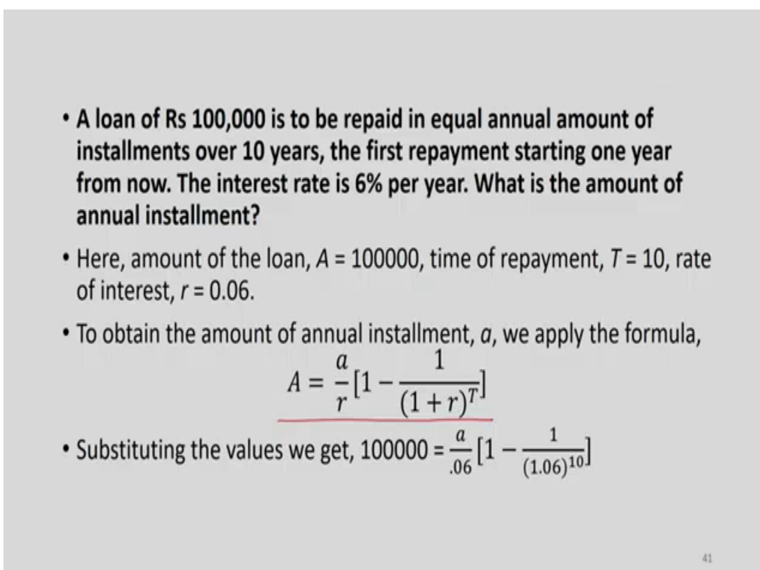
Mathematics for Economics - I
Professor Debarshi Das
Department of Humanities & Social Sciences
Indian Institute of Technology, Guwahati
Lecture 19
Tutorials – 1c

Hello and welcome to another lecture of this course Mathematics for Economics Part I. So, at present we are going through tutorials. What we are doing in this tutorial is we are covering some of the problems. We are discussing how to solve particular exercises with the knowledge that we have from undergoing this course.

So, at present we are talking about differentiation, the applications of differentiation, how differentiation can be useful to solve many exercises and also practical problems. At present we were talking about in particular the application of series. Let me take you to that exercise that we were discussing in the last lecture. So, we were talking about geometric series and how to calculate the present discounted value.

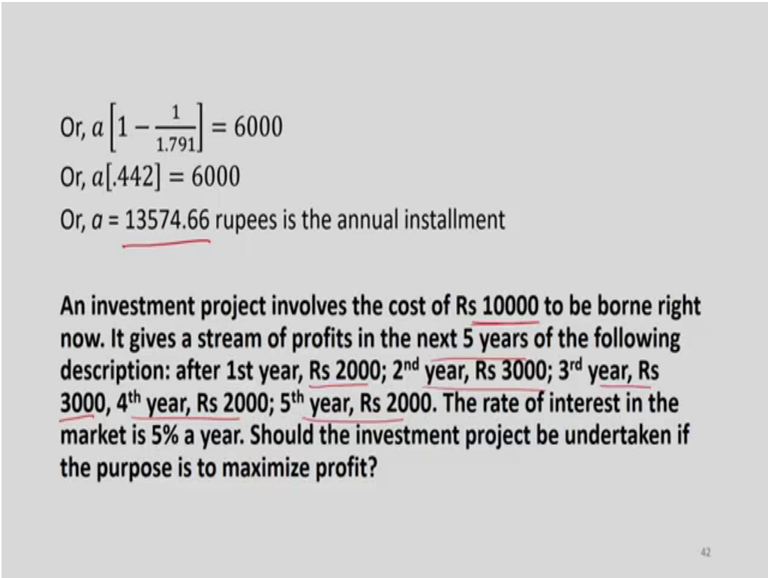
(Refer Slide Time: 01:43)



So, this was the last problem that we discussed in the last lecture. There is a loan of 1 lakh which has to be paid in equal annual installments over 10 years and the first payment is starting from the next year. The interest rate is given as 6 percent. So, we have to find out what is the annual installment? What is the annual amount of money that the borrower has to pay back?

And that we have seen can be solved by taking the help of this formula, $A = \frac{a}{r}[1 - \frac{1}{(1+r)^T}]$. And here A is given, it is 1 lakh, r is 6 percent, capital T is 10 years, 10. So, we just put those values here, and we can solve for a.

(Refer Slide Time: 02:49)

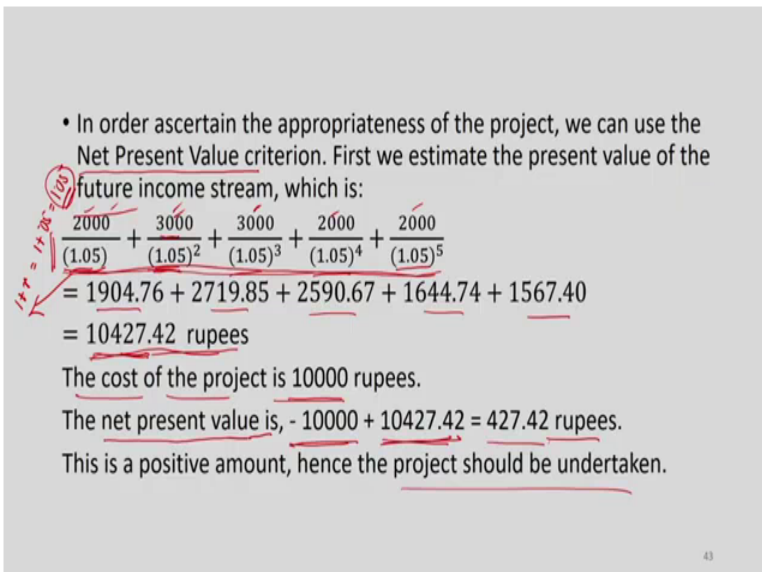


And here it is coming out to be 13574.66. So, basically means that the borrower has to pay back this as the equal annual installment, EAI, you can call it instead of EMI and it is 13,574.66 for 10 years for the loan that he had taken of 1 lakh rupees. So, this is what we discussed, this is something we discussed in the last lecture, last problem.

Now, we are looking at another problem which is using a similar concept. An investment project involves the cost of 10,000 rupees to be borne right now. It gives a stream of profits in the next five years of the following description, after one year 2000 rupees, after the second year 3000 rupees, after the third year 3000 rupees, after fourth year 2000 rupees and after the fifth year 2000 rupees. The rate of interest in the market is 5 % a year. The question is should the investment project be undertaken if the purpose is to maximize profit. So, this is the question.

Whenever a producer thinks whether he will be engaged in an investment project there are two sides to it. One is the cost side. So, here the cost is 10,000 rupees. And this cost has to be borne upfront, that means, in theory this year itself, whereas the profit is coming to him in a stream that is continuous over several years. In this case it is coming to him in a stream of over five years 2000, 3000, 3000, 2000, 2000 and that stream is starting from the next year onwards. Rate of interest is given in the market at 5 percent. So, the question is will the producer being a profit maximizer undertake this investment project.

(Refer Slide Time: 05:49)



So, in order to ascertain the appropriateness of the project we can use the net present value criterion. And what is this net present value criterion? First, we estimate the present value of the future income stream which is given by this. So, here is an application of the present value, the idea of present value. So, this income stream is not occurring today. It is happening, it is occurring to the producer in the future. So, how much is that money which is accruing to him in the future valued from today's point of view. So, that is called the present value.

So, in this case this stream 2000, 3000, 3000, 2000, 2000 this particular stream we have to find the present value of that and that is what we have done here. So, this is the present value. We have taken, for example, 2000 which is accruing going to him in the next year. We have divided that by $(1 + r)$. Here this number 1.05 is nothing but 1 + the rate of interest r. In this case it is $(1 + 0.5)$, which is 1.05. So, that is why in the denominator you can see 1.05.

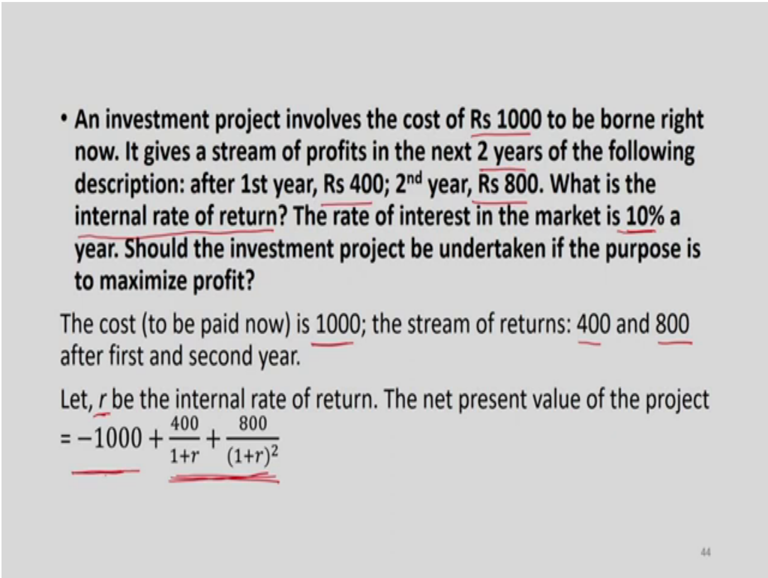
The second income which is coming to him after the second year, so two years from now 3000 that is coming to him after two years that has to be divided by $(1 + r)^2$ and $(1 + r)$ we have just found it is 1.05. So, it should be $(1.05)^2$. So, that is the method that we are applying here. Then we have $\frac{3000}{(1.05)^3}$, $\frac{2000}{(1.05)^4}$ and then to the power 5.

