Mathematics for Economics – I
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Module: Single Variable Optimization 2
Lecture 23: Inflection Points

Welcome to another lecture of this course called Mathematics for Economics part one. So, the particular topic that we have been covering is called single variable optimization. Now, as you can see on this cover screen you can see the name of the topic single variable optimization, but the point where we left in the last lecture was about concave and convex functions.

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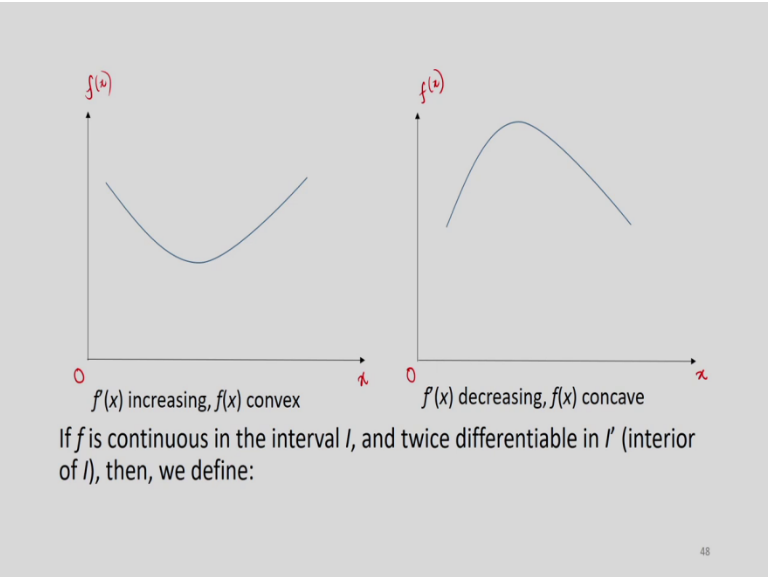

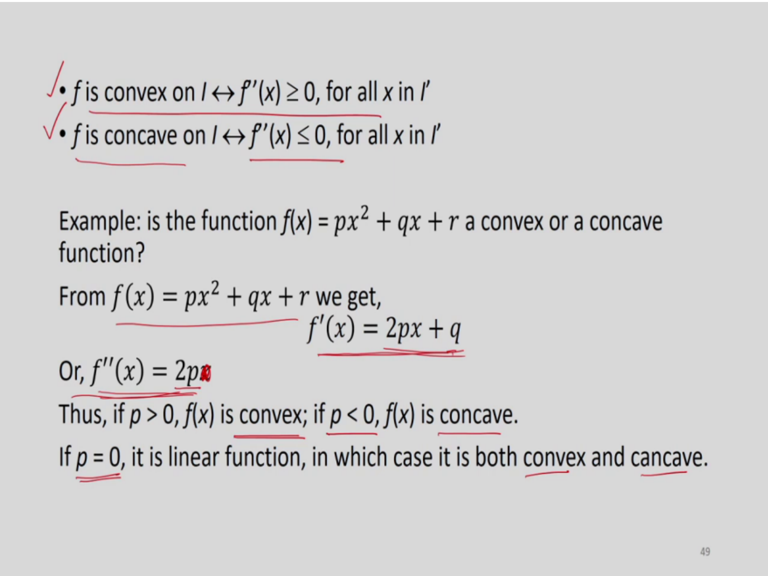

And here is what we have been talking about if a function f is differentiable, twice differentiable in the interior of I, I' is the interior of I and it is continuous in that interval I, then we define f to be convex in I and that statement is ↔ $f''(x) \geq 0$ for all x in I' and f is concave in I, that statement is ↔ $f''(x) \leq 0$ for all x in I'.

So, the point that is being made here is that for convexity, the derivative should go on rising and for concavity the derivative should go on declining; both these things are in a weak sense. It is possible that $f''(x) \geq 0$ which means that the derivative of x is rising and if $f''(x) \leq 0$, which means that the derivative of x is going on declining.

So, this is the second derivative. So, that second derivative is important to understand if the function is convex or concave. If the second derivative is rising, that is in the first case the second derivative is positive that is the first case, then the function is convex and if the second derivative is negative then the function is concave and here is the diagrammatic exposition of that.

So, on the left hand side you have a convex function, here as you can see the derivative is rising. How do I know that? Well look at the slope of the function at two successive points, here this is the slope but at a higher level this is the slope, the slope is rising. So, here the slope was even less, it was 0 and then it was becoming positive and then it is getting more and more positive.

So, this is why I am saying that the first derivative is rising, which means that the second derivative is positive and the function is convex. On your right, you have the opposite case where the function is concave; here the derivative was positive, here at this point, but then it fell.

So, the line is becoming flatter. That means the slope of the tangent is becoming less. So, the first derivative is declining, which means that the second derivative is negative, the function is concave. Here is an example: is the function $f(x) = px^2 + qx + r$. Is it a convex or concave function?

Now, what do we do? We take this function $f(x) = px^2 + qx + r$, where p, q, r are parameters and x is the variable. So, we take the first derivative, if we take the first derivative, it becomes $f^{'}(x) = 2px + q$, by the power rule and we need to find out the second derivative, because this is the property that we are going to use, this is the property.

So we take the secondary derivative, that is we differentiate this function once again with respect to x and if we do so, it becomes, this is simply $f^{''}(x) = 2p$. So it is $2p$, second derivative is $f^{''}(x) = 2p$. Now, if $p > 0$, if p is positive, then the second derivative is also positive, that is $f^{''}(x) \geq 0$ and then we apply this rule, if the second derivative is positive, then the function is convex.

And on the other hand, if $p > 0$, that is p is negative, then this second derivative will be negative, because the second derivative is $2p$. In that case, this applies so the function is concave. So this is how we can actually apply this rule to judge whether a function is convex or concave.

Now one might ask that suppose p is neither positive nor negative, then what is the conclusion? So if $p = 0$, when it is neither positive nor negative, then actually what we see is if $p = 0$, then this second derivative, $f^{''}(x) = 2p$, becomes 0 and in fact, if we look at the first derivative, the first derivative actually is giving us just q, the first derivative is q, which is a constant, that means that the function is a linear function.

So it is a straight line and if it is a straight line then it is both convex and concave. So, this is an application of the rule that we have just talked about, about the second derivative, the sign of the second derivative.

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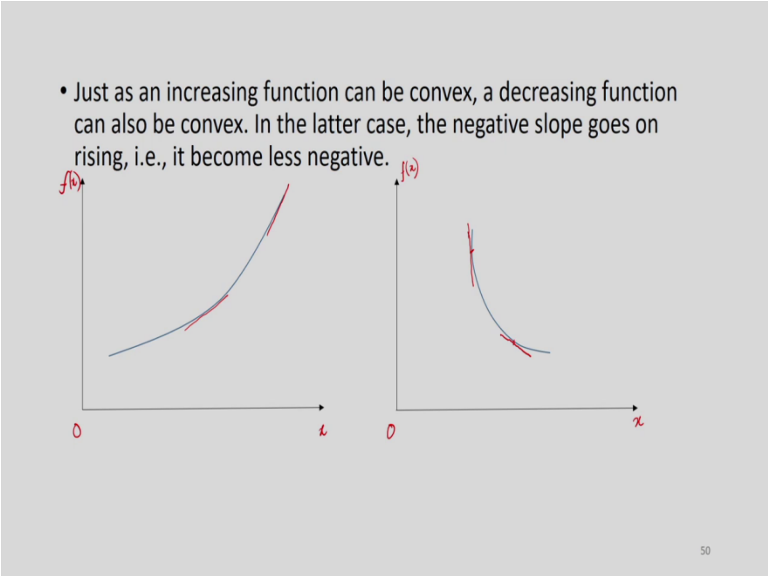

Now, just as an increasing function can be convex, so here is an increasing function which is convex on the left. How do I know? You can just verify that the slope is rising, the slope is rising, that is why it is convex. But this increasingness has nothing to do with convexity; a decreasing function can also be convex.

