Mathematics for Economics - I

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Lecture 21

Global and local extreme points

Good morning, everyone. So, welcome to another lecture of this course Mathematics for Economics Part I.

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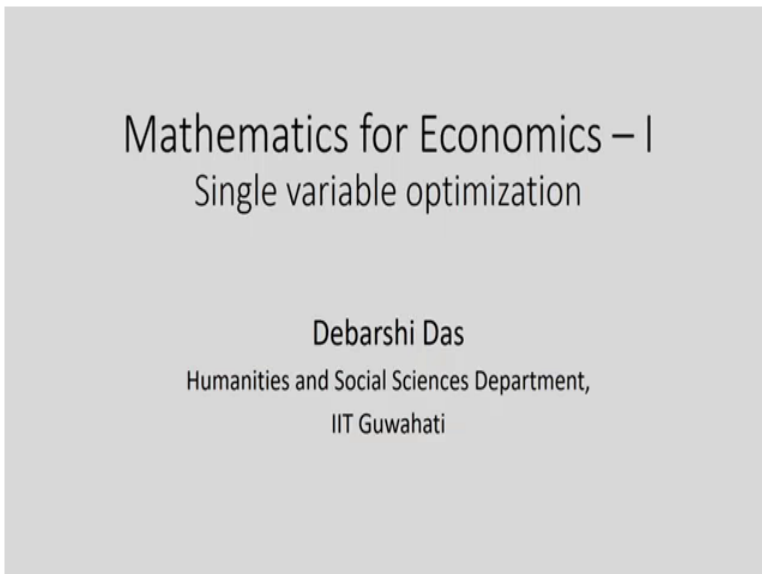

The topic that we have been covering in the last two lectures is called Single Variable Optimization as you can see on your screen. So, what happens in a single variable optimization, let us recapitulate a little bit. So, we have done two lectures on this particular topic. And what we have seen is that, in many cases in economics, in finance one is required to find out at what value of the decision variable a function is attaining its maximum or its minimum. So, we have to find out at what particular value of a variable that a person can control a function, the function can be a profit function, for example, profit of a firm, that profit is getting maximized.

So, as you can see, it has very clear practical implications. For example, a producer might be interested to know how much of the inputs that he or she should employ so as to maximize his or her profit. Now, optimization, in general, could be a problem of several variables, because a decision maker might have control over multiple variables, but this is a basic introduction.

So, we are going to start with single variable optimization. That means we are assuming that the decision maker has control over only one variable and so the function that he wants to optimize it may be a problem of maximization or a problem of minimization that function is a function of a single variable.

Now, in this topic, we have talked about some of the basic ideas of optimization. We have talked about what is called the first derivative test of optimization. Let me take you to that particular part.

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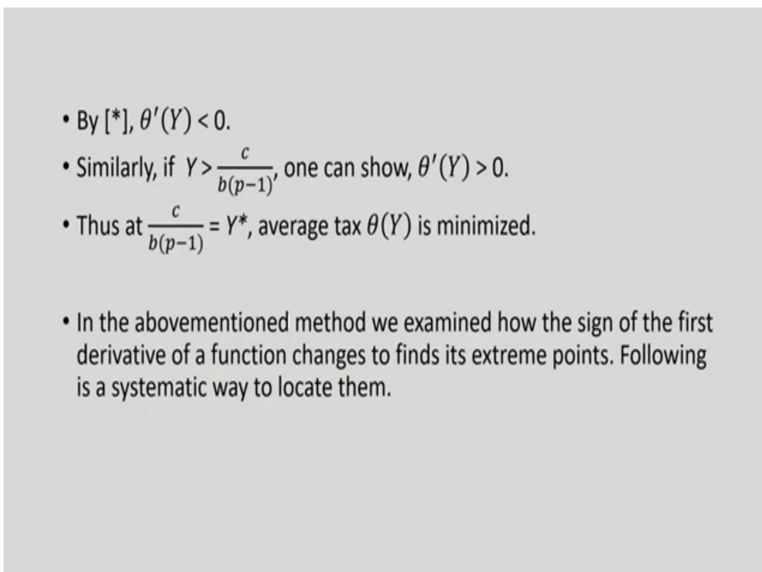

So, here the idea is that if you have a function of a single variable, we look at the sign of the first derivative of the function to find the extreme points. So, the idea is that at what particular value of the variable you have a stationary point. Stationary point means the first derivative is equal to 0. And with that, we have to add the conditions that, before the stationary variable is attained, the first derivative should be positive or at least non negative and after the stationary point is attained, the first derivative should be negative or at least non-positive.

Now, if this is the case, then that stationary point that we are talking about is a point of maximization. So, at that point, the function is attaining its maximum value. And if it is the reverse, reverse means you have a stationary point so at that point $f'(x) = 0$ and before the point is attained, the first derivative is negative and after that point is attained, the first derivative is positive then at that point, the stationary point you have a minimum point. Now, we are going to talk about some other more systematic way to locate these points of extreme. So, that is what we are going to do today.

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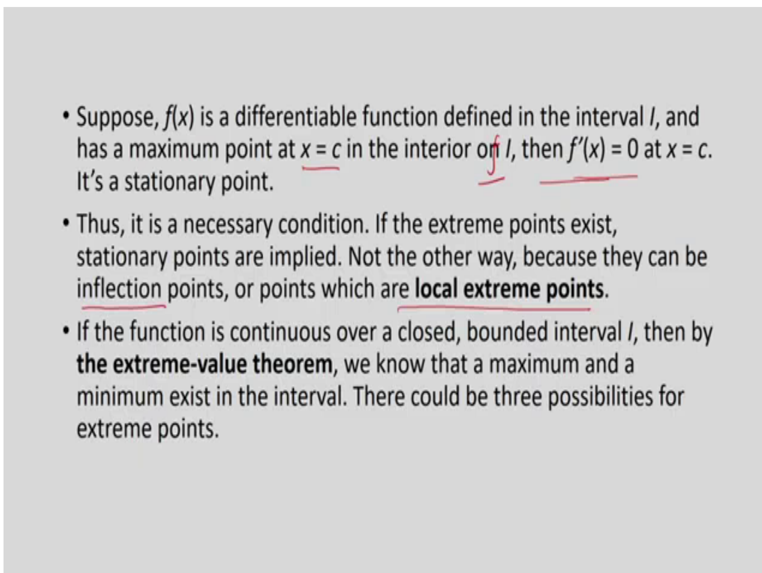

Suppose, $f(x)$ is a differentiable function defined in the interval capital I and has a maximum point at $x = c$ in the interior of I, then $f'(x) = 0$ at $x = c$. It is a stationary point. So, let us try to understand what this is saying. That $f(x)$ is a differentiable function and this function is defined over a certain interval. This is capital I.