So, after we have figured, this out this present value, the next step is just to laboriously calculate these numbers. So, I have found out these numbers. These are, mind you, the approximate values I have taken up to the second decimal place, approximate it up to the second decimal place, and then I have added them up and the submission is coming out to be 10427.42. So, in sum, this is the amount 10427.42 is the present value of this income stream, this income stream 2000, 3000, which is happening in the future. So, this is the present value of the income stream.

On the other hand, there is this cost which is accruing right now, which is 10000 rupees. So, what is this net present value criterion going to say, it is going to tell us that the net present value is the present value of the profit which is this and from that you deduct the cost, the present value of the cost. Now, the present value of the cost is nothing but 10000 rupees, because this is not happening in future. It is in the present itself. So, I do not have to discount this. So, therefore, the net present value is plus 10427.42 and minus 10000. So, the net is coming out to be 427.42.

So, it is coming out to be a positive amount. The net present value is a positive amount. Hence the project should be undertaken. So, that is the idea of net present value. Take the present value of the cost, take the present value of the incomes and look at the difference. If the difference is positive, then the investment is profitable. Therefore, it should be undertaken. So, in this case we have found out the answer.

(Refer Slide Time: 10:56)



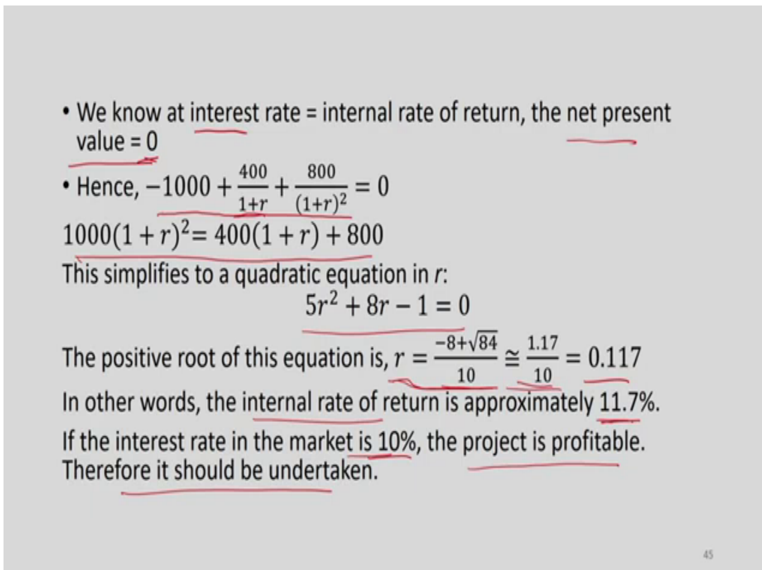
Here is another problem related to investment project. But now we are going to look at another criterion of whether to undertake the investment project or not. An investment project involves the cost of 1000 rupees to be borne right now. It gives a stream of profits in the next two years of the following description, after the first year the profit is 400 rupees, after the second year the profit is 800 rupees. What is the internal rate of return? So, this is the new thing that we are discussing in this question which is by the way something we discussed in the course of the lectures also.

We have to first find out the internal rate of return that is the first part. The rate of interest in the market is given. It is 10 % a year. The question is should the investment project be undertaken if the purpose is to maximize profit? So, this is the second part. So, first part we have to find out what is the internal rate of return of this particular project. And secondly, the market rate of interest is given. It is 10 % a year. So, we have to answer whether the investment project is profitable or not, whether it should be undertaken.

So, how to solve this? First, we note that the cost of the project here is 1000 rupees. This cost has to be paid right now at present and there is a stream of returns which is occurring in the future 400 rupees and 800 rupees after the first year and second year. So, these are the incomes that are occurring in the future. Now, let us suppose r is the internal rate of return that we have to find out. That is the first question. Now, if r is the rate of return, internal rate of return then what is the net present value in this case?

As we have seen in the last problem, the net present value is found out in the following manner. We look at the cost which is 1000 rupees to be borne right now, that is coming as a negative entry here and then we are looking at the present value of the incomes which are occurring in the future and we are taking the present value of that. So, the negative entry is the cost, the positive entries are the present values of the incomes. We are looking at the difference.

(Refer Slide Time: 14:23)



Now, what is the internal rate of return? We know at the interest rate = internal rate of return the net present value is 0. So, that is the idea of the internal rate of return. If we take the rate of interest which we are using to get the present value as the internal rate of return, then the NPV or the net present value will be equal to 0. So, since here r is the internal rate of return, then this should be equal to 0, because this is the net present value. $-1000 + \frac{400}{1+r} + \frac{800}{(1+r)^2} = 0$, because r is the internal rate of return.

Now, next step is just to try to simplify this. I can multiply both sides of this equation with $(1+r)^2$. So, I will get this. And it can be simplified further. And it gives us a simple quadratic equation in r, $5r^2 + 8r - 1 = 0$. Now, as it is known that if we have a quadratic equation then there are two roots general in a quadratic equation and here the negative root we should ignore, because negative rate of interest does not make much sense.

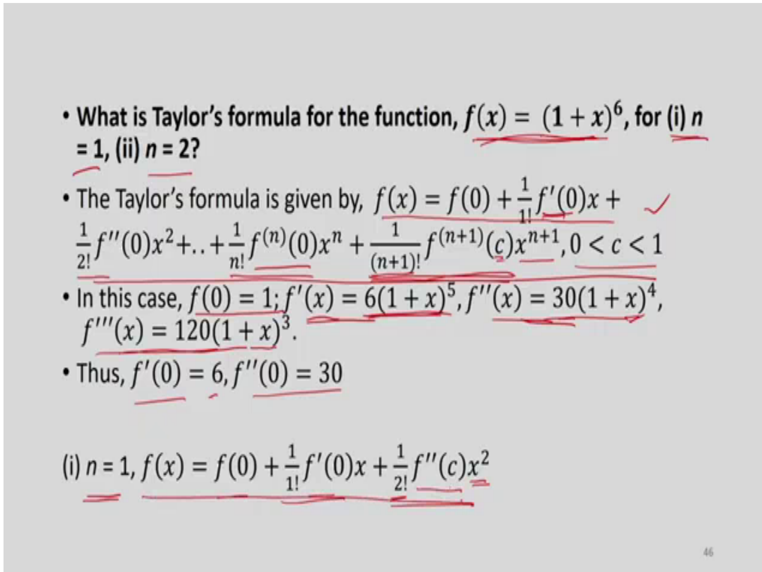
So, we look at the positive root of r and that can be found out by this formula, $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and that is turning out to be $\frac{-8+\sqrt{84}}{10}$. And I take the approximate value after the, up to the second decimal place. So, the numerator is coming out to be 1.17 the denominator is 10 and it is turning out to be 0.117. This is r.

In other words, the internal rate of return is approximately equal to 11.7 percent. So, this is the answer to the first question, what was the internal rate of return. It is coming out to be 11.7 percent approximately. Why, approximately because in this stage we have taken the approximate value.

Now, come to the second part. The second part is saying that the interest rate in the market is given as 10 %. So, the question is will the project be undertaken. Now, we have already found out that the internal rate of return of the project is 11.7 percent and the market rate of interest given as 10 percent which is less than the internal rate of return.

Therefore, the project should be undertaken, because it is profitable compared to the existing market rate of interest. So, that is the criterion that we are using here that if the internal rate of return in a project is more than the market rate of interest then the project is profitable, and therefore, it should be undertaken. So, that is the answer.

(Refer Slide Time: 18:30)



Now, let me come to a separate kind of question which is about Taylor's formula. And again, this is something we discussed in the series of lectures. So, here the function is given $f(x) = (1 + x)^6$ and we have to find out what is the Taylor's formula of this function, for n = 1 and n = 2. So, that is the question. Now, we have to just recall what was the Taylor formula as such in general.

Now, this was the Taylor's formula $f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + ... + \frac{1}{n!}f^{(n)}(0)x^n + \frac{1}{(n+1)!}f^{(n+1)}(c)x^{n+1}, 0 < c < 1$ is the function it is equal to f of 0, so you are evaluating the function at x = 0, plus $\frac{1}{1!}f'(0)x$. That is you are taking the first derivative and then evaluating the first derivative at x = 0 that will give you the $f'(0)$. Then multiplying it with $\frac{x}{1!}$ plus this $\frac{x^2}{2!}f''(0)$and likewise it will go on. You can see the pattern.

The n-th term will be $\frac{x^n}{n!}f^{(n)}(0)$. $f^{(n)}(0)$ means the function has been differentiated n times and then it is evaluated at x = 0. And then there is this balanced term and this balanced term or sometimes it is called a correction term is given by $f^{(n+1)}(c)$.

That means the function has been differentiated n + 1 times and evaluated at c. What is c? c is a value between 0 and 1, multiplied by $\frac{x^{n+1}}{(n+1)!}$. So, this is the formula. We just have to find out if n = 1 then what is the function equivalent to by using the Taylor's formula.