So here is an example of a decreasing function which is convex. In the latter case, the negative slope goes on rising, that is it becomes less negative. So the only thing we have to notice is what is happening to the slope. Is it rising? If it is rising, if the slope is rising, then it is convex.

So it is immaterial whether the function is a rising function or declining function. So on your right, you have a declining function, but the slope is rising. How do I know that? Well, again, take two points and find out what the slope is, here the slope is very high, high means the absolute value of the slope is very high, the line is very steep. But at the same time, this number is negative.

So if the absolute value is very high, and if it is a negative number, it means it is a very small number, like let us say -20 or something. On the other hand, if you go to the right a bit and then you find the slope, here also the slope is negative in sign because the function is declining; the slope has to be negative.

But the absolute value has gone down, which means that the algebraic value of the slope has gone up. So an example could be that at this point, on the left, it could be -20. But here it is, let us say -1. Now -1 > -20 which means that the slope is rising.

Of course the function is a declining function but the slope is rising and if the slope is rising then our conclusion is that the function is convex function. So, both these functions are in fact convex functions.

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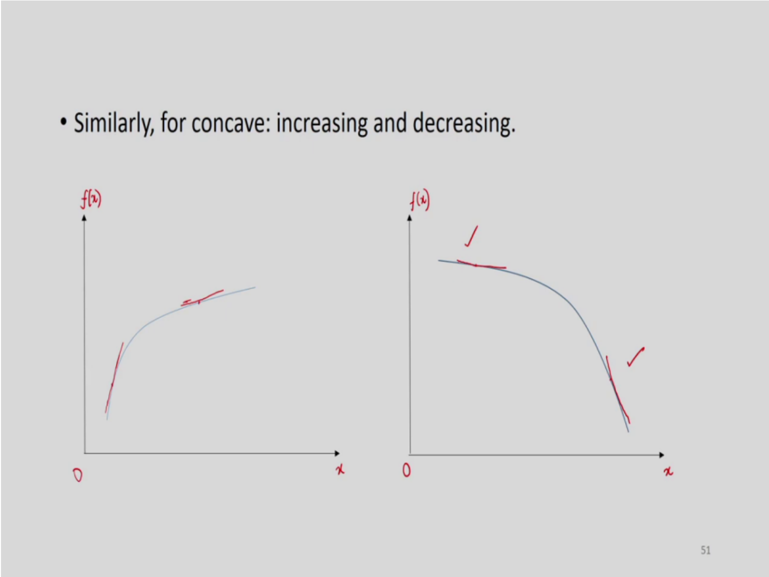

The same thing happens for concave function as well. Here you have two concave functions one is a rising concave function and the other is a declining concave function but both are concave functions. Again let us see the logic. So, you take two values of x here, if you take this value of x the slope is very steep.

So, high slow but if you go to the left then it has become a flatter line, the tangent is now flatter. So, the slope is declining and that is the idea of a concave function but, this is a rising function. Now, let us concentrate on the declining function, if we take a slope here it is a very flat line.

It is a flat line, maybe the slope is -0.2 or something and if you take a point here, here the slope is absolute value of the slope is very high which means its value let us say -4 or -5. So, from -0.2 it has become - 0.5 which means that it is becoming more and more negative. So, this value here is more than this value. So, that means the algebraic value of the slope is declining and that is the idea of a concave function, for a concave function the slope should go on decline.

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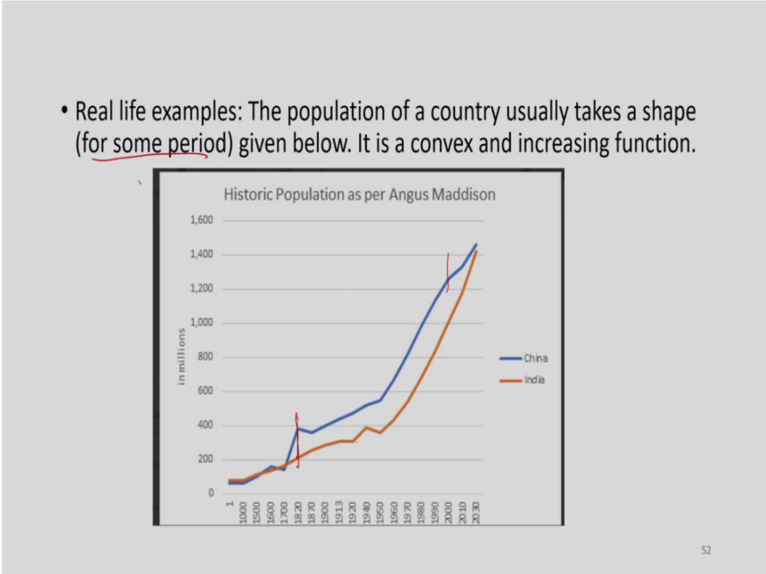

Now, the question that might come to anybody's mind is, that, are these ideas that we have been talking about merely theoretical ideas? I mean are they just a figment of our imagination or do we see some of these ideas getting reflected in reality. So, here are some examples from real life where you have concavity and convexity.

This example is of convexity, convex and increasing that is the combination we have here. The population of a country usually takes a shape at least for some periods given below. It is a convex and increasing function. Here I have taken the two examples, one is of China's population and the other is of India's population.

Now, if you notice what is represented along the horizontal axis, x axis, then you will see that the numbers are not equally spaced. For example, in the beginning you have from 1000 AD to 1500 AD the same gap that is 500 years, this gap is representing 500 years or towards the right, the same gap represents 10 years. So, the horizontal axis is not a simple scale, which you see generally.

Now, India's population figure is represented in the orange line. On the vertical axis however, it is a plain and simple scale. So the numbers are equally spaced. Now, look at the shape of India's population. It is a rising function more or less at some points it is wavering a bit but overall it is a rising function that means population has been rising over the years.

But look at how the function is shaped, it is close to a convex function. As the years are passing by, the rise of population is becoming faster and faster. Similar is the case of China, I mean if we take from this point onwards at least and towards the right of that, the function again It looks like a convex function, but not as clearly as in India's case, after this point of time, you will see that the function has become a little bit flatter.

And maybe that flatness has come about around this time, because of the one child policy that China took from 1979 onwards. So, the Chinese government introduced this policy of penalizing families which have many children from 1979 and that had a severe effect on its population growth.

So, this was something which was out of the, out of the natural progression of the population. Well, you can ask that in India also, there were some policies to control the population, yes, there were some policies which were quite draconian; for example, in the emergency period that is late 1970s. Again, at the same time when China introduced its policies, but India's policies were not as stringent or as draconian as the Chinese policies were.

So, India's population grew at more or less the same way as it was doing before. So, there was no breakpoint as such in India's population growth , it was following the trend and that is why you have a very kind of smooth curve in India's population. So, this is an example where the convex function, you can see that in real life as well.

But notice what I have written here for some period that means that this convexity is there, but it may not last for a long time and that is actually what we have been seeing in case of India and China also that after the point of time, the convexity is not there and the function starts to you know, taper off, it becomes more or less a concave kind of function, it rises but at a declining rate that is a concave function.

