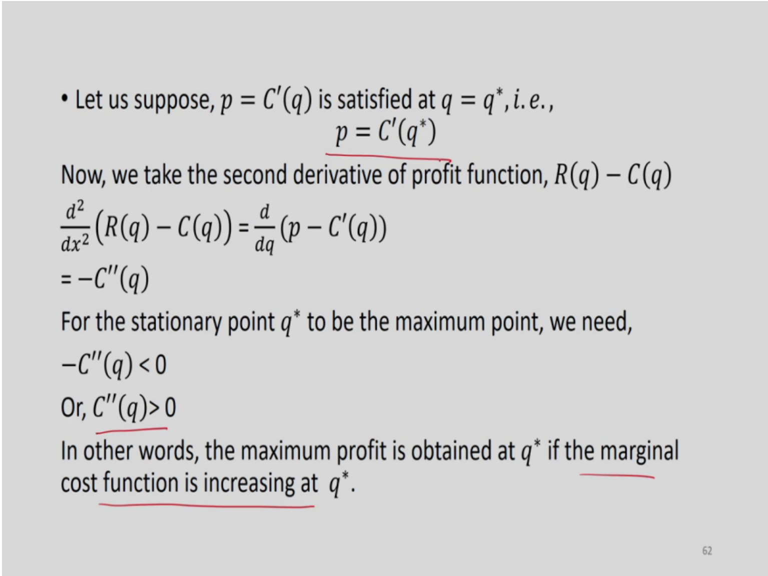

So, this is what we covered in the last lecture, we talked about what happens if you have perfect competition and in a perfect competitive market, there is a single firm not a single firm there are many firms, but we are looking at one firm and that one such firm in a perfect competition market is trying to maximize its profit and we are trying to see what are the necessary and sufficient conditions.

And by applying the necessary and sufficient conditions that we obtained from the previous discussion, we have seen that the necessary condition essentially requires that the price which is constant in the market should be equal to the marginal cost of the firm at the point which gives the firm the maximum profit.

So, this is the necessary condition that is if there is a point where profit is maximized then this particular condition has to be satisfied and the sufficient condition is that at that very point the marginal cost should be rising, the cost function should be convex.

So, this is the condition that we have, the marginal cost function is increasing at q^*. So, this is the second order condition, $C''(q) > 0$ and this is the $p = C'(q^*)$ is the necessary condition and this was the diagrammatic exposition of these two conditions and what we said that at this point, both these conditions are satisfied the $p = MC$ and MC is an upward rising curve at this point. So, the second order condition is also satisfied.

So, q^* is the point where profit is maximized, one can think about the extension of this curve and at this point there will be another point where the first order condition will be satisfied that is $p = MC$, but at this point that is let us suppose $q^{'}$, the second order condition is not satisfied because the MC is declining rather than upward rising. So, $q^{'}$ will not give me a point of maximization of profit whereas, q^{*} is the point where profit will be maximized.

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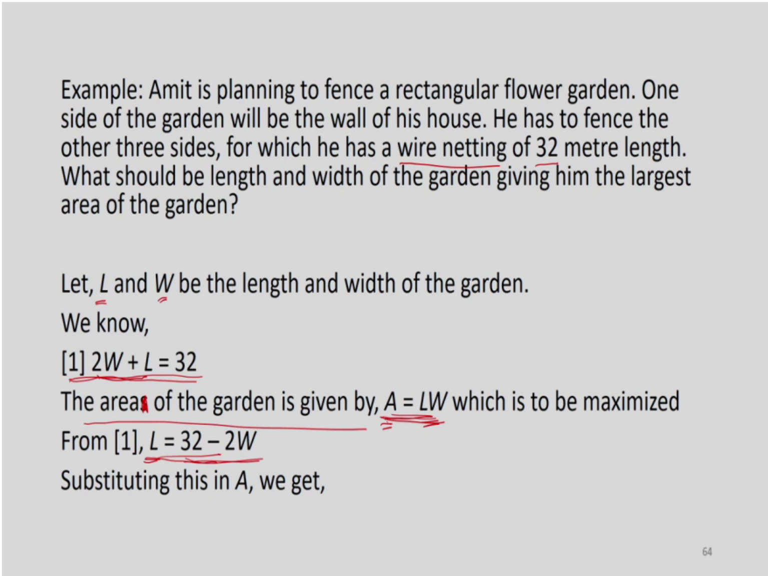

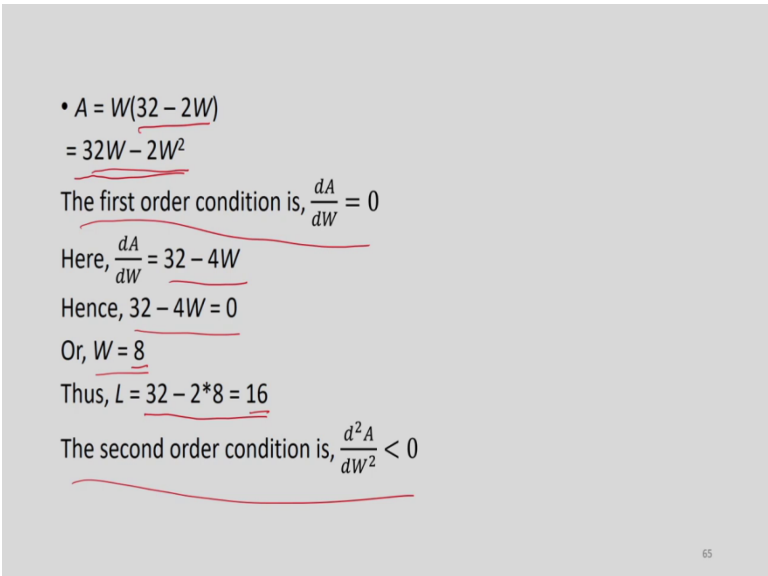

Now here is another example. Amit is planning to fence a rectangular flower garden. One side of the garden will be the wall of his house. He has to fence the other three sides for which he has wire netting of 32 meter length. What should be the length and width of the garden giving him the largest area of the garden?