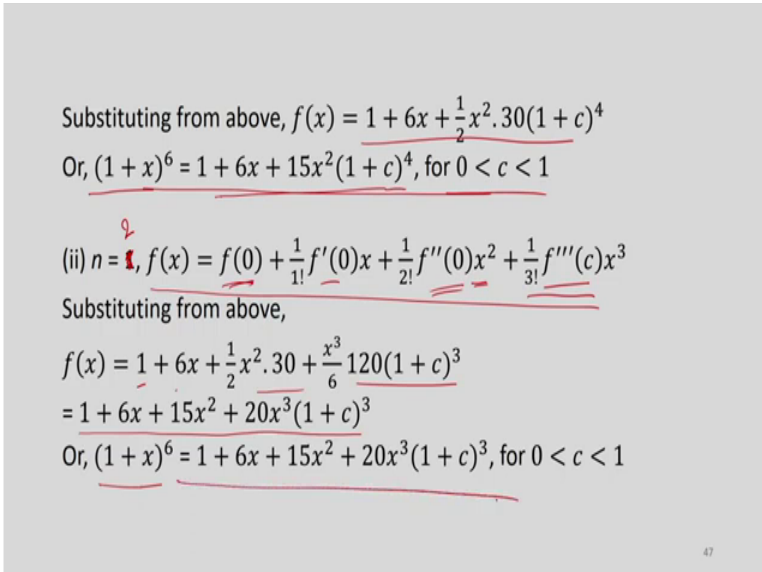
We have to first find out what is $f^{'}(x)$ or $f^{''}(x)$, also $f^{'''}(x)$ that also will be required and then we have to find out what are these functions at x = 0 ? So, that is what we need to evaluate this particular form. Now, if you look at this function $f(x) = (1 + x)^{6}$ then what is $f(0)$. I have to put x = 0 then it becomes $(1 + 0)^{6}$ which is 1. So, that is what I have written here, $f(0) = 1$.

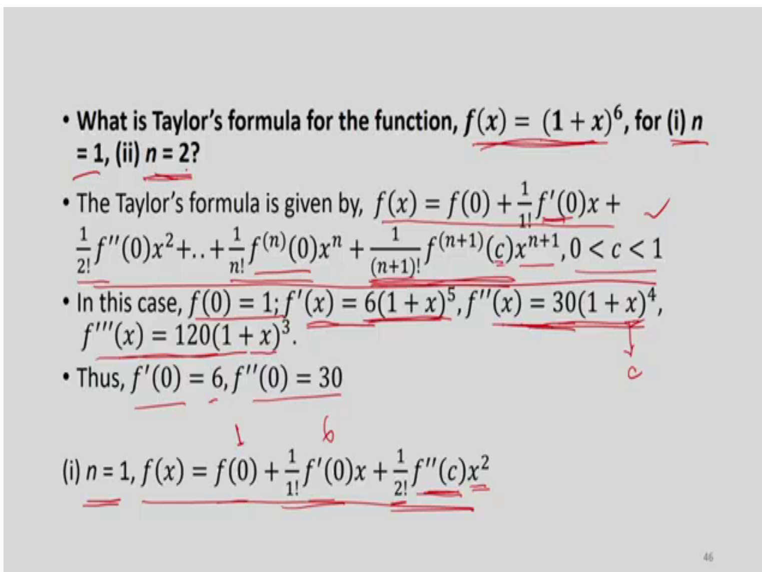
What is $f^{'}(x)$? I take the differentiation of this. So, this will give me $6(1 + x)^{5}$. And what is $f^{''}(x)$. So, I take the differentiation of this function $6(1 + x)^{5}$. And if I take the differentiation of this I get $30(1 + x)^{4}$.. And I also will recreate the third order differentiation of this and that will give me $120(1 + x)^{3}$. And I have to find out what are the values of this and this at x =0.

So, if I put x =0 here it will just give me 6 and if I put x = 0 here it will give me 30. So, I now have everything I need and now I just have to substitute them in the formula and that will give me the answer. Now, at n = 1, what is the formula? In that case, the $f(x)$ the left hand side is equal to the right hand side is $f(0)$ and $\frac{1}{1!}f^{'}(0)x$. So, this is the first term and that is n = 1. So, I have to stop there.

And plus the rest of the, that is the correction term is there which will be like what is n + 1, n + 1 here is 2. So, in the denominator we have factorial 2 and x^{2}, because that is n +1, 2, and $f^{''}(c)$, because n = 1. So, this is how it looks like for n = 1.

(Refer Slide Time: 24:20)





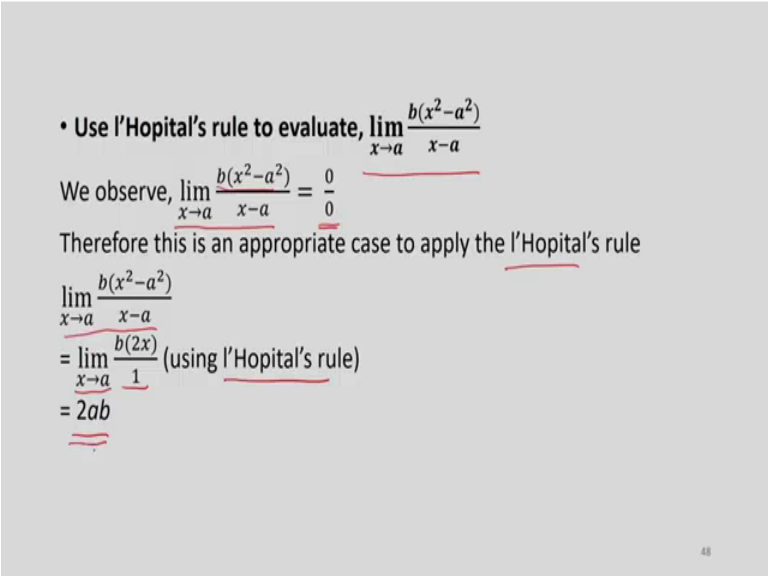
And I substitute the values that I have just found out. I have found out $f(0)$, which is 1, $f'(0)$ has been found out to be 6, $f''(c)$, f double dash is this, so here instead of x I will write c here and that will give me this term. So, those things are substituted in this form. So, I get this. It can be simplified further on the right hand side.

So, on the right hand side I get $(1 + x)^6 = 1 + 6x + 15x^2(1 + c)^4$, where $0 < c < 1$. And just remember that on the left hand side I have $f(x)$. What is $f(x)$? $f(x) = (1 + x)^6$. So, this is the answer. So, $(1 + x)^6 = 1 + 6x + 15x^2(1 + c)^4$, $for\ \ 0 < c < 1$. This is the value of the function.

Now, we look at what the function is at n = 2. So, here this is not 1, it is actually 2, n = 2. Now, just as we have done in the first part, here let us first try to write out the function for n = 2. So, $f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(c)x^3$.

And all these values are known to us $f(0)$ it is 1, $f'(0) = 6$, likewise $f''(0)$ has been put here and $f'''(c)$ which is given by this $120(1 + c)^3$. And this is simplified here, $1 + 6x + 15x^2 + 20x^3(1 + c)^3$ and this is equal to the function, function is $(1 + x)^6$. So, this is the answer, $(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3(1 + c)^3$ $for\ \ 0 < c < 1$. So, that takes care of the problem.

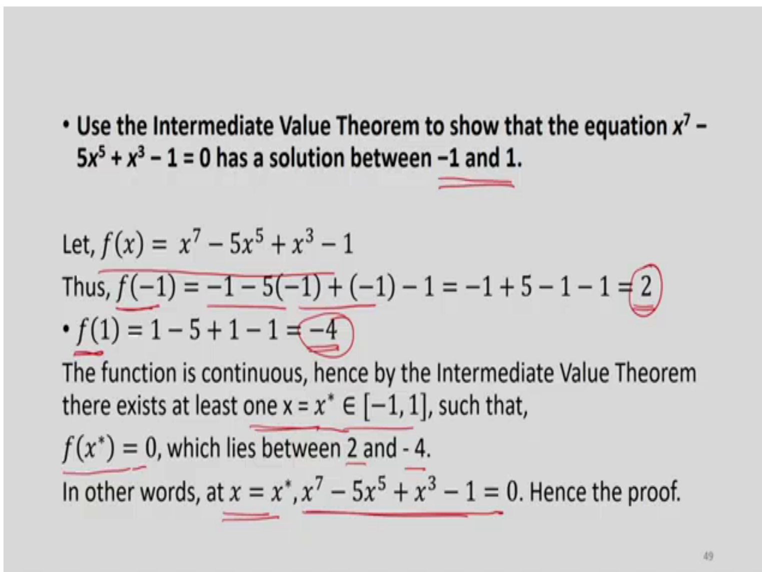
(Refer Slide Time: 27:31)



Now, we come to another question. This is related to the L'Hopital's rule of differentiation. Use L'Hopital's rule to evaluate this, $\lim_{x \to a} \frac{b(x^2 - a^2)}{x - a}$. Now, the specific problem with this particular limit is that as x goes to a, the numerator actually approaches 0 and the denominator also approaches 0. So, the entire function actually is approaching $\frac{0}{0}$ form.

Now, if we have such cases, then we try to use the L'Hopital's rule to find the limit. So, what do we do? We take the problem that is $\lim_{x \to a} \frac{b(x^2 - a^2)}{x - a}$, and then we take the differentiation of the numerator and the denominator with respect to x that is the L'Hopital's rule. And if we do so, in this case, the numerator turns out to be $b(2x)$ and the denominator is 1. So, it becomes just 2bx and as x goes to a, 2bx goes to 2ab. So, simply this is the answer.

(Refer Slide Time: 29:25)



Here we are encountering another problem, but this is an application of the intermediate value theorem. And as we are going to see, intermediate value theorem is extremely useful in certain cases where you want to find the solution of a function or whether a solution exists or not in a particular interval. So, the problem is here this.