Now, suppose we know that there is a point c in this interval where the function is attaining its maximum and the c point is in the interior of I that is it is not the end points of I then this first derivative should be equal to 0 at $f(x)$, that is it is a stationary point.

Thus, it is a necessary condition. What does it mean, that if this condition is not satisfied, then the function is not attaining its maximum. If the extreme points exist, then the stationary points are implied. Not the other way, not the other way means, suppose you say that at a particular point $f^{'}(x) = 0$, that means at that point you have a maximum. That is incorrect, because if you are saying that then you are assuming that this condition is a sufficient condition which it is not.

So, that is why it is written here. Not the other way, because they can be inflection points. We are going to look in detail, what are inflection points. So, in the inflection point also you can have $f^{'}(x) = 0$, but the inflection points are not extreme points, or points which are local extreme points. So, at the local extreme point also this first derivative is equal to 0, but that does not mean that they are in general what are called the global extreme points. So, we are now looking at global extreme points or the extreme point over the entire interval.

If the function is continuous over a closed bounded interval I, then by the extreme value theorem, we know that a maximum and the minimum exist in the interval. So, this is just the extreme value theorem that we have discussed in some previous class that if you have a continuous function over a closed bounded interval then that function will have a maximum or a minimum in that interval. There could be three possibilities for extreme points. So, if you have a continuous function over a closed and bounded interval I, then there could be three possibilities for extreme points.

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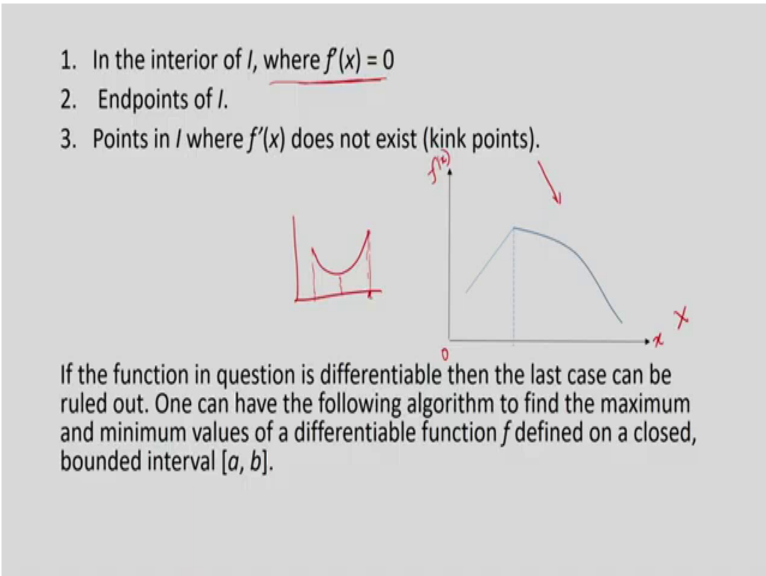

The first possibility is the possibility that we shall often be talking about that point, the extreme point is in the interior of this I. And if it is in the interior of I, then let us assume that $f^{'}(x) = 0$. So, $f^{'}(x) = 0$ is suppose happening, we are assuming that this is correct, you have an extreme point and at that extreme point $f^{'}(x) = 0$ and that point is in the interior of I. So, this is the most well behaved case where you have something like this. So, this point is a point of minimum. an extreme point.

Second possibility that can arise is that the extreme points are endpoints. So, suppose the interval is defined like this then the maximum is actually occurring here that is it is occurring at the endpoint. Now, at this endpoint obviously, you cannot define $f^{'}$ of that point. That is, the first derivative is not defined.

The third possibility is that the points in I, where $f^{'}(x)$ does not exist, so this condition $f^{'}(x) = 0$ does not satisfy because $f^{'}(x)$ simply does not exist, but these points could be extreme points. So, here is an example of that. This is an example. So, you have a continuous function defined over a bounded closed interval, you have a point which is an interior point, but this is a point where you have a kink in the graph of the function and so the first derivative is not defined.

So, this is the third possibility. So, only these three possibilities can arise if you have a continuous function defined over a bounded and closed interval and you have extreme points. So, there could be three cases of extreme points. If the function in question is differentiable then the last case can be ruled out. So, this is something which is not a differentiable function. It is, since you have a kink point, so this function is not differentiable over the entire interval.

So, if we assume that the function is differentiable then this last case is ruled out. Then one has to content with the first and the second cases. One can have the following algorithm in that case to find the maximum and minimum values of a differentiable function f defined on [a,b].

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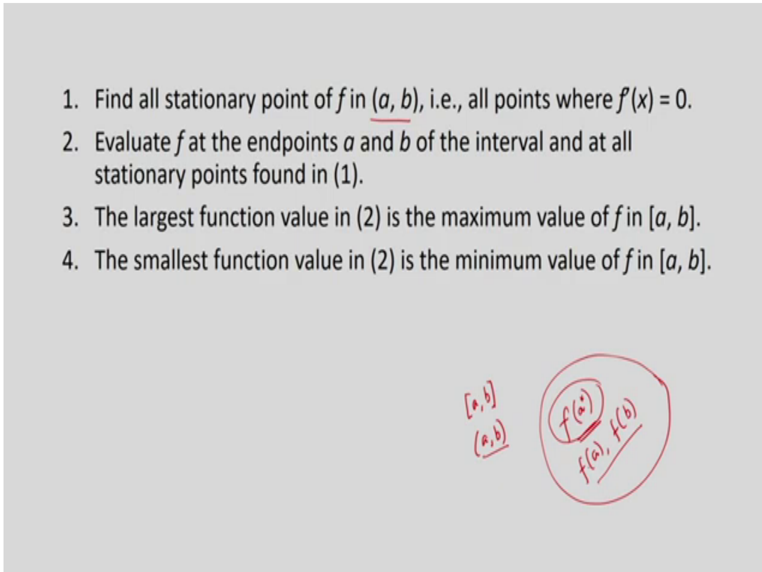