So, this is the question, two things are to be noted; he has to build a fence for the garden and he has to build the fence such that it covers three sides of the garden because on one side he has the wall of his own house. So, that wall will serve as the 4th fence. So, the summation of these three sides must be equal to the length of the wire netting that he has.

The wire netting that he has is 32 meter long. So, this 32 meter long wire netting must be able to cover the three sides that he will be planning for the garden. Our problem is to help him solve this question, that if he wants to maximize the area of the garden, then what should be the length and width of that garden.

Let us assume that L is the length of that proposed garden and W is the width of that garden. Now, we have a relation like this: $2W + L = 32$. Remember, he has to cover three sides of the garden. So, $2W + L$ that basically adds the three sides and so that summation should be equal to 32, that is the condition that we are getting here.

Now, at this point one might ask, why is he choosing to use the wall of his house to cover the length rather than the width. Here when you are writing this then you are assuming that one L is being taken care of by the wall of the house. That is what it means, we are not writing $W + 2L = 32$ we are writing $2W + L = 32$.

So, what we have done is that implicitly we have taken the length of the garden, at least one length of the garden to be covered by the wall of the house and the reason is very simple. Remember, he wants to maximize the area of the garden and so, if it will be optimal for him to cover the length, which is the longer side of a rectangle by the wall of his house, because that will save him more wire netting. If he had covered the width of the garden with his house wall then maybe he will be able to cover one side but that will not be an optimal choice.

So that is why one length is being covered by the wall of his house. So that is why $2W + L = 32$. Now, what he wants to maximize is the area. Now what is the area of this garden that we are talking about? The area is supposed, it is given by capital A, this should be equal to the length multiplied by the width that is, $A = LW$.

So, Amit wants to maximize this A, that is what it boils down to and this $A = LW$. There are two variables here. Now at this point of time we do not know how to maximize a function which has two independent variables. So we have to do something about this form of the function.

What we do is that we use this particular relationship 1, which is $2W + L = 32$. So from this relationship we express the length that is L as a function of the width, so we get $L = 32 - 2W$. So this is the relationship and this will help us to convert this function, the A function into a function of a single variable.

What we do is that, we substitute L from this relation into the area function. So, substituting this into A, we get, $A = W(32 - 2W)$, basically, instead of L I have written $32 - 2W$. Now, that basically equals to $A = 32W - 2W^2$. Now, this has to be maximized. Now, it is a very simple thing, it is a function of a single variable and we know the techniques of maximizing a function which has only 1 variable.

Now, what is the first order condition? I set the derivative equal to 0. So, $\frac{dA}{dW} = 0$, the necessary condition, if you take the derivative of this function with respect to W, you get $\frac{dA}{dW} = 32 - 4W$, so $32 - 4W = 0$ and that gives me $W = 8$. So, the width of the garden should be 8 meters and what is the length? Remember, I know L as a function of W.

$L = 32 - 2W$. So, I will substitute W here, the optimal W here. So, it will give me $L = 32 - 2.8$. So, $L = 16$. So, these are the solutions: the width should be equal to 8 meter, the length should be equal to 16 meter.

However, this is just the stationary point, we have not checked the second order condition that is important because, if we did not do so, it might happen that this solution that we have got, could give us the minimum point or it can give me an inflection point. So, we have to take the second derivative, $\frac{d^2A}{dW^2}$.

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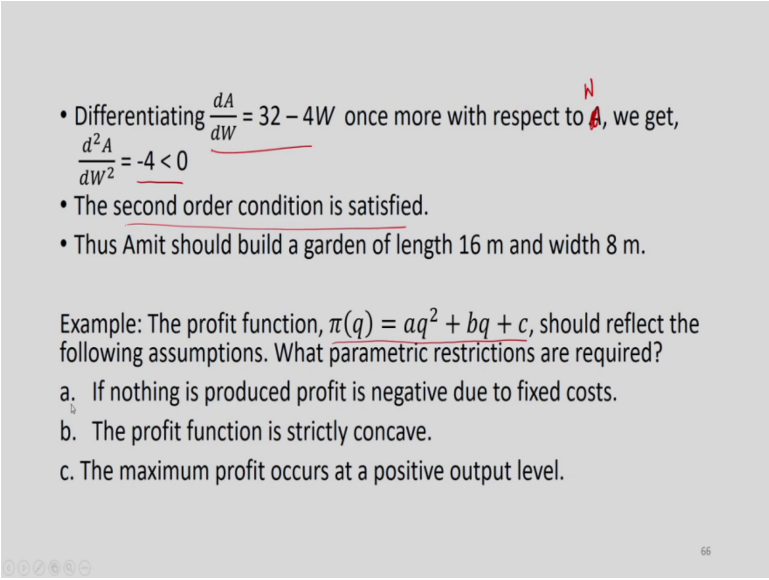

So, $\frac{dA}{dW} = 32 - 4W$, we differentiate this function once more with respect to W and if we do so, it gives me $\frac{dA}{dW} =- 4, - 4$ is always negative irrespective of what W you put there. So, the second derivative is negative which makes the function concave, A is a concave function of W, therefore, the W^* we got the maximum point.

So, the second order condition is indeed satisfied. So, that is our solution that we just obtained from the first order condition is the optimal solution the garden should have a length of 16 meter and width of 8 meters and that will add up to what that will add up to 32 meters because 2 width will be 16 and 1 length is 16, $16 + 16 = 32$ which is the length of the wire netting that he can spare.

So, our solution is adding up. So, here is another example unlike the previous example, this example is from economics, a profit function is given. So, $\pi(q) = aq^2 + bq + c$,. So, this is a general quadratic function. Now, this quadratic function general form should reflect the following assumptions and if it has to reflect these assumptions then what parametric restrictions are required, that is the question.