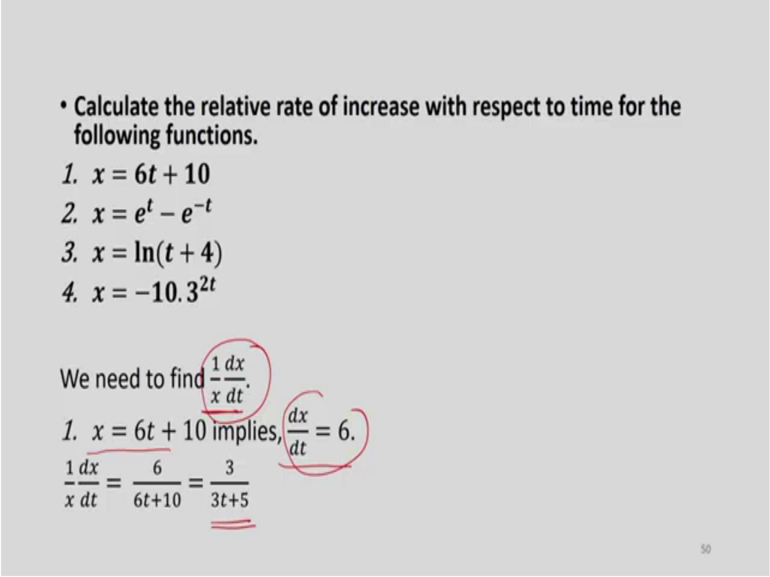
Use the intermediate value theorem to show that the equation $x^7 - 5x^5 + x^3 - 1 = 0$ has a solution between - 1 and + 1. Now, this equation is given to us. We have to show that there is a solution to this equation in a particular interval.

What we do is that we assume that the left hand side of this equation is represented by this function $f(x)$. So, $f(x) = x^7 - 5x^5 + x^3 - 1$. Now, look at this particular interval - 1 and + 1. This is the interval. So, what we do is that we evaluate this function at these two endpoints. So, endpoints are - 1. So, I evaluate this function at the point $x = -1$ and this turns out to be 2. And similarly, at the other endpoint that is $x = 1$ the value of the function turns out to be - 4.

So, here comes the role of the intermediate value theorem. The function is a continuous function. Hence by the intermediate value theorem there exists at least one $x = x^* \in [-1, 1]$such that $f(x^*) = 0$, which lies between 2 and - 4, because 0 is a value which is between these two values, between 2 and - 4.

So, that is how we are using the intermediate value theorem to actually prove that there is at least one $x = x^* \in [-1, 1]$ where the value of the function is 0, which means that the at $x = x^*$ this equation $f(x) = 0$ has a solution that is $x^7 - 5x^5 + x^3 - 1 = 0$. So, hence the proof.

(Refer Slide time: 32:47)

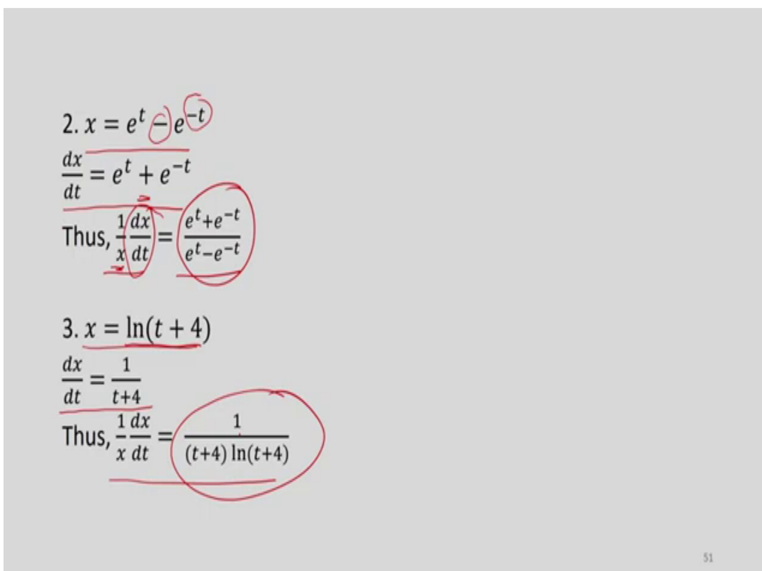


Here are some more applications of differentiation. Calculate the relative rate of increase with respect to time for the following functions. Relative rate of increase that is sometimes it is also called the rate of growth in economics. Here time is the independent variable. So, for example, when we talk about the growth rate of, suppose GDP of a country, then what is the independent variable that one has in mind it is time, time is changing with respect to that, the GDP is changing so one wants to find out what is the relative rate of increase. In this case, it will just be the growth rate.

So, in this case four functions are given to us x is the dependent variable and t is the independent variable. We have to find out this, $\frac{1}{x}\frac{dx}{dt}$. This is the growth rate or the relative rate of increase with respect to time. Let us look at the first function which is $x = 6t + 10$.

So, from this we immediately can find out what is $\frac{dx}{dt}$ that is simply equal to 6. Then I can substitute this $\frac{dx}{dt}$ in this formula and that will give me $\frac{6}{6t+10}$ and which is the same as $\frac{3}{3t+5}$. So, this is the pattern that we are going to follow. In the second case you have $x = e^{t} - e^{-t}$.

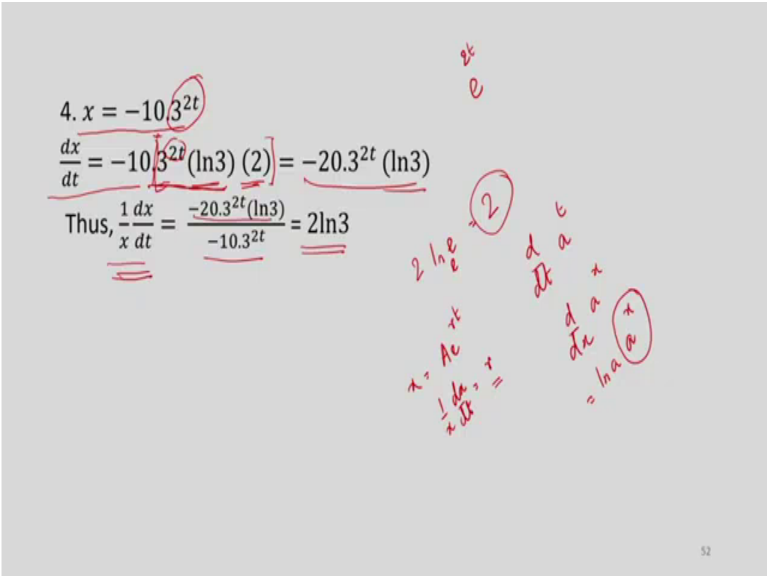
(Refer Slide Time: 34:47)



Now, here again this is given to us $x = e^{t} - e^{-t}$. So, I can find out $\frac{dx}{dt}$ it will be $\frac{dx}{dt} = e^{t} + e^{-t}$, because this minus sign of this power will get multiplied with this minus sign it will become a plus sign. So, $\frac{dx}{dt} = e^{t} + e^{-t}$. Again, I substitute that value here and I will get this, because x as such $x = e^{t} - e^{-t}$. So, you get this particular form, $\frac{1}{x}\frac{dx}{dt} = \frac{e^{t}+e^{-t}}{e^{t}-e^{-t}}$.

The third problem, $x = ln\,(t + 4)$. And first again I find out $\frac{dx}{dt}$, it is $\frac{1}{t+4}$. So, $\frac{1}{x}\frac{dx}{dt} = \frac{1}{(t+4)\,ln(t+4)}$, because $ln\,(t + 4) = x$. So, this is what we are going to get.

(Refer Slide Time: 36:08)



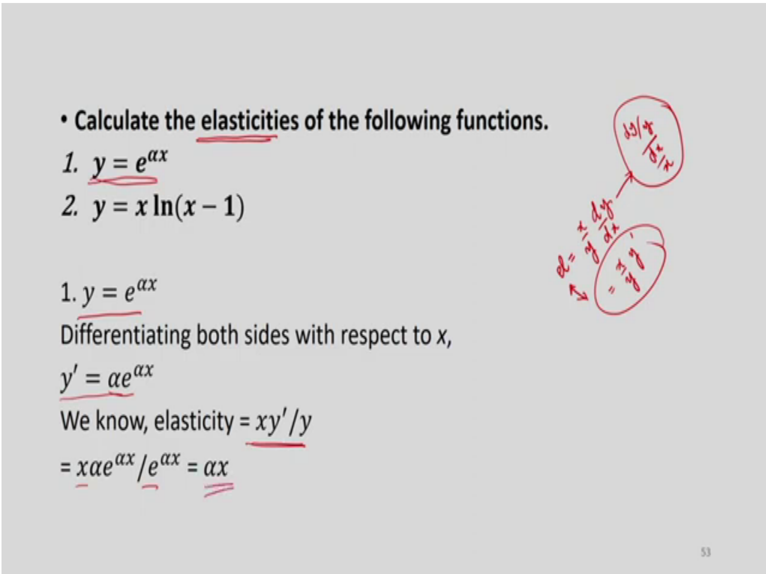
Fourthly, $x = \; - 10.3^{2t}$. Now, in this case again, I have to first find out what is $\frac{dx}{dt}$. And this is going to be a little bit complicated. Firstly, note that -10 is a constant. So, I can just put it in front. So, -10 $\frac{d}{dt}$ of this, $\frac{d}{dt}3^{2t}$. Now, what is that? So, this is in the form of $\frac{d}{dt}a^{t}$ or $\frac{d}{dx}a^{x}$. So, what is $\frac{d}{dx}a^{x}$, it was $loga\; a^{x}$. So, that was the formula and here I am using that formula.