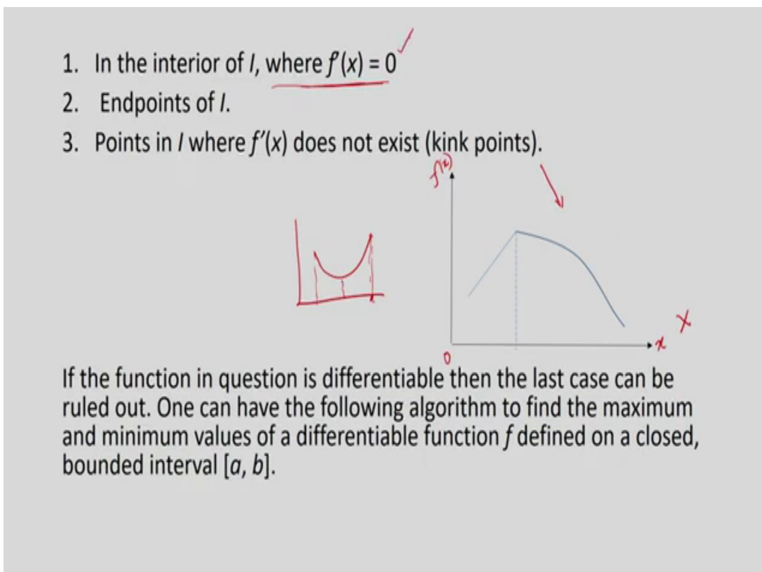

And this is the algorithm. So, there are four steps in this algorithm. Find all stationary points of f in (a, b) i.e., all points where $f^{'}(x) = 0$. Notice, this is the interior. This is not the entire interval because the interval over which the function is defined is [a, b], but here we are taking the open set, (a, b). So, we are talking about the interior and we are finding out which are the stationary points of a over this open interval (a, b) that is all the points where $f^{'}(x) = 0$.

The second step, evaluate f at the endpoints a and b of the interval and at all stationary points found in (1). So, we are going to find out the value of f. So, suppose a^* is a stationary point that we have found in the first step then we first find out what is $f(a^*)$. At the same time, we also find out what is $f(a)$, what is $f(b)$. The largest function value in (2) is the maximum point of f in [a, b] and the smallest function value in (2) is the minimum value of f in [a, b].

Now, this is very intuitive. So, what we are doing is that we are first looking at the interior points and we are trying to find out whether those points are stationary. Then we are finding out the value of the function at the stationary points. These are the values. And basically, we have covered the first case here. This case has been covered. But it is possible that the maximum or the minimum can occur at the endpoints.

So, therefore, we evaluate $f(a)$ and $f(b)$ also and finally we compare all these values to find out where the maximum value is attained. The corresponding point is the maximum point, and similarly, for the minimum. So, this is the very clear way to find out what are the extreme points if a function is differentiable and defined over a closed and bounded interval.

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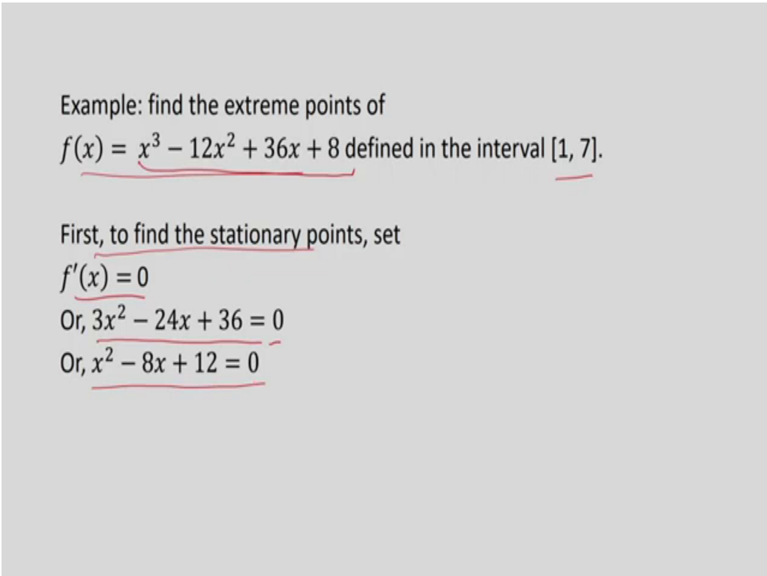

So, here is an example. Find the extreme points of this function and it is written that the interval is [1, 7]. So, we go to step one. Step one, what do I do? I find out the stationary points in this interval. So, how do I find the stationary points? I set $f'(x) = 0$, and so, the left hand side is $3x^2 - 24x + 36 = 0$ is obtained from this function and that is equal to 0. So, this simplifies to this expression $x^2 - 8x + 12 = 0$. I have divided both sides by 3.

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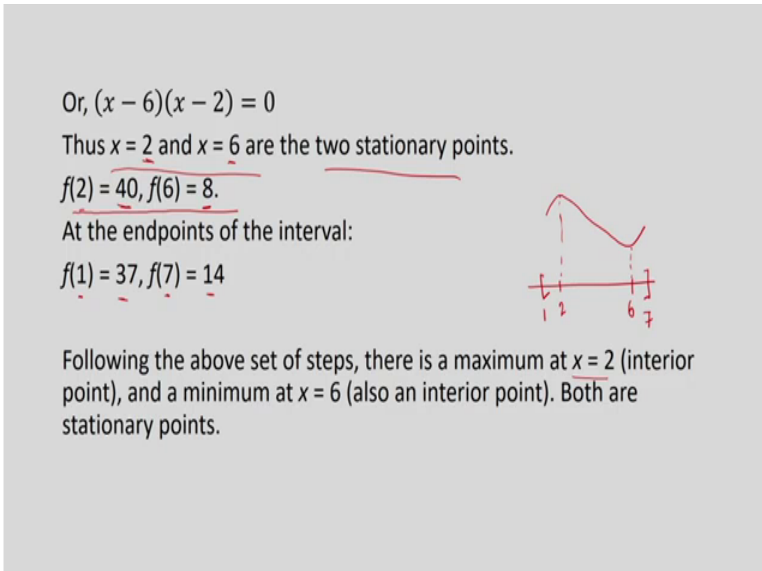

And this basically boils down to, I am skipping some steps, $(x - 6)(x - 2) = 0$. And so, it basically means that there are two values of x for which the first derivative is 0 and these two values are $x = 2$ and $x = 6$. These are the two stationary points. So, we have found the stationary points where the maximum or the minimum could be there, but as we know these are only one of the many possibilities. So, it is possible that extreme values are obtained at the endpoints.

So, what we do next is that we first find out, what the values of the function are at the stationary points. So, I have put $f(2)$, because 2 is a stationary point. So, $f(2)$ is found out to be 40. And $f(6)$, the other stationary point, $f(6) = 8$. So, these are the values of the function at the stationary points. What about the endpoints? We have to evaluate the function at the endpoints.

So, remember the endpoints are 1 and 7. So, $f(1) = 37$, $f(7) = 14$. So, following the above set of steps there is a maximum at $x = 2$, because if you look at all these values the f, the values of the function, then the maximum of these values is 40. So, what is the corresponding value of the x, it is 2. So, therefore, I am saying that $x = 2$ is a maximum. And as you can see that the 2 is an interior point. So, here is how it looks like.