Number 1: If nothing is produced, profit is negative due to fixed costs. So, that assumption should be reflected in this profit function. Secondly, the profit function is strictly concave, that is the second condition and thirdly the maximum profit occurs at a positive output level. So, this is the third condition that should be reflected in the profit function. So, our task is to find out what restrictions should be there on a, b and c, these are the parameters what restrictions should be there on these three parameters so that these three conditions are satisfied.

So, that is the problem. Let us do them one by one. First one; if nothing is produced, profit is negative due to fixed costs, this should not be very difficult to do. So, I have to find out what the profit is when nothing is produced and whatever quantity that we obtain should be equal to less than 0 because of fixed costs. So, that is the thing that we want to apply here.

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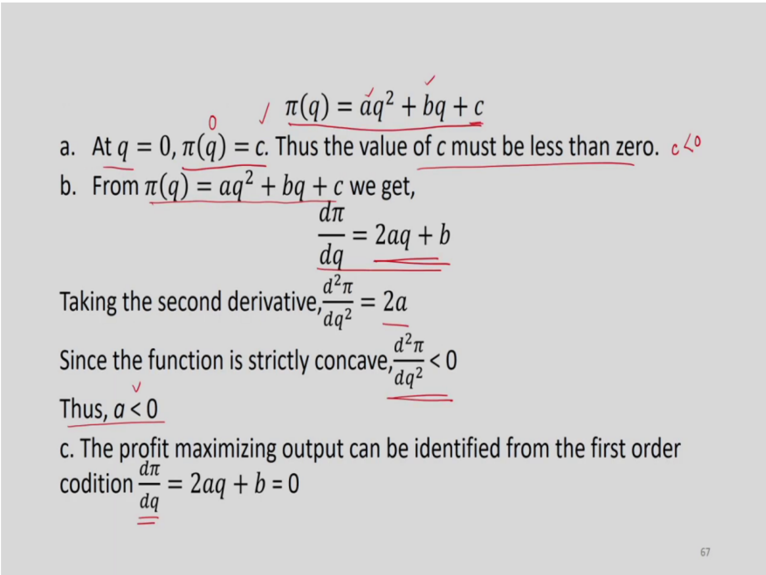

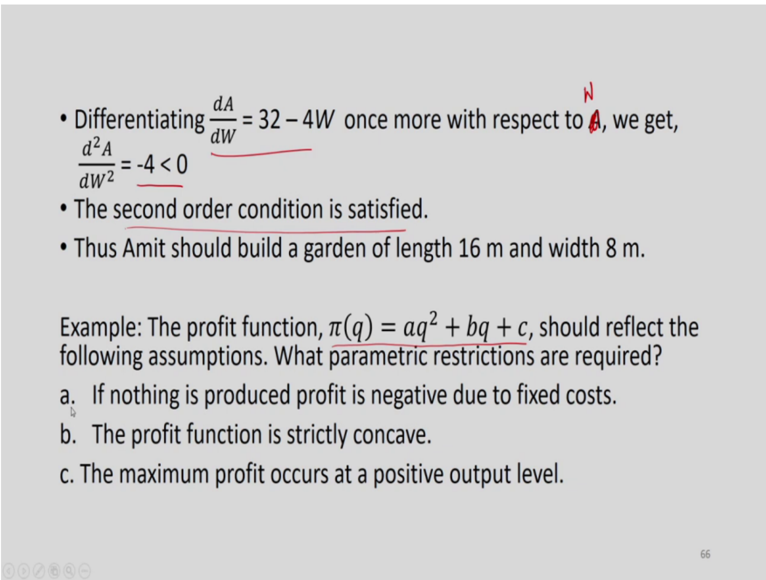

$\pi(q) = aq^2 + bq + c$. So, what we do is that we put $q = 0$ so, that will give me the profit at output 0. Now, if we do so, these first two terms will vanish because these are multiplied with 0. So, only the constant term will remain which is c. So, $\pi(0) = c$ and since in the condition it is mentioned that at $q = 0$ the profit should be negative that means c should be negative. So, therefore, the value of c must be less than 0.

So, this is the solution to the first part. What is the second part? The profit function is strictly concave and here what property we use? We use the property that function is strictly concave if and only if the second derivative of the function is negative.

Second derivative we have to find. So, this is the function $\pi(q) = aq^2 + bq + c$; first we take the first derivative that is, $\frac{d\pi}{dq} = 2aq + b$. We have used the power rule it should be $2aq + b$ and then we take the second derivative that is $\frac{d^2\pi}{dq^2}$ and this again using the power rule this is, $\frac{d^2\pi}{dq^2} = 2a$.

Now, as I have just mentioned, if the function has to be strictly concave then the second derivative should be less than 0, $\frac{d^2\pi}{dq^2} < 0$. So, that is the condition we have applied here and so, $2a < 0$ that simply means that $a < 0$. So, this is the condition that we are getting now.

So, $c < 0,\ a < 0$. So, these are the two conditions we have got so far. What about the third part? Third part is saying that the maximum profit occurs at a positive output level. So, that basically asks us to find out what is the maximum point and that point should be greater than 0.

So, we have to maximize this profit function. The profit maximizing output can be identified from the first order condition that is the necessary condition $\frac{d\pi}{dq} = 0$ and that basically means this $\frac{d\pi}{dq} = 2aq + b = 0$.