And it may so happen that population starts to fall after the point of time in many countries that is actually happening for example, in Japan or Russia, the actual population is declining and demographers say that this will happen in India and China also maybe sometime in the future. So, here was an example of how we can see convex functions, convex and increasing functions in real life.

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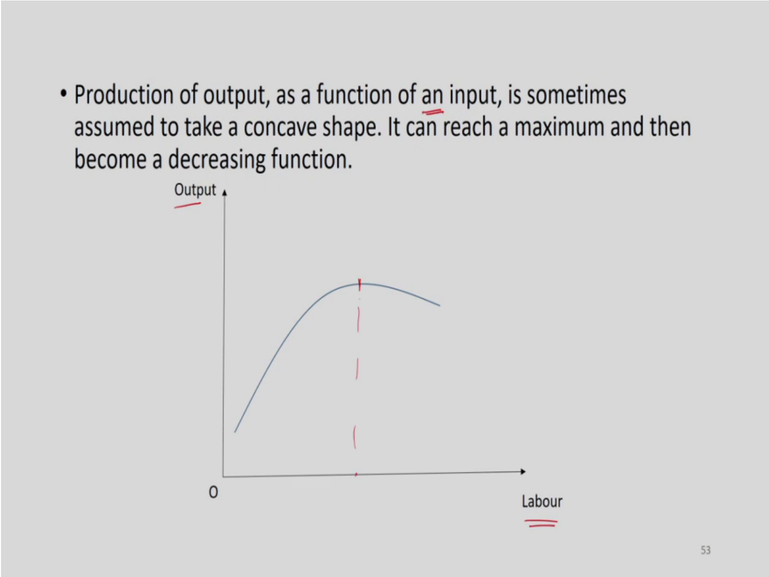

Can there be a concave function in real life? So, here is an example of a concave function in real life. So, this is not based on data unlike the previous function, but what we have written here just pay attention to that. Production of output as a function of an input, an input just one input, is sometimes assumed to take a concave shape; it can reach a maximum and then become a decreasing function in fact.

Notice here the function is like an inverted u, it is rising reaching a kind of maximum here and like an inverted u it is going down and in this particular case, what we have taken in the horizontal axis we have taken labor. So, labor is an input of production, we have seen this example before also and the output is some production that is taking place.

So, maybe labor is being used to produce I do not know wheat or Paddy or it could be industrial production also, you are using more laborers to produce pieces of clothes. Whatever be the case as labor is used more output rises, but one can see in this portion that the output is rising, but the slope is declining, which means output is rising at a declining rate.

And actually, it may so happen that if you are putting a lot of labor in the production without changing the other inputs, suppose the total amount of machines that you are using that is constant, but you are increasing more and more labor then actually beyond the point the production might get hampered because the laborers will cause troubles for each other. They will come in each other's way and that might actually hamper the output beyond the point of time, but that is a very rare case, but it may happen and the output might fall.

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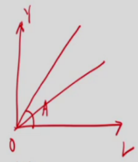
- $Y = AL^{\alpha}$, the value of α is unspecified, other than that it is positive. $A > 0$.
- We get from the above, $\frac{dY}{dL} = A\alpha L^{\alpha-1}$
- And, $\frac{d^2Y}{dL^2} = A\alpha(\alpha-1)L^{\alpha-2}$
- Since $\alpha > 0$, marginal productivity of labour is positive.
- If $\alpha > 1$, then the second derivative is also positive, the production function is a convex function.
- If $\alpha < 1$, the second derivative is negative, the production function is a concave function.
- If $\alpha = 1$, the production function is a simple linear function of labour.



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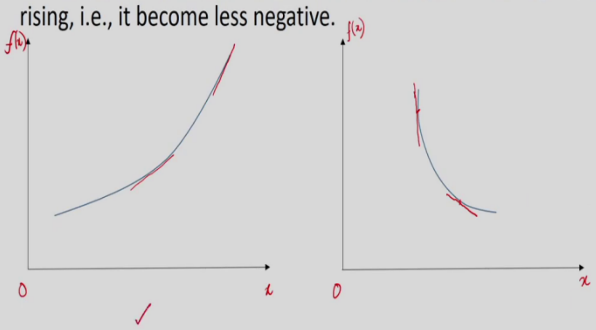
- Just as an increasing function can be convex, a decreasing function can also be convex. In the latter case, the negative slope goes on rising, i.e., it become less negative.



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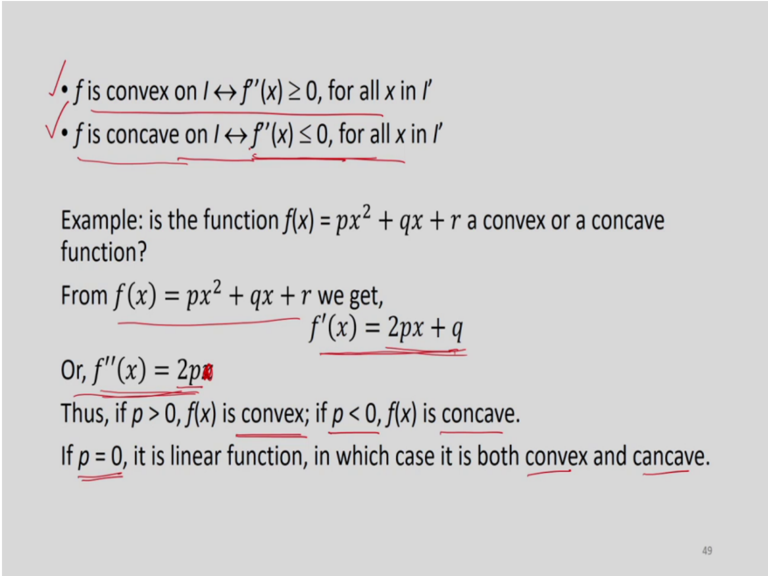

So, for this real life example of production we can take what is called a production function. So, here is the usual production function. This is not the only form of production function one takes, but it is fairly common. So, here you have a power function, $Y = AL^{\alpha}$ Y is the output and which is a function of the input labor in this case, labor is L.

Now, what do you see is that labor has a power here which is alpha, the value of the alpha is unspecified, we do not know what is the value, suppose it is just a constant. Now, L^{α} is the labor term and we are multiplying that with parameter A. What do we know about A?

A is another constant like α, but we do not know what is the value of α, for A at least we can say that $A > 0$, we cannot say more than that. So, that is the general form of the production function that one can consider, $Y = AL^{\alpha}$. Now, if we take the derivative of this output with respect to the input that is $\frac{dY}{dL}$, then what we get is we just apply the power rule and we get $\frac{dY}{dL} = A\alpha L^{\alpha-1}$.

This is the first derivative and from that we get the second derivative it becomes capital $\frac{d^2Y}{dL^2} = A\alpha(\alpha - 1)L^{\alpha-2}$. Now, if $\alpha > 0$ then you look at this form $\frac{dY}{dL} = A\alpha L^{\alpha-1}$, if $\alpha > 0$ then the marginal productivity of labor is positive because A is positive.

So, $A\alpha$ is also positive, marginal productivity of labor is positive. Now, $\alpha > 0$, that is fine, but after that what happens? Suppose $\alpha > 1$ also. Then we can say something about the second derivative, $\frac{d^2Y}{dL^2} = A\alpha(\alpha - 1)L^{\alpha-2}$, if $\alpha > 1$ then $(\alpha - 1) > 0$ which means that the second derivative is also positive like the first derivative.