You have 1 and you have 7. So, this is the interval over which the function is defined. And here you have 2 and here you have 6. So, at 6 the value of the function is 8, which is the minimum of all these f values. So, here is the minimum point and here is the maximum point. So, the function is something like this. So, here you have the minimum value and here you have the maximum value. And as you can see both 2 and 6, the extreme points are actually stationary points in this example.

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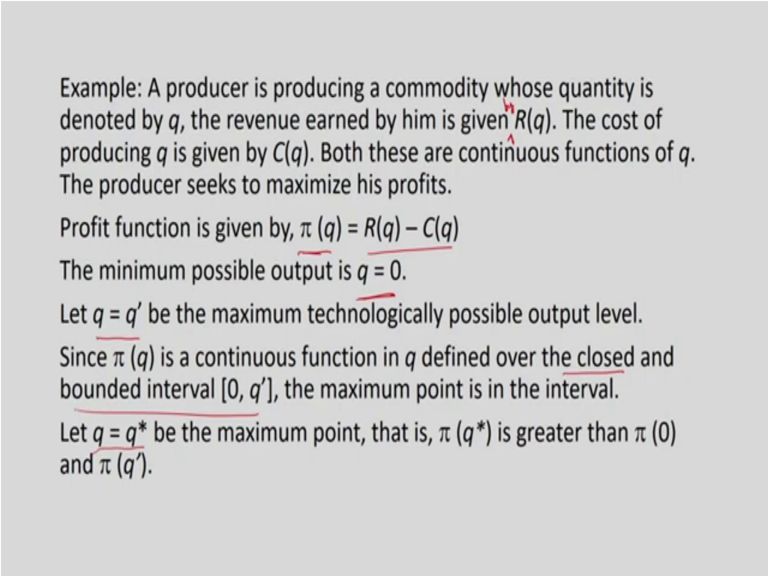

Here is another example. This is an example from economics. A producer is producing a commodity whose quantity is denoted by q. The revenue earned by him is given by $R(q)$. So, R is a function of a single variable q, which is quantity. The cost of producing $q$is given by $C(q)$. So, revenue is just one part of the production. There is the other part which is cost. The cost function is given by $C(q)$. It is also a function of a single variable. Both these are continuous functions of q. The producer seeks to maximize his profits. So, first we have to find out what is the profit function and then we want to maximize that profit function, because the producer wants to maximize the profits.

Now, here is the profit function $\pi(q)$. And what is profit? It is the amount of money you get by selling minus the cost. So, it is $\pi(q) = R(q) - C(q)$. And we have to define the interval over which the production can take place. Suppose the minimum possible output is $q = 0$. This is a reasonable assumption, because what is the minimum output a producer can produce it cannot be negative. So, the minimum possible output is $q = 0$.

On the other hand, $q = q^{'}$. Let us suppose $q = q^{'}$ be the maximum technologically possible output level. So, given the technology that he has at his disposal and maybe the raw materials that he has, what is the maximum output that he can produce, let us suppose that is $q^{'}$. So, these are like the boundaries one is $[0, q^{'}]$. He has to operate between these two.

Now, $\pi(q)$ is a continuous function in q defined over the closed and bounded interval $[0, q^{'}]$. And since it is a closed and bounded interval by the extreme value theorem, the maximum point is there in the interval. And let us suppose $q = q^{*}$ is the maximum point that is, $\pi(q^{*})$ is greater than $\pi(0)$ and $\pi(q^{'})$. So, we are assuming that, there is this$q = q^{*}$ which is the maximum point by the extreme value theorem, but our assumption is that it is inside that interval. So, it is between $[0, q^{'}]$.

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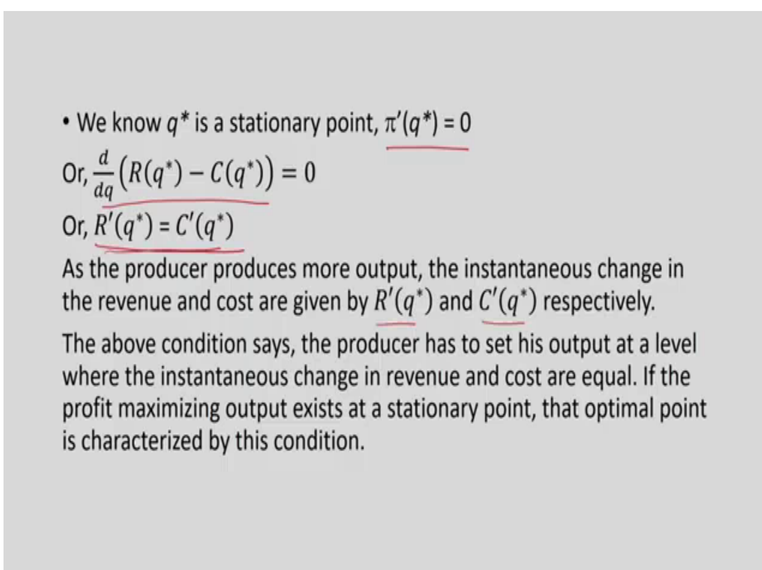

So, if it is inside, inside it is an interior point, then it must be a stationary point, because it is a differentiable function. So, there cannot be cases of kink points. So, there is only one possibility that this is a stationary point. So, q^* is a stationary point. So, if it is a stationary point then we can identify it by this condition. This necessary condition, that $\pi'(q^*) = 0$ and that boils down to this condition, $\frac{d}{dq}(R(q^*) - C(q^*)) = 0$ because $\pi(q) = R(q) - C(q)$. And so we have this condition which is $R'(q^*) = C'(q^*)$.

So, this necessary condition, therefore, gives us a method to find out or isolate that particular output value where the maximum profit is occurring. As the producer produces more, the instantaneous change in the revenue and cost are given by $R'(q^*)$ and $C'(q^*)$. So, that is what this means. $R'(q^*)$ is the instantaneous change in the revenue as the producer produces more output or even less output. So, if the output is changing by a small amount, then $R'(q^*)$ is the change in the revenue at that output level q^*.

Now, what this condition is saying is that the producer has to set the output at a level where the instantaneous change in revenue and cost are equal. So, this left hand side and the right hand side should be equal. The instantaneous change in the revenue should be equal to instantaneous change in the cost. If the profit maximizing output exists at a stationary point that optimal output or optimal point is characterized by this condition $R'(q^*) = C'(q^*)$.