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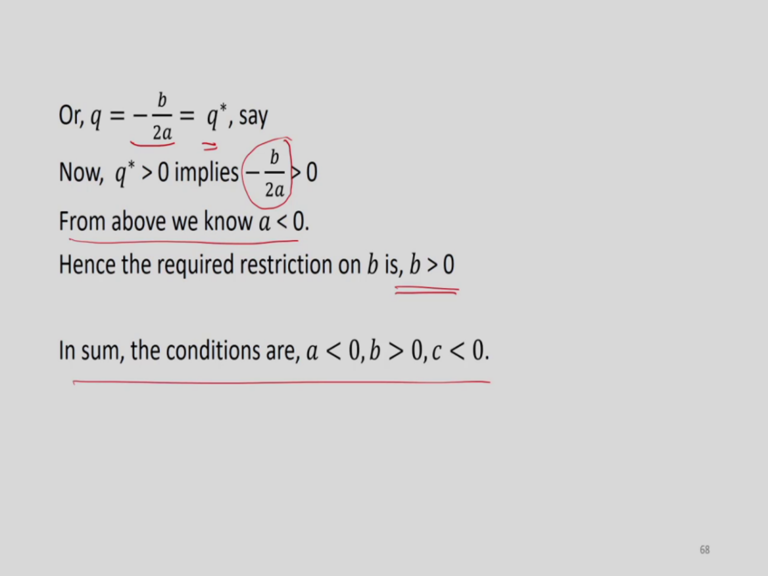

So, from that condition we express q as a function of these parameters. So, $q = \frac{-b}{2a} = q^*$. Now $q^* > 0$ that is what the third part is demanding. Now, $q^* > 0$ means $\frac{-b}{2a} > 0$, but we already know that small $a < 0$ from the second part.

That basically means that if this entire thing has to be positive then $b > 0$ because minus and minus will make it a plus expression, positive expression. So, $+ b > 0$ means $b > 0$. So, we have solved the third part as well. So, putting all these things together this is the conditions, this is the set of conditions that we have got, $a < 0,\ b > 0,\ c < 0$.

So, this is an interesting pattern that we are observing here which is obtained from these three conditions and this pattern is such that the signs are alternating, starting with the minus sign, this is a very interesting property and it will recur in many economic problems.

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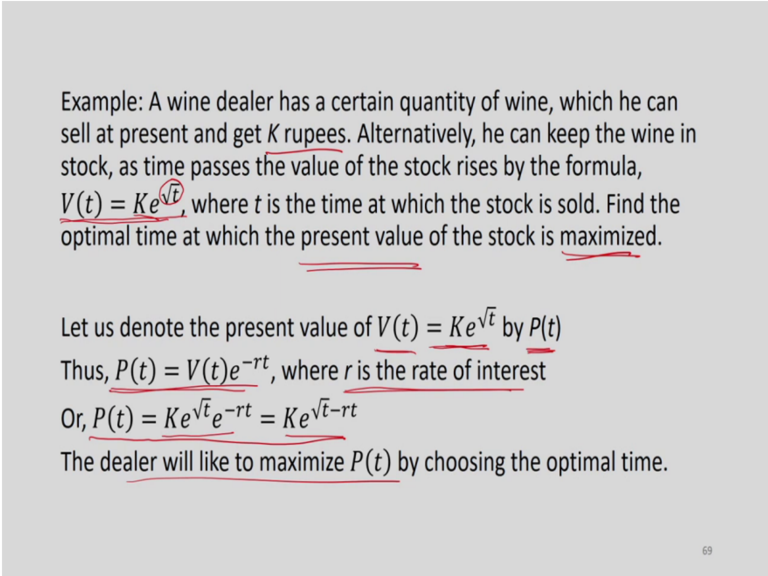

Here is another example again from the world of economics. A wine dealer has a certain quantity of wine which he can sell at present that means now and get K amount of money K rupees. Alternatively, he can keep the wine in stock as time passes, the value of the stock rises by the formula $V(t) = K.e^{\sqrt{t}}$, where t is the time at which the stock is sold.

Find the optimal time at which the present value of the stock is maximized. So, present value is important here, it is a problem of maximization, but at the same time, it is also applying the idea of present value. Let us try to understand the problem a bit. So this person the dealer has a certain quantity of wine.

Now, as we know the value of wine rises with time, if you keep this wine in your cellar, then as time passes, the wine matures and its value rises and so that increasing value is represented by this formula, you can just put $t = 0$ in this formula. So that will give you e^0, $e^0 = 1$.

So that means if $t = 0$ then the value of the wine is $V(t) = K.1 = K$, so that means that if you are selling the wine right now at the point $t = 0$, the value that you will obtain is K rupees that is it. However, suppose you decide to wait for 1 period. So that will give you $t = 1$. So after 1 period the value of the wine will become $V(t) = K.e^{\sqrt{1}} = K.e$.

As we know $e > 1$. It is a value between 2 and 3. So after 1 period actually the value of the wine has gone up by a factor of more than 2. And like that, it will go on rising, as t goes rising, the power will go on rising and e to the power that number whatever that number is root over t; $e^{\sqrt{t}}$ will go on rising. So, that is the idea as time passes the value rises.

Now, the question is the following: if this seller wants to maximize the present value of the stock, his problem is he wants to maximize the value of the stock not at the point where it is being sold but its present value. Present value means standing at the point $t = 0$ there will be something called the present value, this is which we have discussed earlier on how to calculate the present value, that present value will be a function of time small t, t is the point where it is being sold.

He wants to solve the problem as to what optimal time t^*, let us suppose gives him the maximum present value. So, that is the problem. Now, let us suppose the present value of $V(t)$ is denoted by $P(t)$. We know $V(t) = K.e^{\sqrt{t}}$ it is a function of small t.

Now, the present value of $V(t) = P(t)$. Now, we know how to get the present value of anything, it is the value multiplied by e^{-rt} where r is the rate of interest. So, this we shall use to get the complete function of $P(t) = K.e^{\sqrt{t}}.\ e^{-rt} = K.e^{\sqrt{t}-rt}$ because we have to substitute $V(t)$ in this formula

$V(t)$ is given by $K.e^{\sqrt{t}}$, we write it here and we get this expression $P(t) = K.e^{\sqrt{t}-rt}$. So, we have found out the present value. Now, what the dealer wants to do is he wants to maximize it. He wants to maximize this $P(t) = K.e^{\sqrt{t}-rt}$ and he can do that because it is a function of a single variable and what is that variable? It is time, that is at what time the stock will be disposed that he has to find out.