So, I am writing this as 3^{2t}, so that is a^{x} here, and $loga$, here a is 3, so $log3$. But we are forgetting one more term which is that there is a 2, which is the coefficient of t here. So, that 2 will get multiplied. So, all these terms are accounted for. In the next stage it becomes

$-\ 20.3^{\ 2t}\ (ln\ 3)$. So, this was $\frac{dx}{dt}$, but our purpose is not to find $\frac{dx}{dt}$. Our purpose is to find this, $\frac{1}{x}\frac{dx}{dt}$.

So, for that, I use this formula. On the numerator I have $\frac{dx}{dt}$ which is this. In the denominator I have the x which is this and many terms will actually get cancelled out, the minus signs will get cancelled out and I am left with $2ln\ 3$. So, that is the answer. Notice instead of 3 if I had $e^{\ 2t}$, then it would just have been 2 $ln_e e$ and that would have been just 2, if instead of 3 I had e. And that is expected. If you have this kind of function $A\ e^{\ rt}$ and if you want to find out $\frac{1}{x}\frac{dx}{dt}$, then the answer is always r, r means the power of e where rt is the power. So, r becomes the relative rate of increase of x with respect to t.

(Refer Slide Time: 39:25)



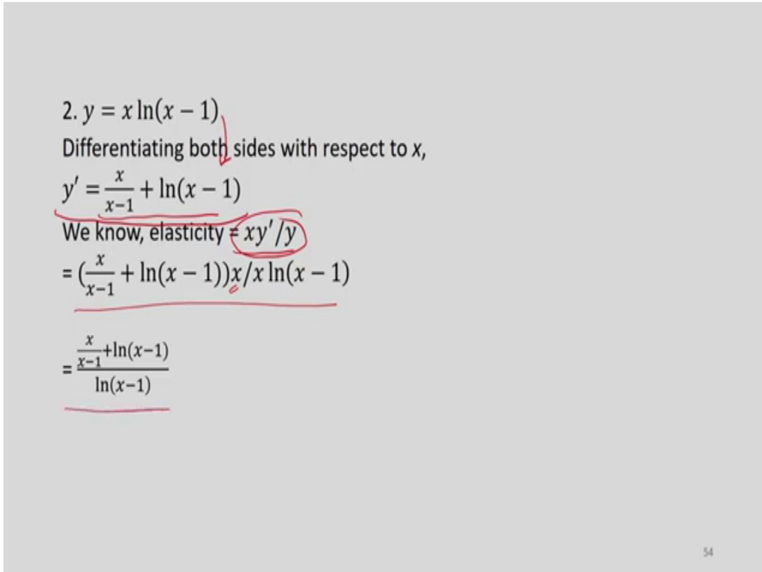
And now we come to another application of differentiation and this is elasticity. So, we have talked about elasticities. What is the importance of calculating elasticities, especially in economics. So, let us try to find out the elasticities of certain functions which are given to us. So, $y\ =\ e^{\ \alpha x}$. I have to find out the elasticity. Now, this is written in a very succinct form. So, in this case, I have to first understand what is the independent variable, what is the dependent variable here. x is the independent and y is the dependent variable.

And in this case, the elasticity is given by this. Let us call it el, elasticity, or $el = \frac{x}{y}\frac{dy}{dx}$ or you can write it as this $el = \frac{x}{y} y'$, because y' is the general notation for $\frac{dy}{dx}$, the first derivative of the function. And you can actually explain this further by saying that this is nothing but $\frac{dy}{y} / \frac{dx}{x}$. What is the numerator, the percentage change or the relative change of y and the denominator is the percentage change or the relative change in x.

So, elasticity tells us what is the percentage change in the dependent variable with respect to the percentage change in the independent variable. So, that is why you are actually getting this form as the elasticity $el = \frac{x}{y} y'$. Now, we are going to use this simplified form for these functions. So, first, we have to find out what is the y', the first derivative of the function, and that is simply, in this case, $\alpha. e^{\alpha x}$.

Once y' is found out, I substitute that here, and this is going to give me $x\alpha. e^{\alpha x}/y$, $y = e^{\alpha x}$. So, e^{ax} will get cancelled from the numerator and the denominator. Therefore, the answer is simply αx.

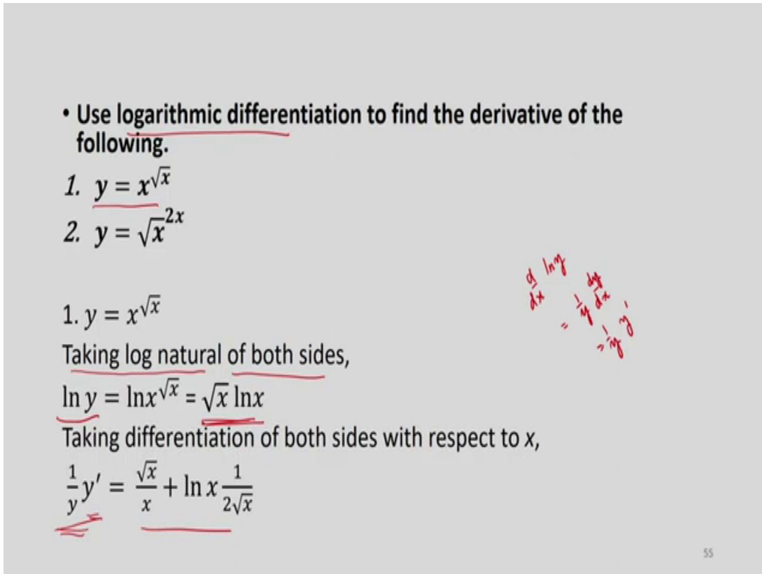
(Refer Slide Time: 42:38)



Here is the second problem. So, once again, I have to find out what is the elasticity. And again, I am going to take the help of this form $el = \frac{x}{y}y^{'}$. So, I have to find out what is $y^{'}$ that is easily found out by differentiating y with respect to x. So, I have used the product rule here, because y is a product of two functions, $y = xln\,(x - 1)$. So, using the product rule, I get $y^{'} = \frac{x}{x-1} + ln\,(x - 1)$.

And that I substitute here and ultimately, I am going to get this form $y^{'} = (\frac{x}{x-1} + ln\,(x - 1))x/x\,ln\,(x - 1)$ and that turns out to be this $\frac{\frac{x}{x-1}+ln\,(x-1)}{ln\,(x-1)}$. So, this is the answer.

(Refer Slide Time: 43:58)

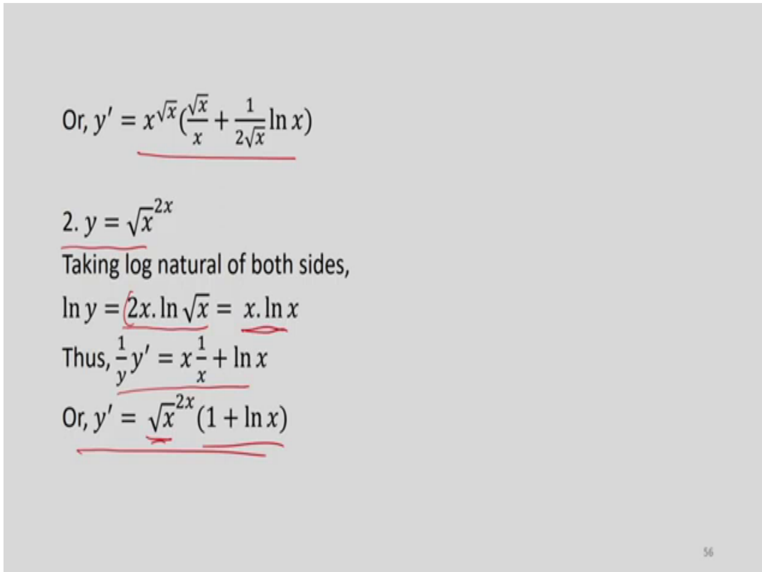


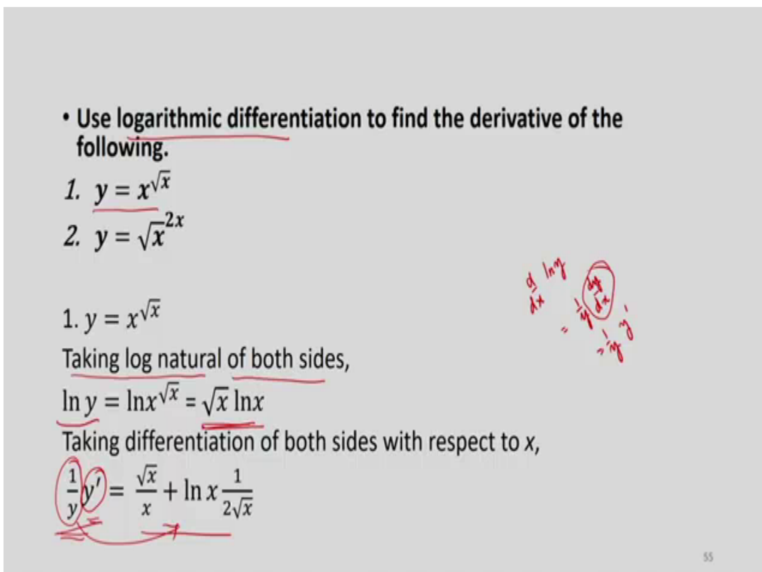
Use logarithmic differentiation to find the derivative of the following. So, here also I have to take the help of differentiation. I have to find the differentiation of certain functions, but I have to use logarithmic differentiation. The first function is $y = x^{\sqrt{x}}$. So, what happens in logarithmic differentiation?