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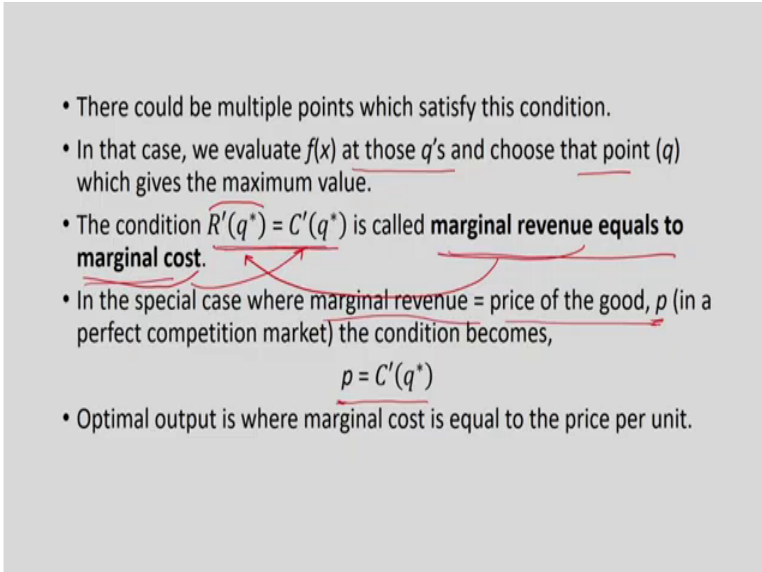

Now, it is possible that there could be multiple points which satisfy this condition. So, it is not necessary that you have only one q^* for which this condition is satisfied. So, if there are multiple points for which this condition is satisfied, what do we do? We evaluate $f(x)$ at those q's and choose that point or that q which gives me the maximum value.

The condition $R'(q^*) = C'(q^*)$ is called the marginal revenue equals to marginal cost condition. Why I am calling this marginal revenue equals to marginal cost condition. The reason is on the left hand side you have the marginal revenue and on the right hand side you have the marginal cost. So, that is why the optimal point is characterized by this condition where the marginal revenue is equal to marginal cost.

Now, in this special case where the marginal revenue is equal to the price of the good p. This happens in a perfect competition market. We have talked about this case in some previous lectures. So, in a perfect competition market, if a producer produces more, then the revenue changes by the value of the price. The price remains constant.

So, the implication is that if a producer wants to produce more, then he does not need to reduce the price in order to sell his goods. He can sell any amount of commodity at the existing price. So, here the marginal revenue will be constant. It is given by the price of the good and that is let us suppose it is given by p.

Now, in this special case of perfect competition, then we can replace this marginal revenue by this simple p which is a constant amount. So, the condition boils down to this, $p = C^{'}(q^{*})$. So, price is equal to marginal cost in this case. Optimal output is where marginal cost is equal to the price per unit. But this is a special case. The general case is this case, the marginal revenue is equal to marginal cost.

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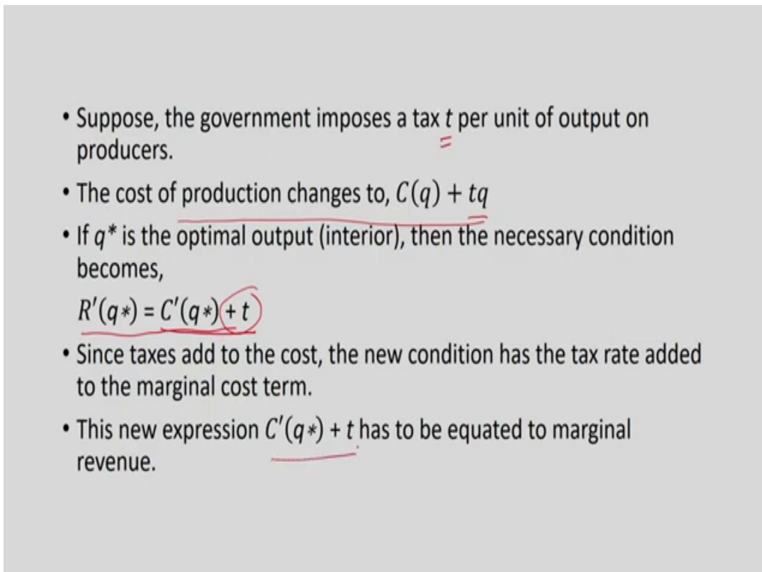

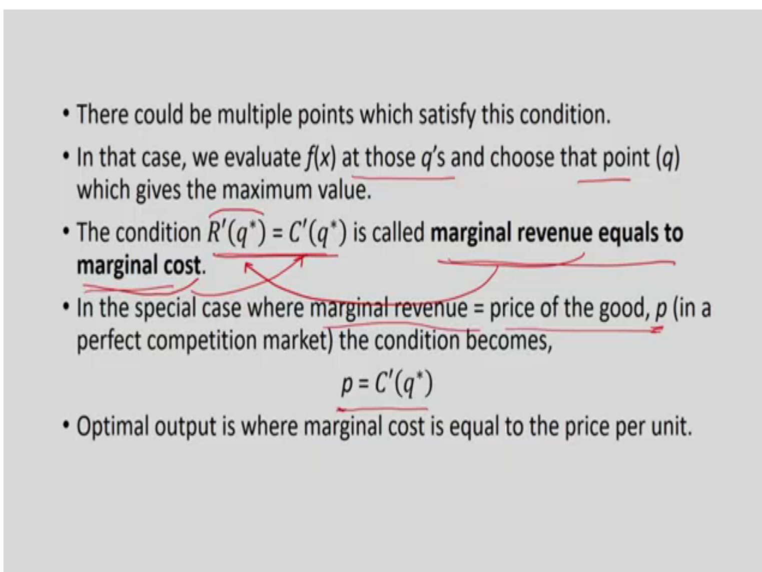

Now, let us introduce government tax in this scenario. Suppose the government imposes a tax of small t per unit of output on the producers. So, if the producers produce q units of output then for 1 unit of output they have to produce small t to the government.

So, suppose you are a cold drink producer and suppose you are producing a bottle of cold drink and selling it in the market for INR20 rupees then before the tax was there you were just getting those INR20, in this case, you have to pay, let us suppose, INR2 per bottle to the government also. So, per bottle you are going to pay INR2 extra. So, if you are producing 100 bottles, the total tax is going to be 100 multiplied by 2 that is INR200.

In algebraic terms what is the cost now? The cost is this, $C(q) + tq$ it is $C(q)$, which is the earlier, plus the cost component. What is the cost? Cost = tq. t is the tax per unit and the producer is producing in general q units. So, tq is the total tax and that adds to the cost.