That means the production function is rising and since the second derivative is positive it is rising at an increasing rate which means that the slope is rising and the function is a rising and convex function like this, here $\alpha > 1$. On the other hand, it may happen that $\alpha > 0$, so this condition is satisfied but at the same time $\alpha < 1$.

So, basically alpha lies between 0 and 1, in this case, look at the second derivative, $\frac{d^2Y}{dL^2} = A\alpha(\alpha - 1)L^{\alpha-2}$ in the secondary derivative we have a term $(\alpha - 1)$. Now if $\alpha < 1$ then $(\alpha - 1) < 0$. So, the second derivative is negative, $\frac{d^2Y}{dL^2} < 0$ and what happens if the second derivative is negative, we have seen that it becomes a concave function if the second derivative is negative.

First derivative positive, second derivative negative and basically, you have this form. So this is a probable shape of the function in that case, it is a rising function because the first derivative is positive. But since the $\alpha < 1$, the second derivative is negative, $\frac{d^2Y}{dL^2} < 0$. So, the function is concave.

The last case is alpha is neither greater than 1 nor less than 1, that is the third possibility which means $\alpha = 1$. Now, if $\alpha = 1$, then again we can go back to the second derivative it becomes actually 0, $\frac{d^2Y}{dL^2} = 0$. Second derivative is 0, first derivative is positive, which means the production function is a linear function, as the labor input rises the output rises in a linear manner.

And actually what you are going to get if $\alpha = 1$, $L^{\alpha-1} = L^0$. So, this, $A\alpha L^{\alpha-1}$ is a constant term the whole thing becomes a constant it simply becomes equal to A. So $\frac{dY}{dL} = A$, which is a constant parameter.

So, that is how the output is going to look like it is going to rise but in a linear manner and this slope is A. So, depending on what is the value of A it could be a steep line or it could be a flat line, we will not know as long as we do not know the value of A.

So, this is a very common form of production function one uses where you have output and it is a function of one input and basically you take the input put a power to that and that power in general, that $0 < \alpha < 1$ and the function becomes a concave function. That basically demonstrates diminishing marginal productivity of labor.

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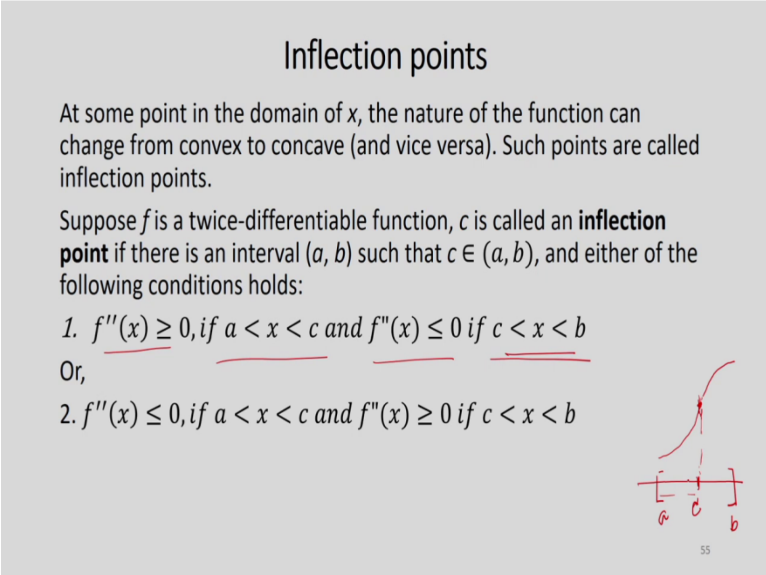

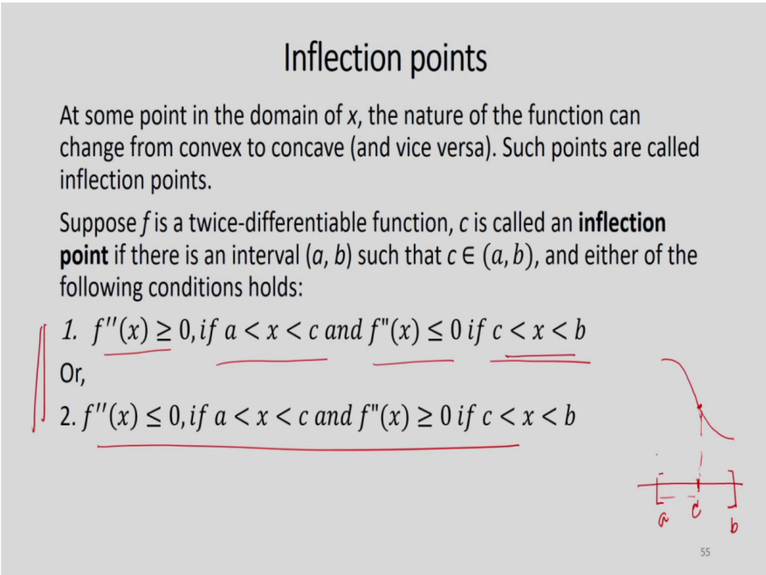

Now, we introduce another concept which is called inflection point. What are inflection points? At some point in the domain of x, the nature of the function can change from convex to concave and vice versa, that means it can change from concave to convex as well. Such points are called inflection points.

So, this is the definition of an inflection point. And now, we come to the technicalities, suppose, f is a twice differentiable function, then c is called an inflection point, if there is an interval (a, b) such that $c \in (a, b)$ and either of the following conditions holds.

Number 1: If $f''(x) \geq 0$, for $a < x < c$ and $f''(x) \leq 0$ for $c < x < b$. So how do I picturize this? So you have a particular interval (a, b), and point c here, x is a general point.

Now, c is an inflection point, what happens to the left of c, so from a to c, the second derivative is positive.

So as we know if the second derivative is positive, the function is a convex function, we are assuming that it is an increasing function to draw the picture. And to the right of c, when x is between c and b, the second derivative is negative which means the function is a concave function may be of this shape.

And the function is a continuous function; it is a twice differentiable function so, it must be continuous. So, these two things are connected here. And as you can see at this point the function is changing its nature from convex it is becoming a concave function. So, that is what we have seen before that at the inflection point the nature of the function changes, it could be from a convex function convex to the left to a concave function concave to the right of c.

This was the first case actually. What happens in the second case? In the second case it is just the opposite. So, you take any $a < x < c$ that is to the left of c then the second derivative is negative and what happens to the right of c? The second derivative is positive. So, I can write it as an example I can write it like this second derivative is negative means what?

It is something like this, is a concave function, it is becoming more and more steeper in a negative way and to the right of c, it becomes a convex function. How does a convex function look like? It looks like this. So, this could be an example of what is being said here. So, here also c is an inflection point, at c we can see that the function's nature has changed.

The second derivative sign has changed; actually that is a more precise way of saying it because to the left of c, the second derivative is weakly negative, to the right of c, the second derivative is weakly positive. So, in this case also c is an inflection point. So, this is very important to note, if we want to identify inflection points, then these two properties have to be kept in mind.