First, we take the natural log of both sides. So, $ln\, y = ln\, x^{\sqrt{x}}$. And this turns out to be, the right hand side turns out to be $ln\, y = \sqrt{x}\, ln\, x$. Then we take the differentiation of both sides with respect to x. So, if I do so then what happens?

In the next stage on the left hand side, I am going to get $\frac{1}{y}y'$. This is very standard. You have $ln\, y$ and you are taking differentiation of that with respect to x. So, it becomes $\frac{1}{y}\frac{dy}{dx}$ and that is $\frac{1}{y}y'$. So, that is how we are getting $\frac{1}{y}y'$ on the left hand side. On the right hand side, I am taking the derivative with respect to x of this function. So, I am using the product rule. So, $\frac{\sqrt{x}}{x} + ln\, x\, \frac{1}{2\sqrt{x}}$.

(Refer Slide Time: 45:56)



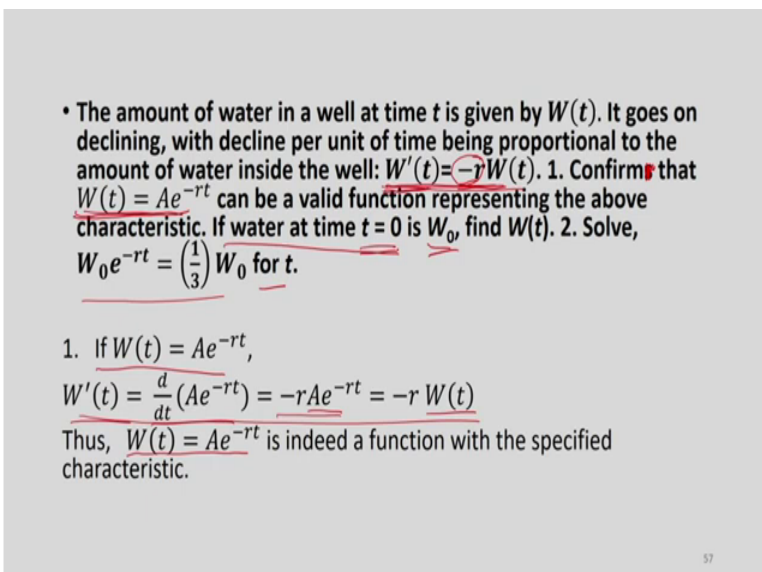


So, in the next stage, I just take y to this side, because I have to find out y', I have to find out $\frac{dy}{dx}$. So, on the left hand side, I cannot have this term which is getting multiplied with y dash, so I multiply both sides of the equation with y and that will give me only y' on the left hand side. So, that is what I do here. And therefore, $y' = x^{\sqrt{x}}(x^{\sqrt{x}} + \frac{1}{2\sqrt{x}}ln\ x)$. This is the answer.

The second part is a different problem. I mean, you have root over $y = \sqrt{x}^{2x}$. Earlier you had $x^{\sqrt{x}}$. However, this can be easily found out again like before I take the log of y on the left hand side, in the right hand side again, I have taken the log. And if I have taken the log, I get $ln\, y = 2x.\ ln \sqrt{x}$. And this is simplified as $x.\, ln\, x$, because this 2 is there and here $x^{1/2}$. So, 2 multiplied by 1/2 will give me 1. Therefore, I get $x.\, ln\, x$.

And like before, I take the derivative of both sides with respect to x, and I will get $\frac{1}{y} y' = x.\frac{1}{x} + ln\, x$. And then in the next stage, I multiply both sides by y and I will get $y' = \sqrt{x}^{2x}(1 + ln\, x)$. Actually, $\sqrt{x}^{2x}$ this comes out to be x^{x}, because root over that is there it is nothing but $x^{1/2}$and half and 2 will get multiplied. So, you are going to get x^{x}. But that is the same thing as saying $\sqrt{x}^{2x}$.

(Refer Slide Time: 48:32)



Here is an application of I think differentiation. Here is the problem. The amount of water in a well at a time t is given by $W(t)$, W as a function of t. It goes on declining with decline per unit of time being proportional to the amount of water inside the well. So, this is the decline per unit of time $W'(t)$, it is proportional to the amount of water inside the well.

So, it is written as, $W'(t) = -rW(t)$. $W(t)$ is the amount of water inside the well. So, it is getting multiplied with some constant and that constant is $-r$. Why minus, because there is a decline in water. So, $W'(t)$ is negative. So, that is why they have taken the minus sign here.

So, there are two questions related to this setting. Confirm that $W(t) = A.e^{-rt}$ can be a valid function representing the above characteristic. And if water at time $t = 0$ is W_0, find $W(t)$. So, that is the first part. Second part is, solve this equation $W_0.e^{-rt} = (\frac{1}{3})W_0$ for t.

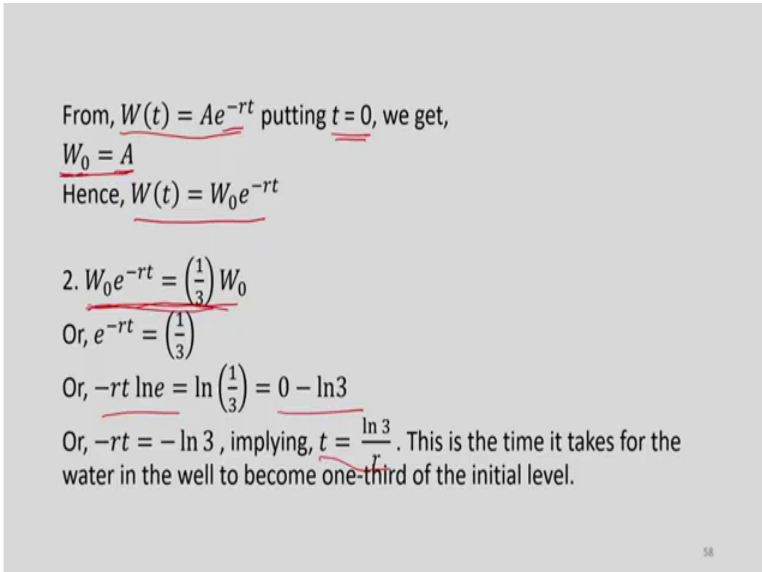
So, the first part is we have to confirm that this function is a valid function representing the above characteristic. What is the above characteristic? This is the above characteristic that $W'(t) = -rW(t)$. So, that is the property here. We have to confirm that if $W(t) = A.e^{-rt}$ then this particular characteristic is maintained.

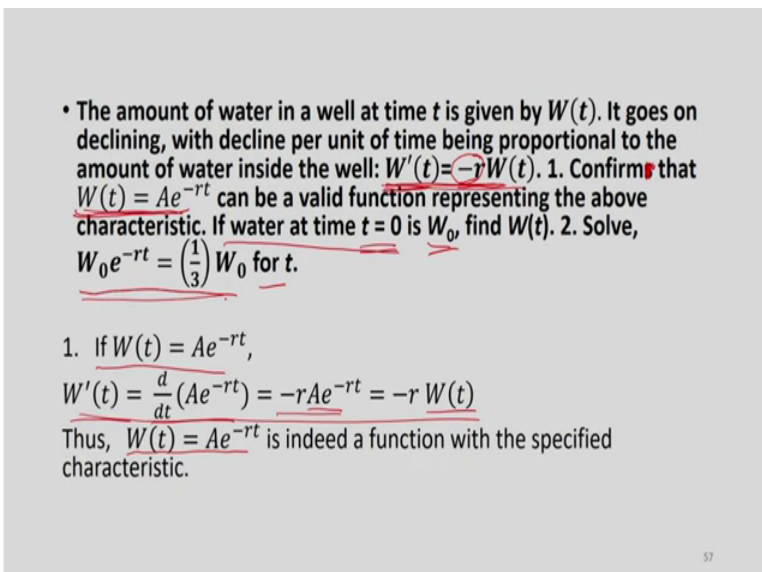
That is what we have to find out if that is maintained. Now, this form $W(t) = A.e^{-rt}$ this is given to us. So, to ascertain whether this particular function is following this characteristic we have to do what, we have first find out what is $W'(t)$, that is the left hand side.

So, that is easily found out $\frac{d}{dt}W(t)$ and that is giving me $-r.A.e^{-rt}$ and $A.e^{-rt}$ is nothing but $W(t)$. So, actually you are getting this, $W'(t) = -rW(t)$ and that is given in the question. So, that characteristic is maintained. So, therefore, $W(t) = A.e^{-rt}$ is indeed a function with the specified characteristic. So, we have confirmed that.

However, here, in this particular form A is not known to us. Capital A is not known to us. We have to find out what is $W(t)$. Since A is not known to us, we have to use this information. If water at time $t = 0$ is W_0 in that case let us see what happens.