Now, like before, if q^* is the optimal output which is an interior point, then the necessary condition becomes this, $R'(q^*) = C'(q^*) + t$. Notice, it is similar to the previous condition. What was the previous condition? This was the previous condition: $R'(q^*) = C'(q^*)$, marginal revenue is equal to marginal cost at the maximum point. Here $R'(q^*) = C'(q^*) + t$. So, this is the additional component that has been added to the right hand side, to the marginal cost part.

And what is the idea, since taxes add to the cost the new condition has the tax rate added to the marginal cost term. So, this is quite intuitive, because now the marginal cost has changed. Because of the tax to produce one additional unit of output, the cost has become this much. So, this new marginal cost has to be equated to the marginal revenue. This new expression $C'(q^*) + t$ has to be equated to the marginal revenue.

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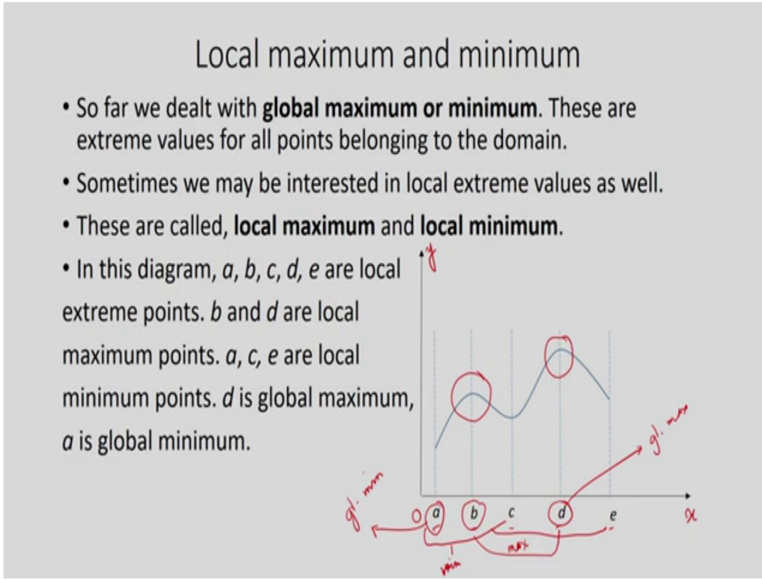

Now, we talk about something else. So far we dealt with global maximum and global minimum, now we are going to talk about local maximum and local minimum. Now, in global maximum and global minimum, what are these, these are extreme values for all points belonging to the domain. So, I just said that. So, you have a domain on which there could be a maximum and a minimum, so those are called global maximum and global minimum. But sometimes we may be interested in local extreme values as well and these are called local maximum and local minimum.

So, here is an example, a diagrammatic example. So, just look at the graph. In this diagram, there are five local extreme points. So, these are the five local extreme points a, b, c, d, e are the local extreme points. Out of these five b and d are local maximum points. So, these are the maximum points. And a, c, e are local minimum points. So, these are the minimum points.

So, this is how global and local are different that these two are the local maximum points b and d. But out of these two local maximums there is only one global maximum which is d. So, d is the global maximum, because between b and d at d you have a higher value of the function. And on the other hand, between the three local minimums that is a, c and e if I think about the global minimum, it will be a. This is the global minimum.

Now, this is intuitive and this is diagrammatic. You can see the rational more or less. So, in case of a local maximum, for example, this b is a local maximum, so in this locality b gives you the highest value of the function and d is also a local maximum. So, in this locality, in this neighborhood, you get the highest value at d. But b is not global maximum, because over the entire range b does not give you the highest value. It is d which is giving you the highest value, similarly, for the minimum.

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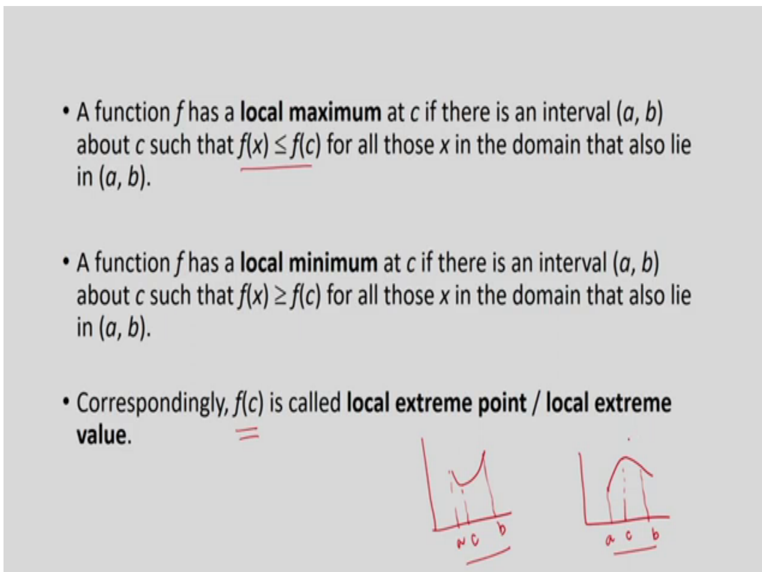

A function f has a local maximum at c if there is an interval (a, b) about c such that $f(x) \leq f(c)$ for all those x in the domain that also lie in (a, b). So, this is what I am trying to talk about here that suppose you have a maximum here, now, this is a local maximum in the sense that if you take an interval here, this is (a, b), then in this interval from a to b, the value of the function at c is always higher than the value of the function at any other value in this interval.

And similarly, for the minimum, a function f has a local minimum at c if there is an interval (a, b) about c such that $f(x) \geq f(c)$ for all those x in the domain that also lie in (a, b). So, this is just the opposite of the previous case. Here you have c here and here suppose a and b. So, $f(x)$, x can be anywhere from a to b, $f(x) \geq f(c)$, in this case. Correspondingly, $f(c)$ is called the local extreme point or local extreme value.