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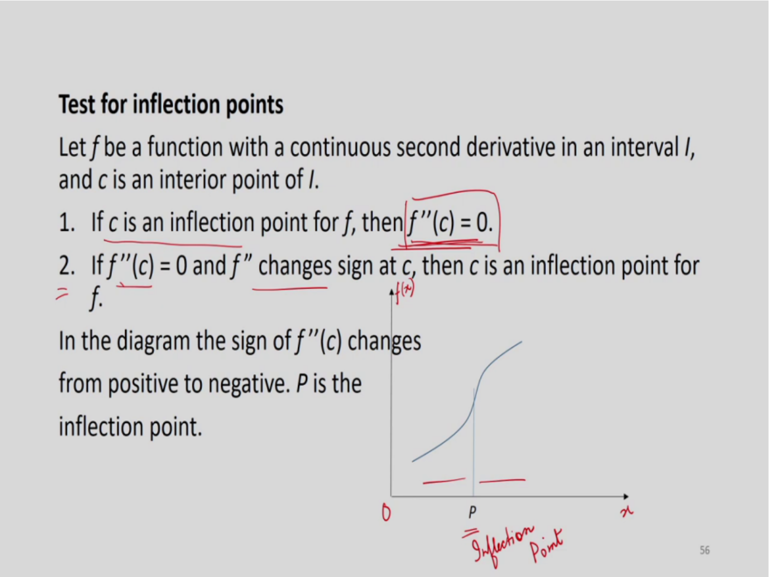

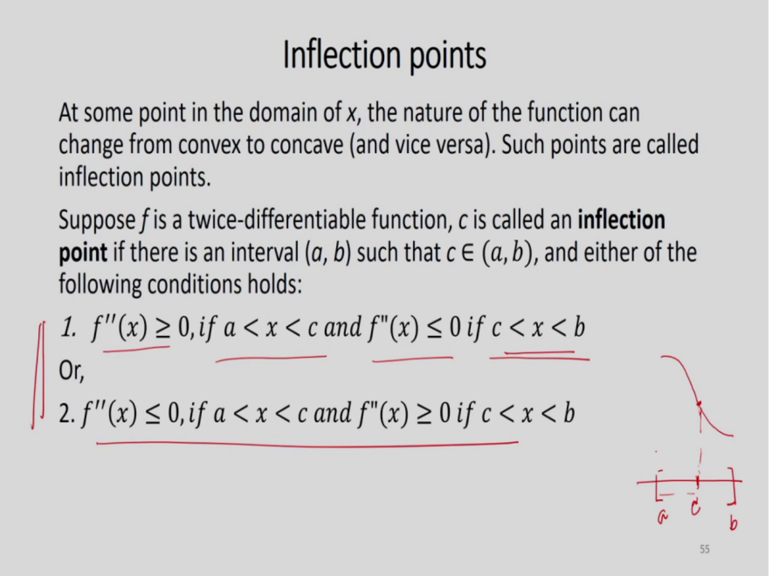

Here is a more precise way to test inflection points. Let f be a function with a continuous second derivative in an interval I. c is an interior point of I. Number 1, if c is an inflection point for f, then $f''(c) = 0$. So, notice this is a necessary condition, if c is an inflection point, then $f''(c) = 0$ which means, if $f''(c) \neq 0$, then c is not an inflection point for f.

That makes this one that is a condition 1 that is written here it is a necessary condition. So, this is a necessary condition, this is not a sufficient condition. Now, you might be wondering why we are focusing on this particular form of a condition $f''(c) = 0$. Well, that should have come to you intuitively, because, you see here if we go back to the first principle of an inflection point, then at c that is the inflection point the sign of the second derivative either changes from positive to negative or changes from negative to positive.

That means, at c there is a possibility that it is equal to 0. So, that is the intuition that we are applying here because, you know it is a twice differentiable function. So, this is a necessary condition, $f''(c) = 0$ but what is a sufficient condition? So, the second thing here is specifying the sufficient condition.

If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f. So, this is the sufficient condition. Here is a diagrammatic exposition of that what we have written and this we have seen before also that at this point P, P is the inflection point here, at P the second derivative is likely to be 0 because you see one way to understand that is here look at the function at point P.

Here the function, actually at the neighborhood of this point P, becomes a linear function and if it is a linear function then obviously the second derivative is 0. Now to the left of P, the function is a convex function here. So, the second derivative is positive on the right of P, you have a concave function so, the second derivative is negative.

So, at this point the sufficient condition is satisfied if you have both these things to be valid that at $f''(c) = 0$, that means, it becomes a linear kind of function around that point and the sign of the second derivative changes.

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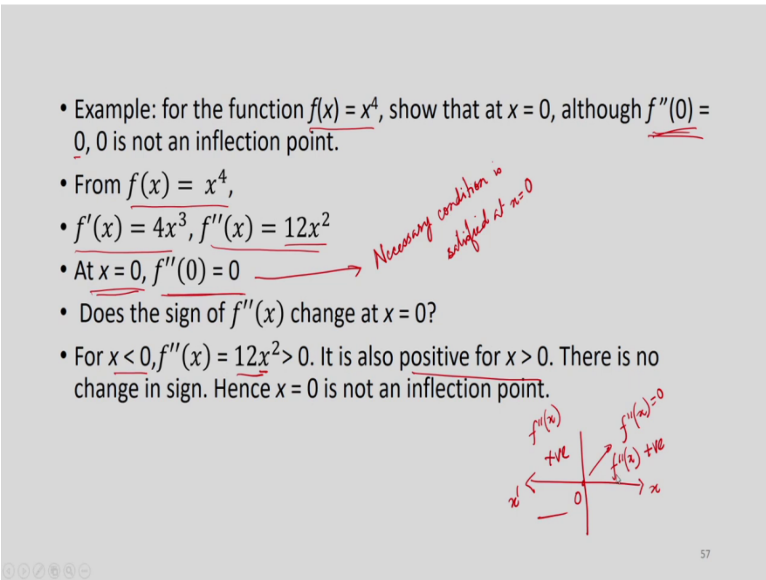

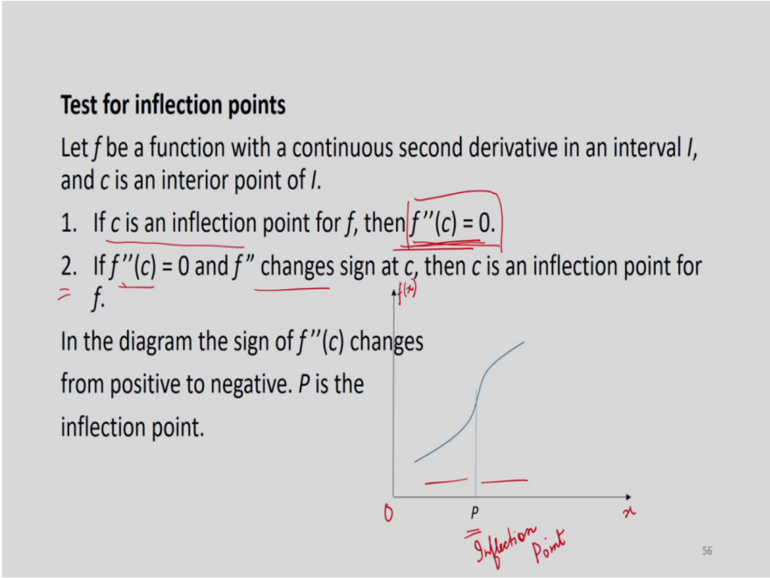

Here is an example for the function $f(x) = x^4$, show that at $x = 0$ although $f''(0) = 0$, 0 is not an inflection point. So, what is being done here is that we have the satisfaction of the necessary condition, this is the necessary condition, $f''(c) = 0$ that is being satisfied at $x = 0$.