(Refer Slide Time: 52:31)





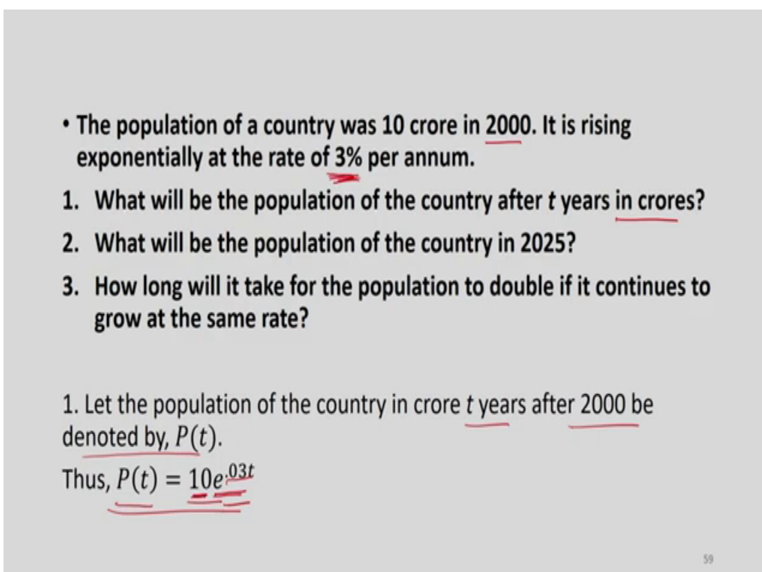
So, this is the form, putting $t = 0$ what do we get, e^0. On the right hand side, you are going to get e^0, because $t = 0$. $e^0 = 1$. So, $W_0 = A$. Why W_0 on the left hand side, because that is given to us. At time $t = 0$ the water in the well is W_0. So, W_0 turns out to be A. Therefore, the form that we are getting that is $W(t) = W_0 . e^{-rt}$. So, the first part is done.

Second part, second part is saying that suppose this is given to us then solve for t. So, this is given to us that $W_0 . e^{-rt} = (\frac{1}{3})W_0$. So, let us try to solve this. So, W_0 will get cancelled from both sides, $. e^{-rt} = (\frac{1}{3})$. I can take the log of both sides. So, I get minus $- rt \ ln \ e = ln \ (\frac{1}{3})$. What is $ln \ (\frac{1}{3})$? It is $ln \ 1 - ln \ 3$. So, it becomes $- ln \ 3$.

On the left hand side, I just have $- rt$, on the right hand side I have $- ln \ 3$ and that means $t = \frac{ln \ 3}{r}$. So, this is the time. What is the meaning of this answer? This is the time it takes for the water in the well to become one-third of the initial level. Why I am saying that, because look at the thing that we started with.

So, this is telling me, what this left hand side is the water in the well. And on the right hand side I have one-third of the initial level. So, solving this I am getting t so that t is nothing but the time it will take for the water in the well to become one-third of the initial level.

(Refer Slide Time: 54:58)



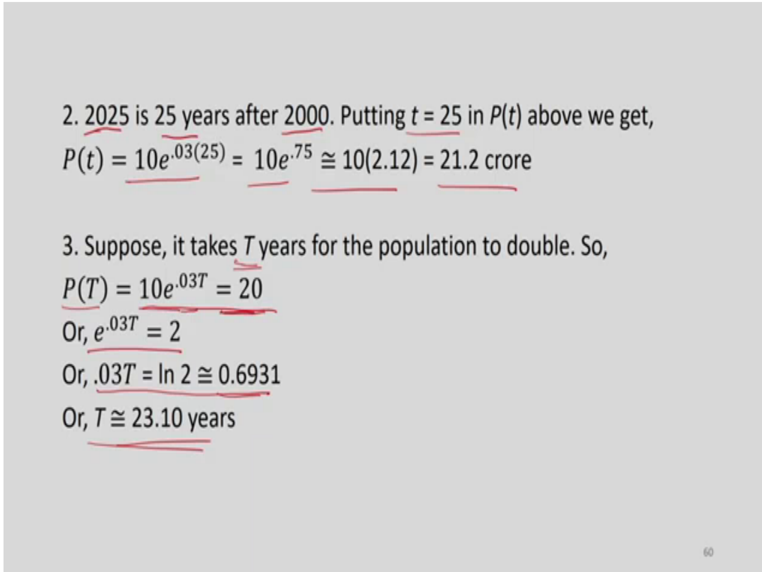
Population of a country was 10 crore in the year 2000. It is rising exponentially at the rate of 3 % per annum. What will be the population of the country after t years in crores? That is the first question. What will be the population of the country in the year 2025? How long will it take for the population to double if it continues to grow at the same rate, the same rate means 3 % per annum. So, this is again a practical application of what we have learned in this course.

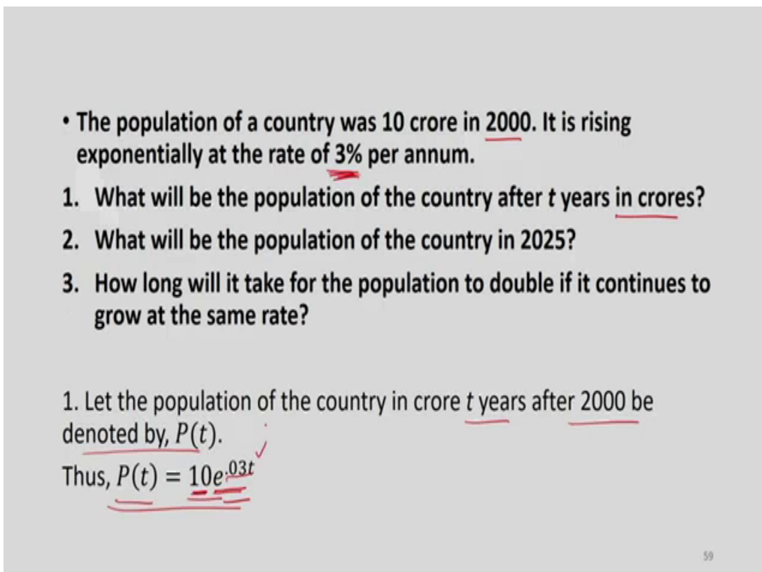
For example, if we think about India, India's population in 2000 was around 100 crore. So, one might ask the question that after 2000, the year 2000, suppose t more years have elapsed. Now, what will be the population of the country after these t more years? So, that is the first question. And likewise, the second question is what will the population of India in the year 2025? And after how many years the population will double. So, in 2000 it was 100 crore. In which year it will become 200 crore assuming, obviously, that the rate of growth of population is 3 percent, it is rising exponentially at 3 percent per annum.

Let the population of the country in crore t years, number of years that have elapsed after 2000 be t, after 2000 be denoted by $P(t)$. Thus, $P(t) = 10. e^{.03t}$. So, this is the simple formula that how many time or how many years have elapsed let us suppose that is t, so that is coming as the power of e and there is a coefficient of that t also it is the growth rate which is 3 %. 3 % turns out to be 0.03.

So, you are getting this neat formula $e^{.03t}$. But that is not all I have to multiply with the population of the country at the point of origin. So, here the point of origin is the year 2000. So, what was the population 10. So, this gives me the full formula $P(t) = 10. e^{.03t}$.

(Refer Slide Time: 58:13)





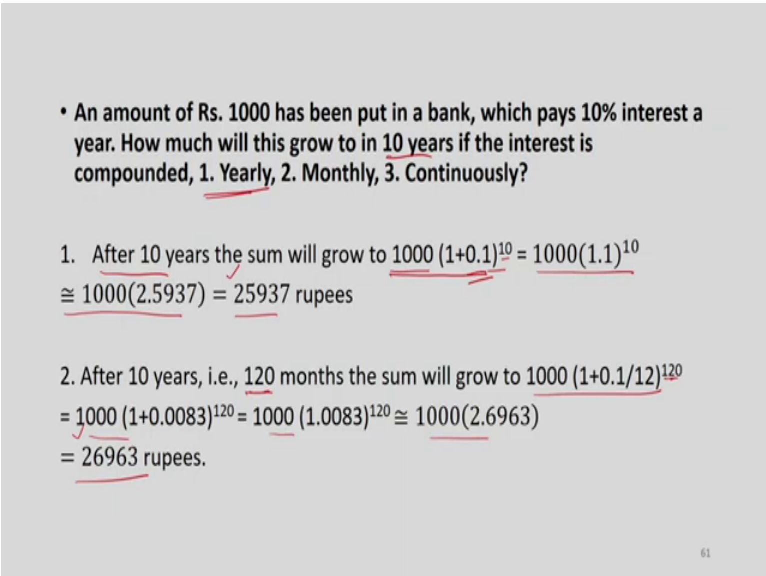
The second part was what the population will be in the year 2025. Now, 2025 is 25 years after 2000. So, in this case actually t that we are considering here this t is 2025 - 2000 that is t = 25. So, I put that t here in this previous formula. So, this turns out to be $P(t) = 10.e^{.03(25)} = 10.e^{.75}$ and that is approximately 21.2 crore.

So, in this case, the population of the country in 2025 is 21.2 crore. Now, obviously, we are assuming, we are making an implicit assumption that 3 % population growth rate is going to be maintained. If that is not maintained then this formula will give me a wrong answer.

And the third part was how much or how long will it take for the population to double if it continues to grow at the same rate. So, again, I will use the same formula. Suppose, it takes capital T years for the population to double, so I can write $P(T) = 10. e^{.03T}$. But that will be equal to 20, i.e $P(T) = 10. e^{.03T} = 20$, because the population is doubling. So, it was 10 before, now, it will become 20. So, I have to solve this for capital T and that is not very difficult.

It becomes, as you can see, $0.03T = ln\,2 \approx 0.6931$. Therefore, $T \approx 0.6931/0.03 = 23.10$ years. And so, in short, the population of that country will double in a little bit more than 23 years if it continues to grow at 3 percent. Now, the relevant question for India will be how long will it take for India's population to become 200 crore. It was roughly 100 crore in 2000. So, how long will it take for the population of India to become double that is 200 crore.