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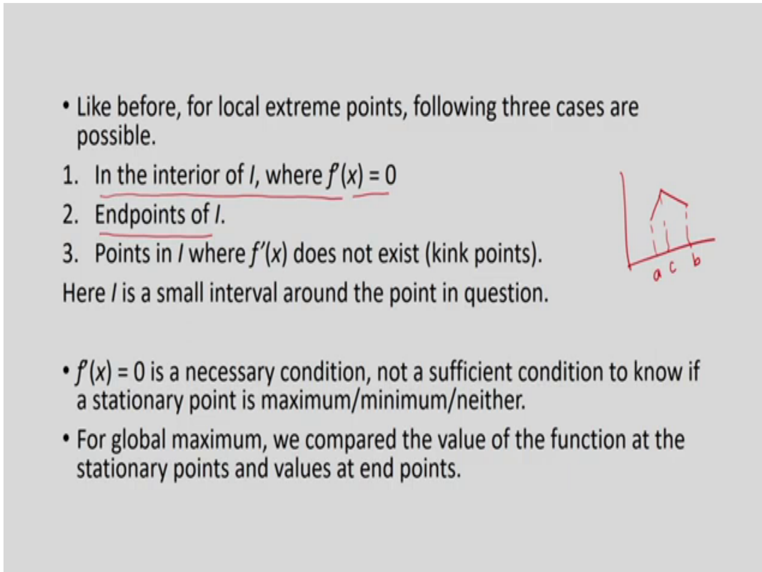

Now, like before, like in the case of global maximum, for local maximum or minimum also or local extreme points, the following three cases are possible that those local maximum or minimum those could be in the interior of I, where $f^{'}(x) = 0$. So, you have stationary points here. Or those could be endpoints of I, so I is that interval a to b that we are considering. And thirdly, it is possible that points in I where $f^{'}(x)$ does not exist. So, these are kink points.

So, here is the example of three. So, you have a to b and here you have c. So, you have a local maximum, but $f^{'}(c) \neq 0$, because $f^{'}(c)$ is not defined. So, there are three cases like in the case of global maximum or global minimum. So, this basically means that $f^{'}(x) = 0$ is a necessary condition. It is not a sufficient condition to know if a stationary point is maximum or minimum or neither. Because we have seen earlier that $f^{'}(x) = 0$, it is not going to imply that you have a maximum point there. It could be a point of inflection as well.

For the global maximum what we did? We compared the value of the function at the stationary points and the values at the endpoints. So, that is what we did. Endpoints means the endpoints of the domain, but that method will not apply in the case of local maximum or minimum.

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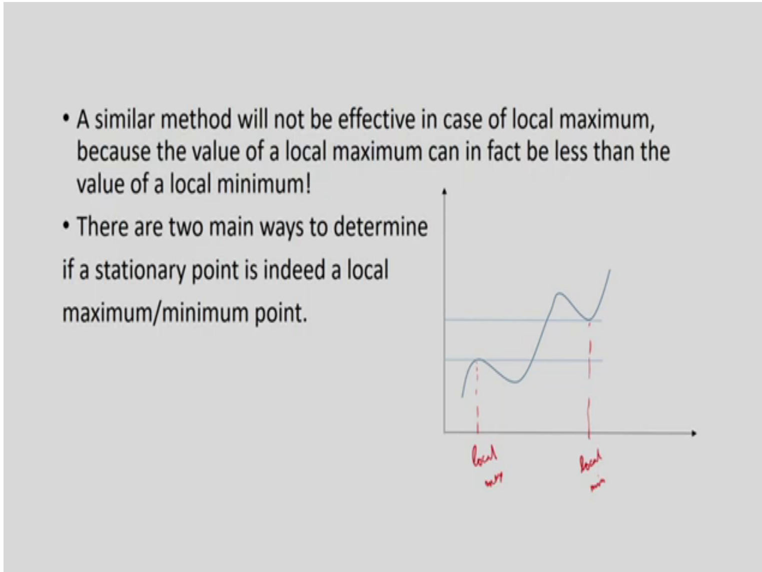

A similar method will not be effective in case of local maximum, because the value of local maximum can in fact be less than the value of a local minimum. So, here you have a local maximum. So, you have a local maximum. And here you have a local minimum. The value of the local minimum is greater than the value of the local maximum. So, it is futile to identify this particular local maximum by comparing their value with the endpoints of the interval. So evaluating the endpoints is not going to work for local maximum or minimum. There are two main ways to determine a stationary point if a stationary point is indeed a local maximum or a minimum point.

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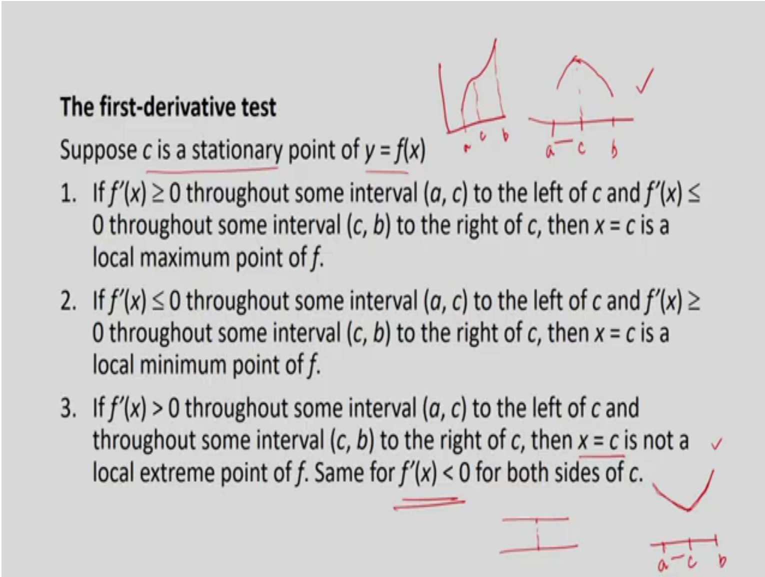

So, this is just like the previous case of global maximum or minimum. So, in case of local extreme points also there could be two main methods to determine the maximum or minimum. Suppose, c is a stationary point of $y = f(x)$, that means $f'(c) = 0$. Now, number 1, if $f'(x) \geq 0$throughout some interval (a, c). a to c means what.

So, you have c here and you have a here, so in this interval $f'(x) \geq 0$. So, you have something like this, to the left of c. And $f'(x) \leq 0$ throughout some interval (c, b), so suppose b is here, to the right of c. So, it is like this. Then $x = c$ is a local maximum point of f. So, this is the intuition.

At this point the derivative is equal to 0, to the left the derivative is non-negative, to the right the derivative is non-positive and that will guarantee that at c you have a maximum point, local maximum. You have to basically find out some interval around c which satisfies this condition. And what happens if you are trying to get the minimum points, this is the second point, if $f'(x) \leq 0$ throughout some interval (a, c) to the left of c and $f'(x) \geq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local minimum point of f. So, this can be easily seen. It is just the opposite of what we have discussed.

Here suppose you have the $f'(c) = 0$ and to the left the derivative of the function is non-positive that means it is something like this, it can be something like this and to the right it is something like this, then you have a minimum, local minimum. Notice one thing, this does not rule out the case where $f'(x) = 0$ to the left or to the right.

So, if you have $f'(c) = 0$, then you are going to have a horizontal line. But that does not mean that you do not have a local maximum or local minimum. So, these are the two cases. You have a local maximum here and local minimum here. So, this is basically the first derivative test.