But this is a necessary condition that does not guarantee that at $x = 0$ you are actually going to get an inflection point. So, this is an example of that. So, let us see if we can show that. So, $f(x) = x^4$, we take the first derivative it becomes, $f'(x) = 4x^3$, second derivative becomes, $f''(x) = 12x^2$.

Now, what is the necessary condition for an inflection point? It is f'' should be equal to 0. Now, at $x = 0$, what happens to the f''? It becomes equal to 0. So, the necessary condition is satisfied; but we do not know about the sufficient condition. For that we have to find out what happens to the sign of the second derivative.

Does it change at $x = 0$? So, that is the sufficient condition. Now, $f''(x) = 12x^2$. Now suppose $x < 0$. So, that is to the left of $x = 0$. So, think about this, you have $x = 0$. At this point $f''(x) = 0$, but what happens to the left, it is $12x^2$ but x is negative that means this becomes a minus term, but, square over minus term is positive so, it is positive.

Second derivative is positive and if you take $x > 0$ so here but there also $f''(x) > 0$. So, the sign actually is not changing, the sign was positive to the left, it became 0 at $x = 0$ and it has become once again positive to the right of $x = 0$ the sign is not changing, therefore, the sufficient condition is not being satisfied. Therefore, 0 is not an inflection point although the necessary condition is satisfied.

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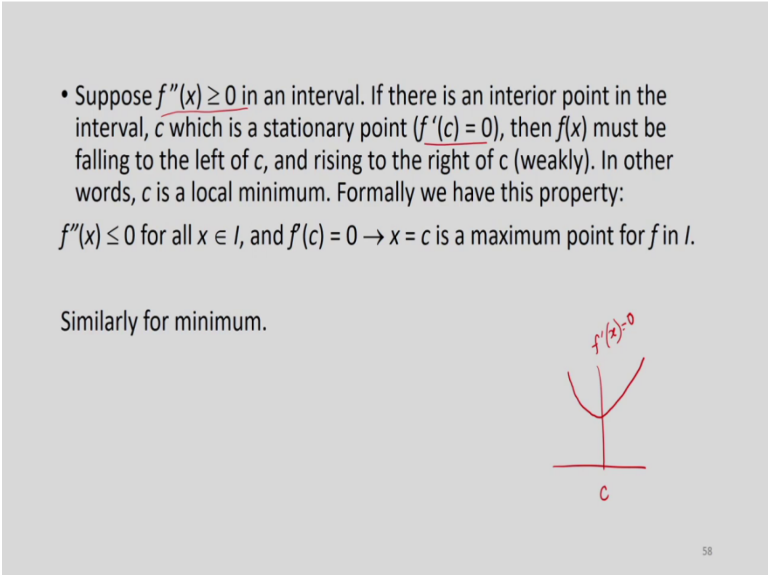

Here is a definition, suppose $f''(x) > 0$ that means it is a convex function, if there is an interior point in the interval, c which is a stationary point; which is a stationary point that is $f''(c) = 0$ then $f(x)$ must be falling to the left of c and rising to the right of c weakly.

In other words, c is a local minimum. So, this is something which is quite intuitive, you have a convex function because the second derivative is positive and at some point c suppose $f''(c) = 0$. Then what does it mean? It means that here $f''(c) = 0$ but $f''(x) > 0$.

So, I am sorry, the second derivative is positive, which means the function will have a shape like this. It is falling to the left of c and rising to the right of c. In other words, c is a local minimum and we can state this formally as follows; $f''(x) < 0$ for all x in an interval I and $f''(c) = 0$ that implies $x = c$ is a maximum point for f in I and similarly for a minimum point.

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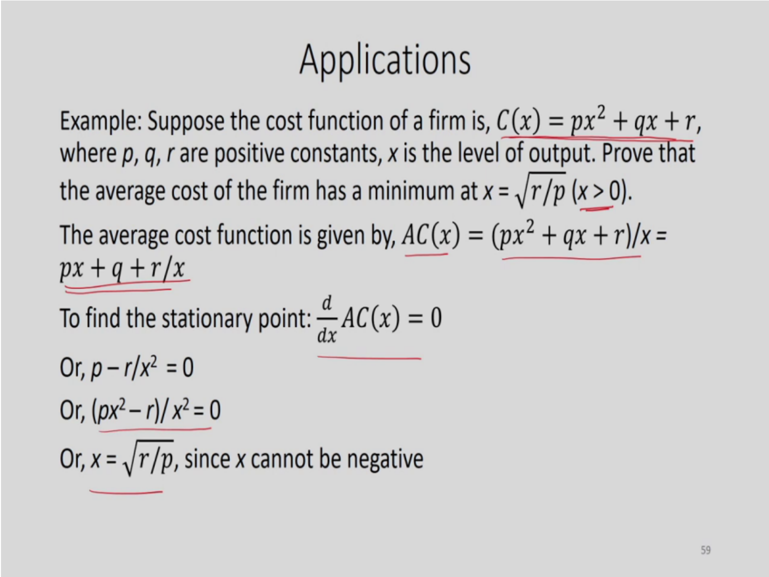

Here are some applications of these properties that we have been talking about. Suppose, the cost function of a firm is given $C(x) = px^2 + qx + r$, where p q and r are positive constants, x is the level of output. Prove that the average cost of the firm has a minimum at $x = \sqrt{r/p}$, where x is assumed to be positive, $x > 0$, because you know output level cannot be negative.

Now, the cost function is given which is $C(x) = px^2 + qx + r$, then we can find out what is AC or the average cost this will once again be a function of x. So, what I need to do is that I take the cost function that is $C(x) = px^2 + qx + r$ and divide that whole thing by x. And if we do so, it becomes this expression, $AC(x) = px + q + \frac{r}{x}$.

Now, we have to prove that the average cost of the form has a minimum at $x = \sqrt{r/p}$. So, we have to find the minimum point and the minimum point should be $= \sqrt{r/p}$. For that, first we applied the necessary condition; the necessary condition is that the first derivative of the function should be equal to 0.

So, $\frac{d}{dx}AC(x) = 0$, that is the necessary condition and if you take the derivative of this it becomes $p - \frac{r}{x^2} = 0$ and this left hand side becomes $(px^2 - r)/x^2 = 0$.

So, the numerator should be equal to 0 and if you simplify this, it becomes $x = \sqrt{r/p}$. There is a negative root also but, we are ignoring this because x is assumed to be positive. So, this is a stationary point but we have not proven so far that it gives us a minimum, for that we have to check the second order condition or the sufficient condition.