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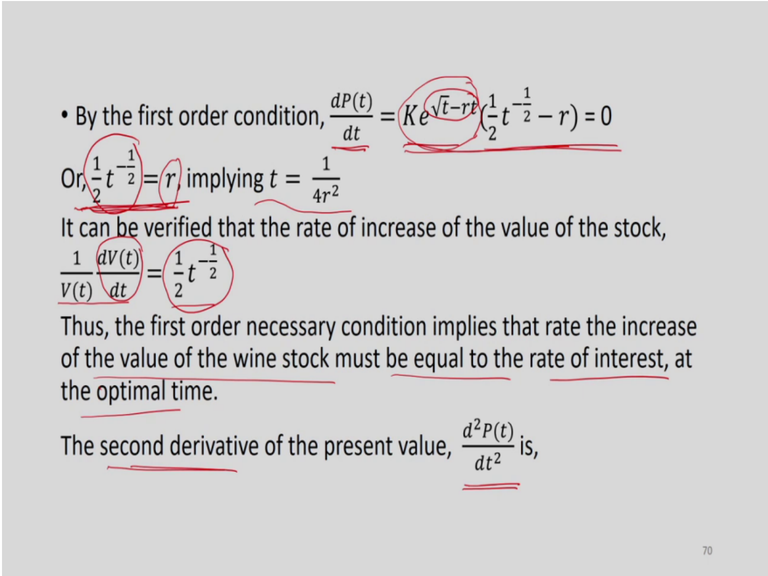

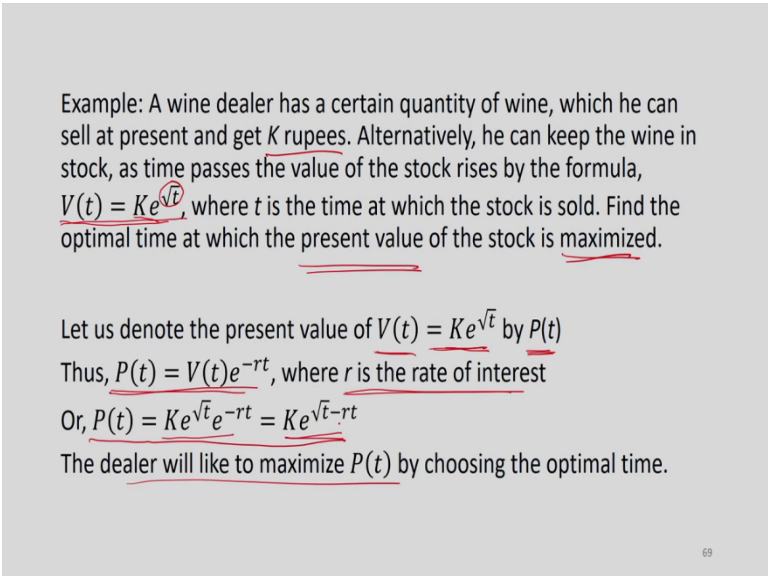

Now, how we solve the usual method, we first use the first order condition. So, I take the derivative of the $P(t)$ with respect to time and this is what I shall get, $\frac{dP(t)}{dt} = K.e^{\sqrt{t}-rt}(\frac{1}{2}t^{-1/2} - r)$ because $P(t) = K.e^{\sqrt{t}-rt}$.

$\frac{dP(t)}{dt} = K.e^{\sqrt{t}-rt}(\frac{1}{2}t^{-1/2} - r)$. So, this is a substitution method and if you take the derivative of $\sqrt{t} - rt$ then you get $(\frac{1}{2}t^{-1/2} - r)$. So, the whole thing should $\frac{dP(t)}{dt} = K.e^{\sqrt{t}-rt}(\frac{1}{2}t^{-1/2} - r) = 0$ by the first order condition. Now, this part cannot be equal to 0 because, you know t is limited, it is a finite number. So, the first part cannot be 0.

The second part that means will be equal to 0 to satisfy the first order condition. Second part is this $(\frac{1}{2}t^{-1/2} - r) = 0$ which if we simplify it then it becomes this, $t = \frac{1}{4r^2}$. So, I have to express this as a function of r because I have to find out the optimal t.

So, optimal $t = \frac{1}{4r^2}$. What does this condition mean, this condition I am talking about this condition, what does this mean? Now, it can be verified that the rate of increase of the value of the stock, rate of increase of the value of the stock is this $\frac{1}{v(t)}\frac{dv(t)}{dt}$ because this is the instantaneous increase of the value of the stock, .

So, rate of increase is $\frac{dv(t)}{dt}$ divided by $V(t)$ that is $\frac{1}{v(t)}\frac{dv(t)}{dt}$ is the rate at which it is rising. Now, this should be it can be verified $\frac{1}{v(t)}\frac{dv(t)}{dt} = \frac{1}{2}t^{-1/2}$ and I mean this can be seen from that formula itself, what was $V(t)$? $V(t) = K.e^{\sqrt{t}}$ was this, this is $V(t)$. So, if you take the derivative with respect to t and divide it by $V(t)$ itself you are going to get $\frac{1}{v(t)}\frac{dv(t)}{dt} = \frac{1}{2}t^{-1/2}$.

So, this is the rate of increase of the value of the stock, the rate of increase of $V(t)$. Now, the first order condition implies that the rate of increase of the value of the wine stock must be equal to the rate of interest at the optimal time, this is what this mean, this means, on the left hand side you have the rate of increase of the value of the stock the rate at which the value of the stock is increasing with time because as I said as time goes by the stock is rising in value because the wine is maturing.