Third case, if $f'(x) > 0$ throughout some interval (a, c) to the left of c and throughout some interval (c, b) to the right of c then $x = c$ is not a local extreme point of f. Same for $f'(x) < 0$ for both sides of c. So, basically what you have here is something like this that you have this.

At this particular point $f'(x) = 0$, but to the left say a to c it is always positive. So, it is a rising function. And to the right of c also, it is a rising function. Then clearly this function is not getting any maximum or minimum at the point c. And similarly, if you have $f'(x) < 0$ to the left and to the right so then also you do not have a maximum or a minimum. There are no local extreme points.

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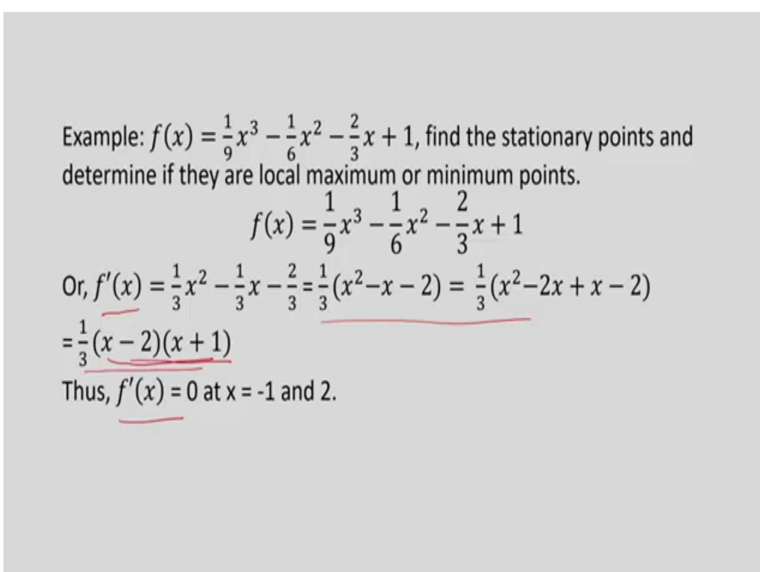

Here is another example. Suppose $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$, find the stationary points and determine if they are local maximum or minimum points. So, it is a cubic function. What we do. We first find out the stationary points. For that we take the first derivative. The first derivative is giving me, $f'(x) = \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3}$. So, this boils down to $f'(x) = \frac{1}{3}(x - 2)(x + 1)$. I am just skipping some steps.

Now, how to find the stationary points? We set this $f'(x) = 0$. That means this, $f'(x) = \frac{1}{3}(x - 2)(x + 1) = 0$ which means that either $x = 2$ or $x = -1$. So, these are the stationary points. So, the first part is done. Stationary points are -1 and 2.

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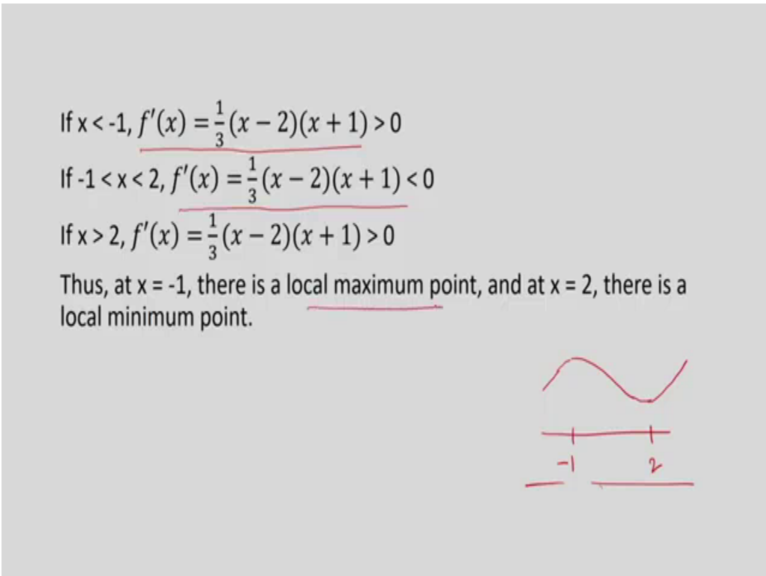

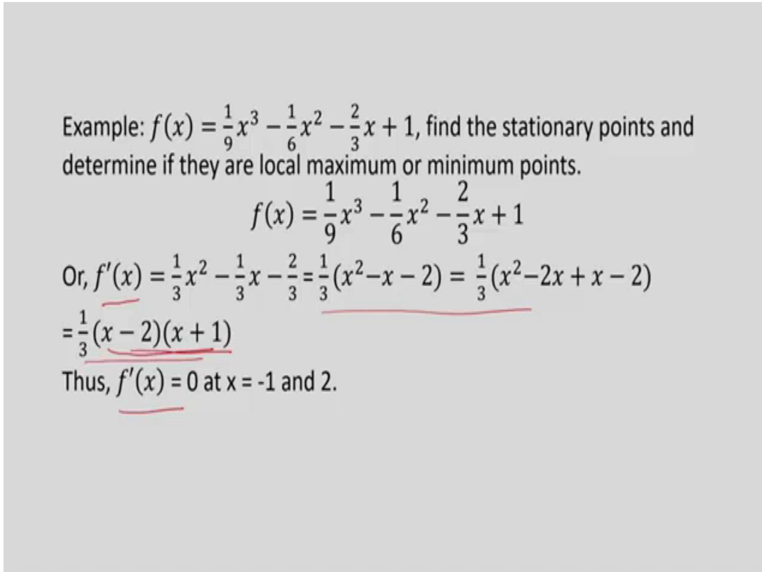

Now, we have to use the first derivative test to understand whether you have a local maximum or local minimum. So, to do that, I have to find out what happens to the left of $x = -1$. So, you have -1 here and 2 here. To the left of -1, i.e., $x < -1$, let us see what is the value of the function. The value of the function turns out to be positive. Value of the function means $f'(x)$.

So, here to the left of -1 the function is rising. Here obviously it is a stationary point so the derivative is 0. Between -1 and + 2 the value of the function that is $f'(x) < 0$. So, it is a declining thing. And at 2 it is a stationary point, to the right of $x = 2$, i.e., $x > 2$ the $f'(x) > 0$. So, something like this.

Therefore, just looking at the diagram that I have drawn in a rough manner, one can see that at $x = -1$ you have a local maximum point and at $x = 2$ there is a local minimum point. So, that is what we were supposed to answer and determine if there are local maximum or minimum points.

So, you have two stationary points -1 and +2. In -1 you have a local maximum and at +2 you have a local minimum. I think I will call it a day for the time being. And I shall see you in the next lecture. Thank you.