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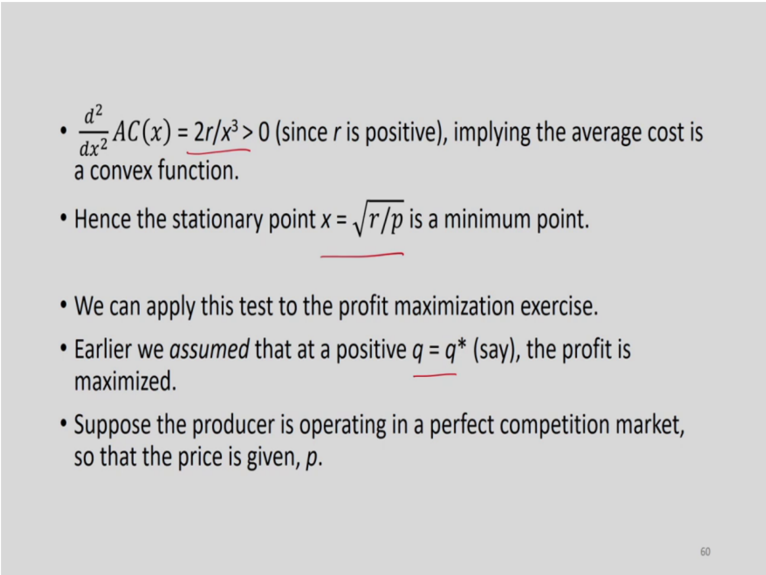

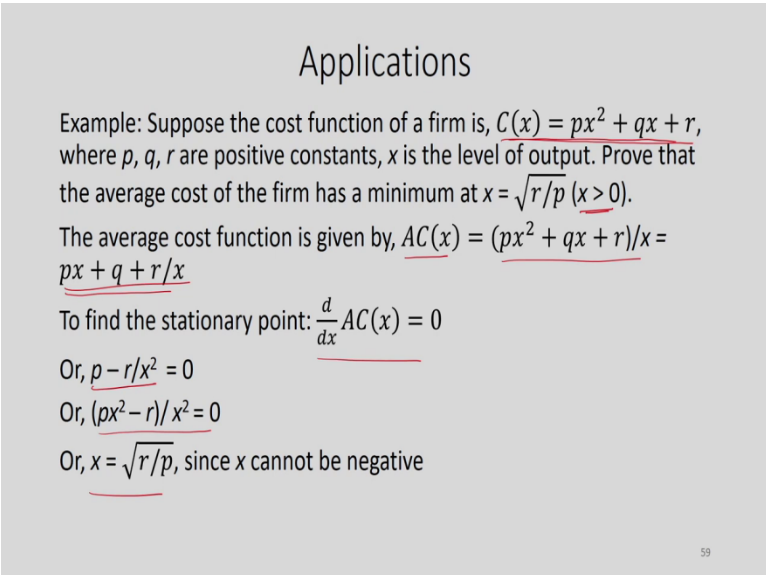

So, for that we take the second derivative of the average cost function. Now, the first derivative was this, $\frac{d}{dx}AC(x) = p - \frac{r}{x^2}$. So if you take the derivative of $p - \frac{r}{x^2}$ it becomes, $\frac{d^2}{dx^2}AC(x) = 2r/x^3$. Since r is positive which is given in the question itself, so, the second derivative is positive. At that point where the first derivative is satisfied implying that the average cost is a convex function and if you have a convex function then we just apply that formula.

If you have a convex function and if you have a stationary point, then the stationary point should give us the minimum. So, $x = \sqrt{r/p}$ is a minimum point proven. We can apply this test of finding maximum and minimum to the case of profit maximization exercise.

Here earlier we assumed that at a positive q that is output level $q = q^*$ say, the profit is maximized and then we apply the necessary condition. If there is a maximum at a particular $q = q^*$, then the necessary condition will be satisfied and then that condition will give us the value of the q^*.

Suppose the producer is operating in a perfect competition market, so that the price is given at p. So, it is a perfect competition market which means the producer cannot affect the price in the market, he is a price taker and that price is given by p.

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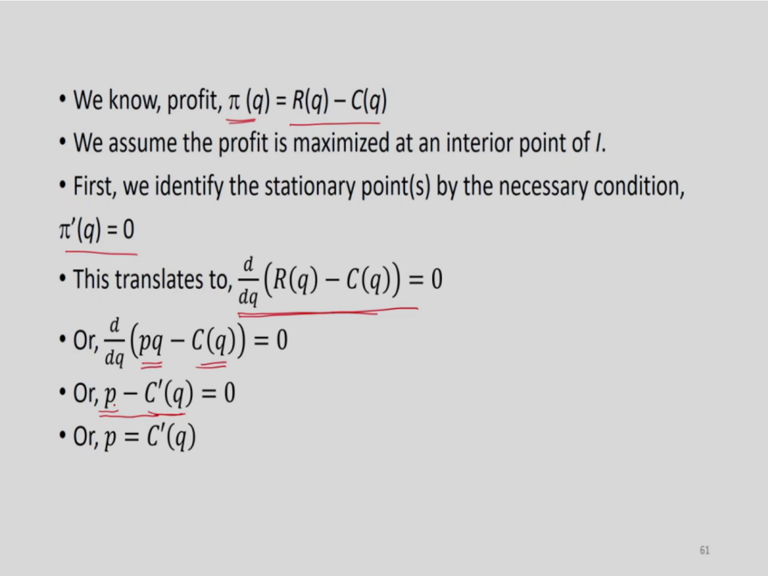

What is the profit function? Let us suppose that the profit function is given by π, $\pi(q)$, q is the output level. $\pi(q) = R(q) - C(q)$, R is the revenue function, C is the cost function. We assume that the profit is maximized at an interior point of I. I means, the range of q that we are considering interval.

First we identify the stationary point or points by the necessary condition that is $f'(q) = 0$ and if we apply this condition $f'(q) = 0$, that means, I have to take this first derivative of the profit function and set that equal to 0 that is $\frac{d}{dq}(R(q) - C(q)) = 0$. This is our necessary condition.

And $R(q)$ is simplified as $R(q) = pq$, p is the price, q is the quantity. So, this is the revenue minus this cost function is there. So, this is $\frac{d}{dq}(pq - C(q)) = 0$ and if I differentiate this with respect to q, I get $p - C'(q) = 0$, $C'(q)$ is the derivative of the cost function with respect to q.

Here the perfect competition market condition is coming into effect here because, you know , p is fixed. It is a constant, it is not a function of q. That is why we get a simple expression on the p as the first term on the LHS. So, this becomes $p = C'(q)$.

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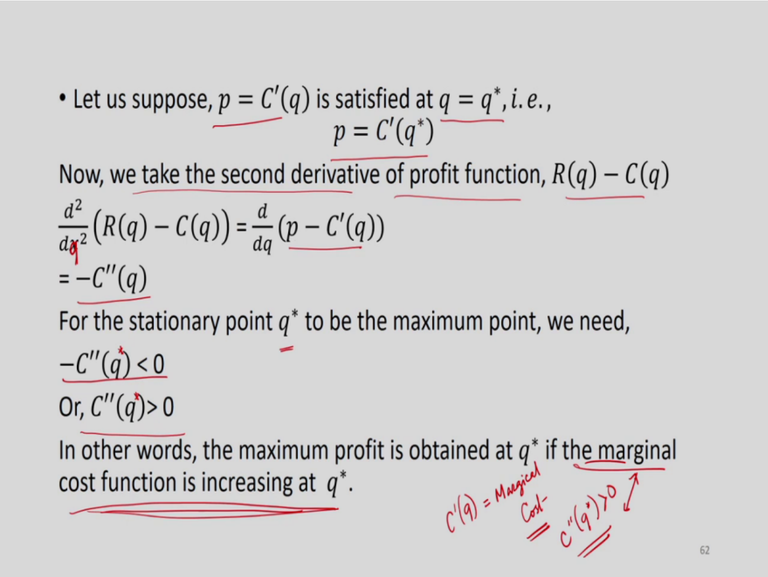

And let us suppose that this condition, the necessary condition is satisfied at a particular output level given by q^*, $q = q^*$ is that output level at which this condition is satisfied and we are implicitly assuming that, this q^* is unique, that is there is a single output level at which this condition is being satisfied.