So, this is the rate at which it is increasing the value is increasing on the other hand there is this rate of interest and rate of interest can be thought of as the opportunity cost instead of putting your money in stocking the wine if you put it in the bank then r is the rate of interest that you get.

So, at the optimal time these two sides must match with each other $\frac{1}{2}t^{-1/2} = r$ the rate at which money grows in the bank and the rate at which the value of the stock, wine stock grows in the cellar; these two must be matching with each other but that is only the necessary condition, that is the first order condition if this does not happen then you do not have maximum that is what this essentially means.

But that is not a sufficient condition, it is possible that at this point you are getting a minimum point and by the way, this sort of condition we obtained previously also when we were talking about the optimal time at which trees must be cut down. There also this time element was involved, it was a dynamic choice.

So, there also a present value was involved and we obtained a similar condition in that problem also. So, that is why I said that whenever dynamic choices are involved, similar conditions will recur. This was a necessary condition. Now, we come to the sufficient condition or the second order condition. Now, that will demand that the second derivative of $P(t)$, $P(t)$, remember was the present value of the stock. So, if you take the second derivative of the stock with respect to time it becomes $\frac{d^2P(t)}{dt^2}$ and it should be negative $\frac{d^2P(t)}{dt^2} < 0$.

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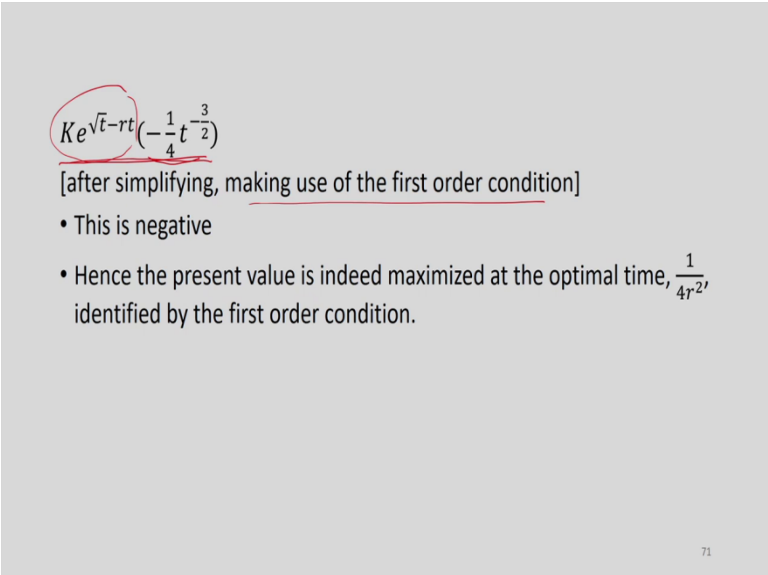

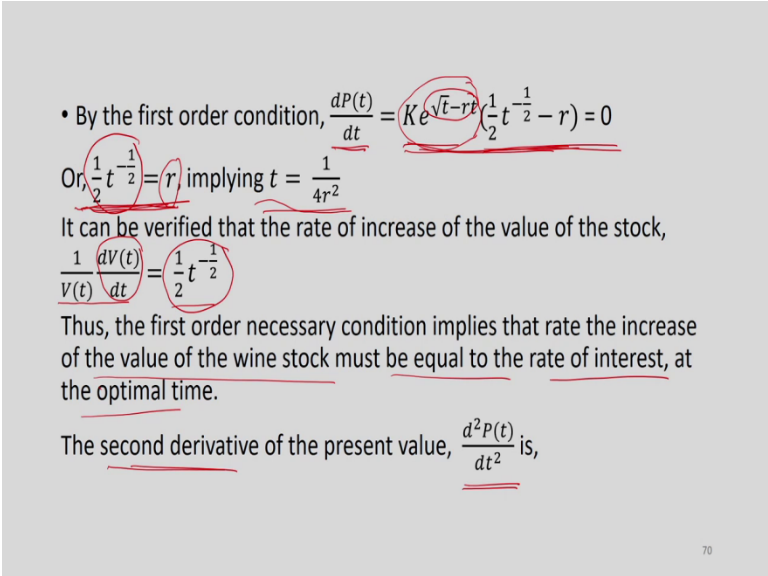

This is the first derivative, $\frac{dP(t)}{dt} = K.e^{\sqrt{t}-rt}(\frac{1}{2}t^{-1/2} - r)$ if you take the derivative once more with respect to t then actually it will turn out to be this quantity, $\frac{d^2P(t)}{dt^2} = K.e^{\sqrt{t}-rt}(\frac{-1}{4}t^{-3/2})$ and this has been obtained after some jumping off steps, I have simplified the expression and while simplifying I have made use of the first order condition, what was the first order condition that this $\frac{dP(t)}{dt} = K.e^{\sqrt{t}-rt}(\frac{1}{2}t^{-1/2} - r) = 0$.

So, if I use that then I simplify the second order condition to this simple expression $\frac{d^2P(t)}{dt^2} = K.e^{\sqrt{t}-rt}(\frac{-1}{4}t^{-3/2})$. Now this is the second derivative, this is the second derivative and this should be negative and it is obviously negative because the first part is not negative and neither is 0. For the reason I have just told you and whatever the second part well as long as t is nonzero and so this should be the whole thing should be negative.

So we are through actually the t that we have obtained by the First Order condition which is $t = \frac{1}{4r^2}$is actually the point where the maximization is occurring. As the present value is indeed maximized at the optimal time that we have obtained, which is $t = \frac{1}{4r^2}$ so that is the solution.