(Refer Slide Time: 61:17)



Here is another problem. An amount of 1000 rupees has been put in a bank and this bank pays 10 % interest rate per year. How much will this grow to in 10 years if the interest rate is compounded yearly or monthly or continuously? After 10 years how much 1000 rupees will become if the rate of interest is 10 percent.

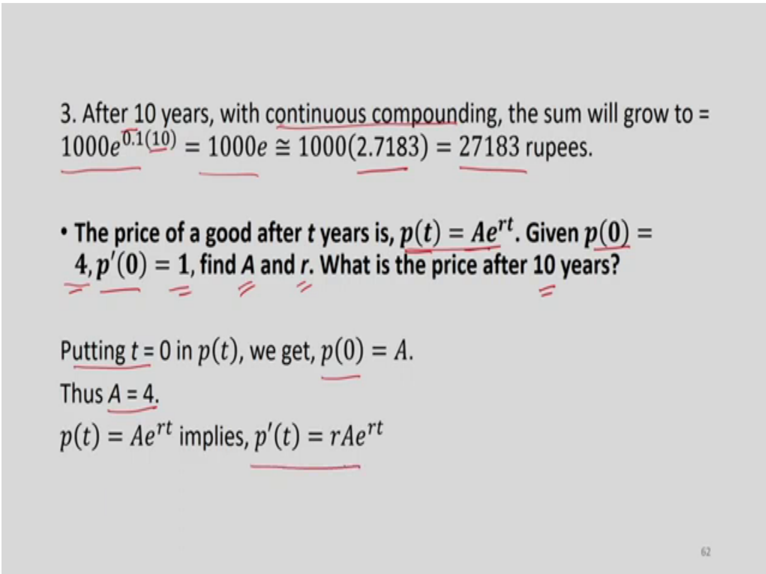
10 percent as we know is 0.1 and here the yearly rate of interest is taken to be 10 percent and it is compounded yearly, annually. This is the first part. So, the formula is $1000(1 + 0.1)^{10}$. 10 is

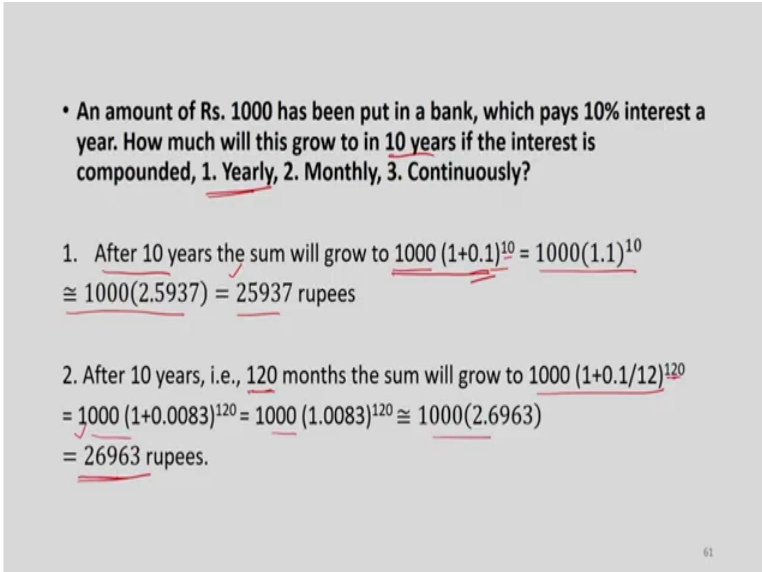
the number of years. And the rest is just to calculate that and this is approximately equal to 25937 rupees after 10 years. So, this was the first part.

In the second part what happens, here we are compounding it, but not annually, it is compounded monthly. In 10 years, how many months are there? There are 120 months. So, t here is 120. So, that is coming as the power like before. However, I have to be careful with the rate of interest. Rate of interest is per month. It is, whatever the rate of interest is annually divided by 12. So, it is $\frac{0.1}{12}$. Therefore, this is the formula that I have $1000(1+\frac{0.1}{12})^{120}$ and this is simplified in the next following steps and I am taking the approximate value and answer is 26963 rupees.

The interesting thing to note here is that this number, 25937 is less than this number, 26963. So, if the interest is compounded annually, then the amount of money that you will get is going to be less compared to the case where the interest is compounded not annually but monthly. And the difference is quite a lot. It is more than 1000 rupees.

(Refer Slide Time: 64:03)



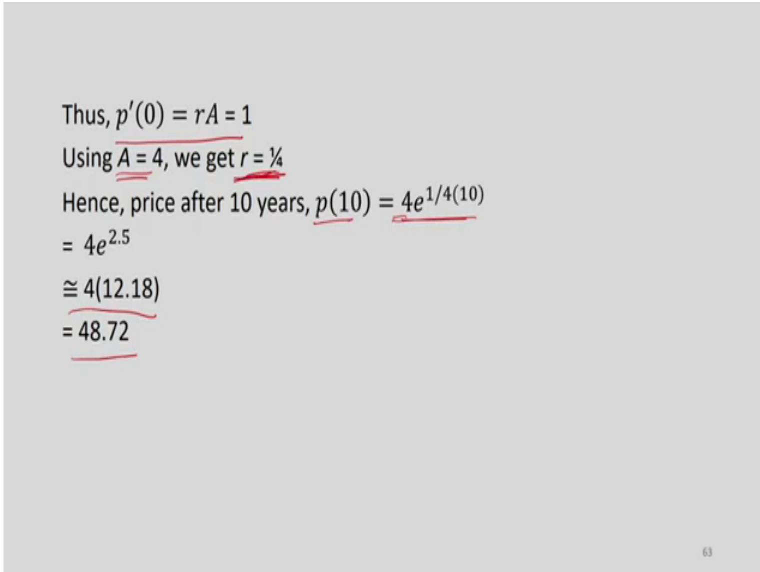


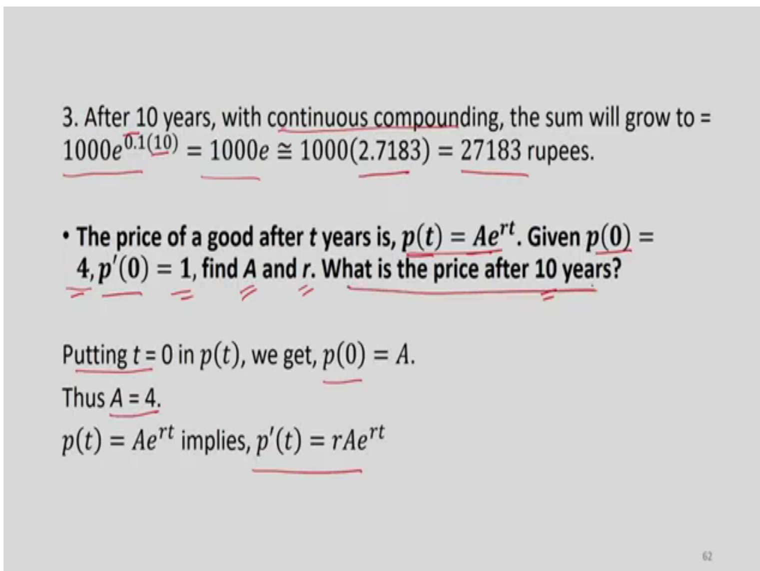
And in the third case, the principal sum remains the same, the annual rate of interest remains the same, the time remains the same, but here the compounding is being done continuously. And here the formula is this, $1000e^{rt}$, r is 0.1, t is 10. So, we just have to find out what is $1000e^{0.1.(10)} = 1000e$. e as we know is given by this number, 2.7183 approximately again and this comes out to be 27183 rupees and this is higher than the amount that is obtained when the compounding is done monthly. So, that is the answer.

I think I will pick up the last question in this lecture now. The price of a good after t years is given by $p(t)$ and it is having this form, this functional form, $p(t) = Ae^{rt}$. We are given this information that $p(0) = 4$, that is price at time 0 is equal to 4, $p^{'}(0) = 1$, that is a derivative of the price with respect to time at $t = 0$ is equal to 1. I have to find out what is capital A and small r. These are not known to us. What is the price after 10 years, that also is a question.

So, I put $t = 0$ here. So, $p(0)$, because $t = 0$, $p(0) = 4$, simply A because $e^{0} = 1$. Therefore, $A = 4$, because $p(0) = 4$. Secondly, I use the second information that $p^{'}(0) = 1$. So, I first find out what is $p^{'}$. $p^{'} = rAe^{rt}$.

(Refer Slide Time: 66:28)





Therefore, $p'(0) = rA = 1$ according to the question. We further know that $A = 4$, therefore, $r = 1/4$. That is what we are getting here that is 0.25. So, r is known to us, A is also known to us. So, the first part is done. What about the second part? What is the price after 10 years?

So, price after 10 years is equal to $p(10) = 4e^{1/4(10)}$. So, I have just used the fact that $A = 4$, $r = 1/4$, so this is the price after 10 years, $p(10) = 4e^{1/4(10)}$ and that is $p(10) = 4e^{2.5}$ and that turns out to be approximately equal to 48.72.

So, just to put it in context, the price was 4 at $p(0)$. So, at the point of origin the price was 4, after 10 years it becomes 48.72, quite a lot, after 10 years or you can take t to be in years, it could be in months also, but whatever it is. If it is years then after 10 years the price becomes 48, which is like 12 times the original price. If it is growing at what rate in this case, it is growing at 25 percent per year. I will stop here, this particular tutorial. And in the next tutorial maybe I will take up some other problems from the course. Thank you for joining me. Have a nice day.