Now, this only gives us a stationary point it does not tell us whether the profit is getting maximized or minimized and to get a hang of that, we take the second derivative of the profit function $\frac{d^2}{dq^2}(R(q) - C(q))$; and if we do so. So, it basically is taking the derivative of the first derivative $\frac{d^2}{dq^2}(R(q) - C(q)) = \frac{d}{dq}(p - C'(q))$, p is a constant and so the first term will drop out, it will now boil down $\frac{d^2}{dq^2}(R(q) - C(q)) =- C''(q)$.

Now, what we need is that for the stationary point q^* to be the maximum point we need, that this should be satisfied, $- C''(q) < 0$, because we need the profit function to be concave, for a concave profit function, the second derivative should be less than 0, that is what I have written here.

And if we multiply both sides by minus 1, I get $C''(q) > 0$. In other words, the maximum profit is obtained at q^* if the marginal cost function is increasing at q^* that is what it boils down to, the marginal cost function is increasing at q^*. Why I am saying that, because $C'(q)$ is the marginal cost.

$C(q)$ is the cost function, so $C'(q)$ is the marginal cost and when you are saying that $C''(q) > 0$, it means that the marginal cost function is increasing at q^*. This is what this means. So, the second order condition of profit maximization in case of perfect competition boils down to the condition that the marginal cost function should be rising at the point where the first order condition is satisfied.

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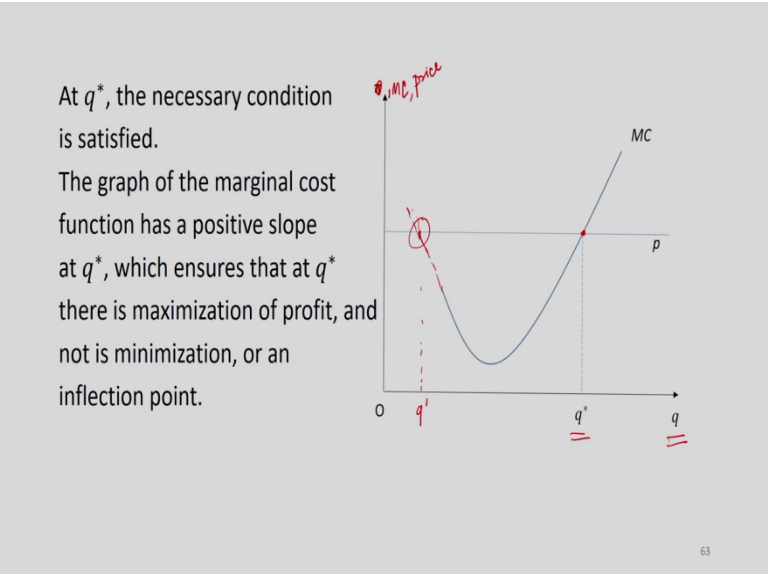

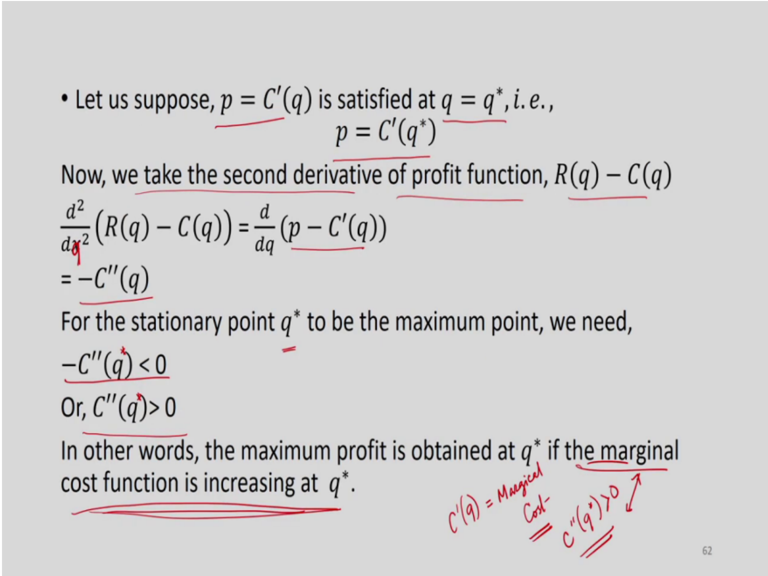

Here is a diagrammatic representation of what we have been talking about. So, in this diagram, this is a very standard diagram in a perfect competition market. Along the horizontal axis, you have the output level. On the vertical axis, you have the different variables like marginal cost, price, etc., etc., not really profit. A profit is not being represented here, at least in this diagram.

So, remember profit, the price is constant, it is given. So, it is a horizontal line and I have drawn a marginal cost curve MC, it has been purposefully drawn to be a convex function. So, you have a declining portion first U shaped and then it is a rising function to the right of that minimum point. Now, this MC function actually if we extend this to the left it can intersect the price line at this point as well.

But here also there is another point of intersection. Now at q^*, the necessary condition is satisfied. Remember in this mathematical exercise that I have just done, I have implicitly assumed that there is a single point at which the first order condition is satisfied, this condition is satisfied, $p = C'(q)$.

Now, that means that I am not considering this point in this mathematical exposition, but in general one can think of another point of intersection here. Now, at this point $p = MC$ marginal cost is equal to a price that is the necessary condition is satisfied. Is the sufficient condition, the second order conditions satisfied at q^* ? and the answer is yes, because, remember what was the sufficient condition that the marginal cost function should be increasing at q^* and that is clearly satisfied because the MC has a positive slope at this point of intersection q star.

The graph of the marginal cost function has a positive slope at q^* which ensures that at q^* there is maximization of profit and not minimization or neither is this an inflection point. So, that is how it looks like diagrammatically. Notice as a side note that had I considered this case where you know MC is declining as well and there is a point of intersection at the falling part of the MC.

Here the first order condition is satisfied that is $MC = p$, intersection is there, but the second order condition is not satisfied. So, this will not give you, let us suppose this is q', q' is not the point of profit maximization whereas, q^* is the point of profit maximization. Both of them are however stationary points.

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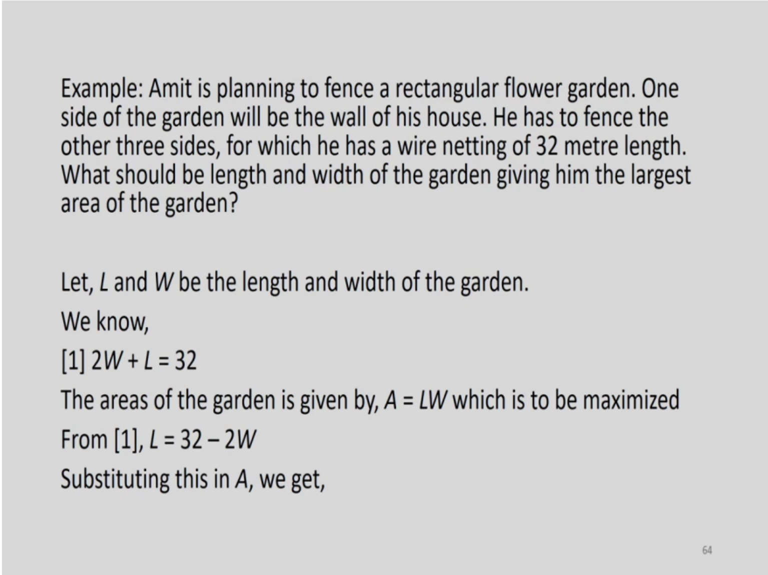

Here is another example, but let us keep it for the next lecture. I hope that in the next lecture I shall be through with this topic of optimization with a single variable. Thank you for joining me and I shall see you in the next lecture. Thank you.