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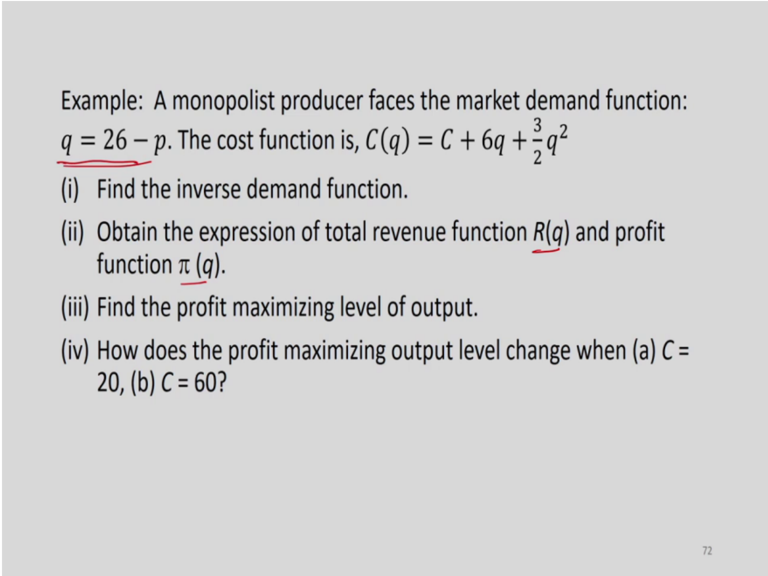

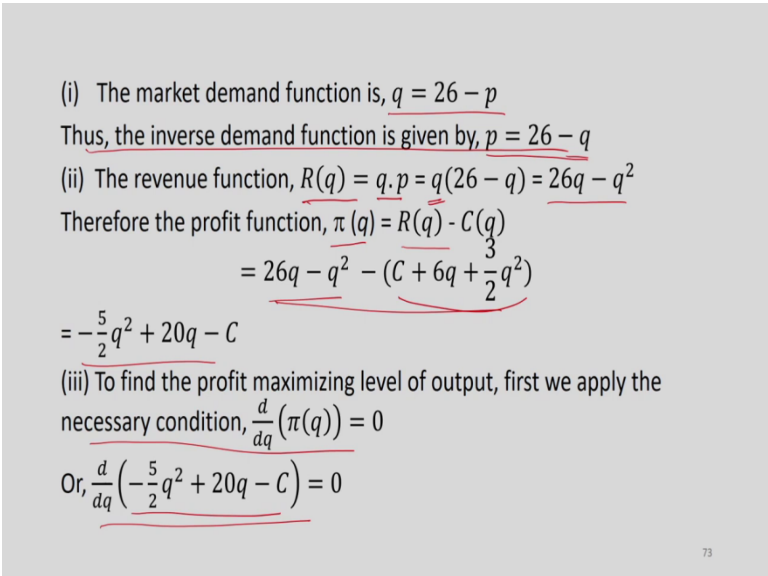

Here is another problem and this is again from the world of economics. A monopolist producer faces the market demand function $q = 26 - p$, the cost function is $C(q) = C + 6q + \frac{3}{2}q^2$. Find the inverse demand function. Obtain the expression of total revenue function $R(q)$and profit function $\pi(q)$.

Find the profit maximizing level of output. How does the profit maximizing output level change when $C = 20$ and $C = 60$? The first three parts should not be difficult, the last part could be a little bit tricky; you have to look into that. So, first we have to find out the inverse demand function. The demand function is given, $q = 26 - p$. We have to find $p(q)$ that is the inverse demand function.

The market demand function is $q = 26 - p$, the inverse demand function is simple I take the inverse of this and this is $p = 26 - q$, just one small point to be noted here is that this function is valid as long as q is not greater than 26, if q is greater than 26 then this function will give me negative price and which is not feasible. The second part was obtain the expression for total revenue, function $R(q)$and profit function $\pi(q)$.

What do we know about revenue? Revenue is equal to the total amount of money that the producer gets by selling his goods. So, it is $R(q) = pq$. Now, here we make use of the inverse demand function and that is $p = 26 - q$ that then is multiplied by q. So, that, the expression becomes a function of q only.

So, it is $R(q) = pq = 26q - q^2$ so this is the revenue function, from this we will get the profit function; what is profit? Profit is revenue minus cost, $\pi(q) = R(q) - C(q)$. So, revenue we have found out it is $R(q) = pq = 26q - q^2$ and cost is given, cost is given to be $C(q) = C + 6q + \frac{3}{2}q^2$ that we substitute here in the cost function and then this is simplified as $\pi(q) =- \frac{5}{2}q^2 + 20q - C$. So, this is the profit function.

Third part obtain the profit maximizing level of output, this is the usual method we are going to apply to find the profit maximizing level of output first we apply the necessary condition that is the derivative of the profit function with respect to quantity is equal to 0 and basically we have to differentiate $\pi(q) =- \frac{5}{2}q^2 + 20q - C$.

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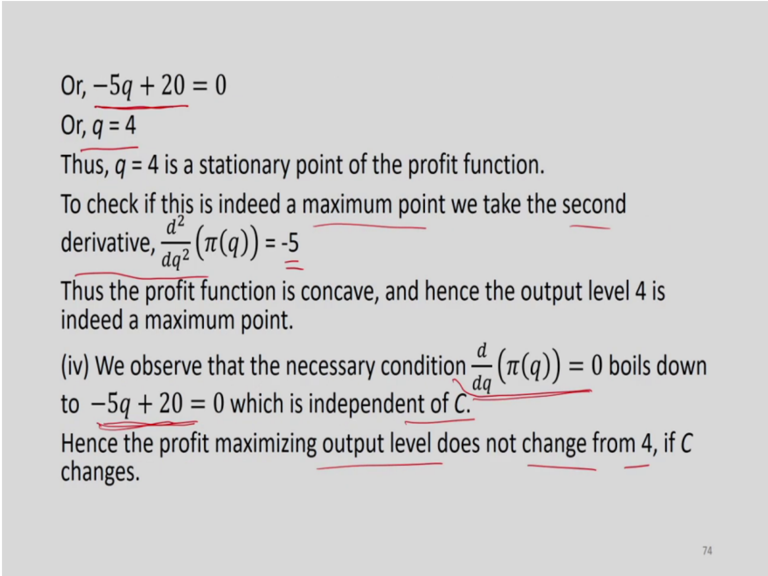

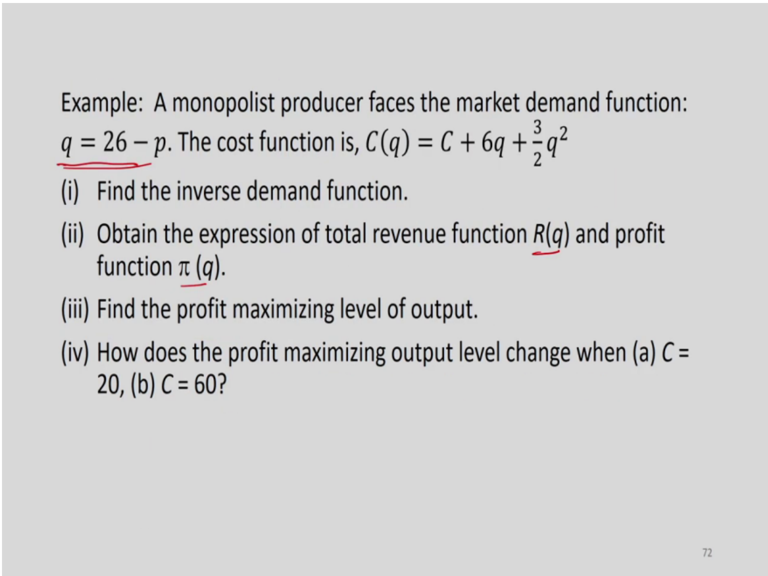

If I do so, I get $\frac{d\pi(q)}{dq} = -5q + 20 = 0$ and that basically boils down to $q = 4$. Now, $q = 4$ is a stationary point, at this point the first order condition is satisfied, we do not know whether it is being maximized or minimized. So, to check for that, whether we actually have a maximum point we take the second derivative of the profit function and if we do so, this basically means that this expression $\frac{d\pi(q)}{dq} = -5q + 20$ has to be differentiated once more and we get $\frac{d^2\pi(q)}{dq^2} = -5$.

If it is negative, profit function is concave and therefore, we are actually getting maximum point at $q = 4$. Now the fourth part, fourth part is how does the profit maximizing output level change, when C is equal to, C means the fixed cost, there is a fixed cost here C, when $C = 20$ and $C = 60$.

So, the question basically is does the profit maximizing output level change? If it does, how?, If the fixed cost of production change? Now, we observed that the necessary condition was giving me the optimal q and this is the necessary condition it boils down to $\frac{d\pi(q)}{dq} =-5q + 20 = 0$ and this condition is independent of capital C.

So, C is not making its presence in this condition. So, it is independent of C hence the profit maximizing output level does not change from 4 if capital C changes. So, if the fixed cost changes as it is written in the question from 20 to 60 it does not make any difference to the profit maximizing output level, it remains at 4.

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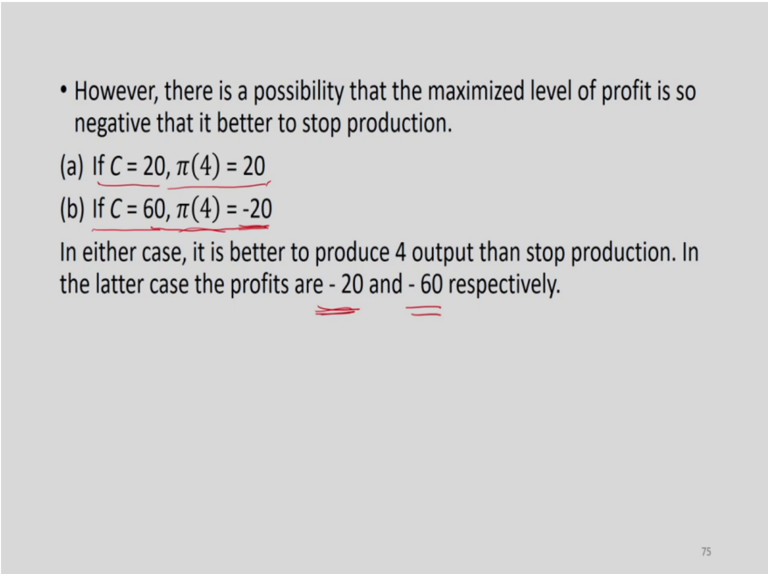

However, there is one caveat; however, there is a possibility that the maximized level of profit is so negative that it is better to stop production. Let us try to understand what I am trying to say if the fixed cost becomes very high, it is possible that even after maximizing the profit; the profit that you obtain becomes extremely negative.

At that point of time it may be possible that it is better to stop production. Rather than producing $q = 4$ if you produce $q = 0$ maybe the profit will not be that bad, that is a possibility. Let us see whether that is correct or not in this case. If $C = 20$, what is the profit? The profit is turning out to be 20, $\pi(4) = 20$.

If $C = 60$ the profit is turning out to be -20, $\pi(4) = -20$. And what happens if this producer does not produce, if he does not produce when the fixed cost is 20, he is getting a profit of -20, it is written here. So, if he does not produce, he gets -20 profit, if he maximizes profit and produces something, his profit is 20. So, it is optimal to produce rather than not produce.

In the second case, if he maximizes profit, and you know produces 4, then he is getting a profit of -20, which is negative, actually a loss. But if he does not produce at all, the loss actually rises. So, in either case, he is making a loss, but if he produces 4, the loss is less.

So, therefore, in either case, it is optimal to produce rather than not produce. So, that gives us the answer that, as the fixed cost changes from 20 to 60. In either case, the profit maximizing output level remains at 4. I think I will call it a day now, our topic of optimization of functions with a single variable that is finished, we have done quite a few exercises from that topic. From the next lecture onwards we are going to take up a new topic. Thank you.