Mathematics for Economics - I
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Lecture 15
NPV, evaluation of investment projects

Welcome to another lecture of this course Mathematics for Economics Part I. And the topic that we are covering is continuity, differentiability and series. We have done one lecture of this particular topic. Now, we are going to take up from that that topic.

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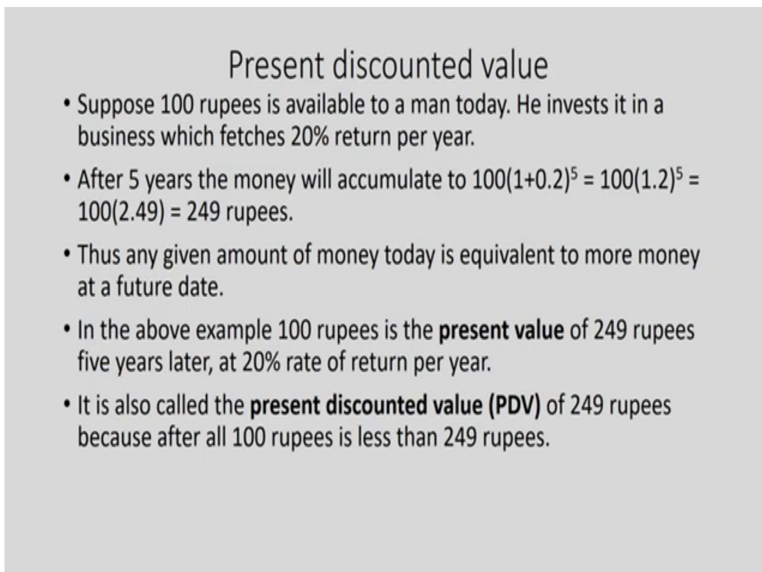

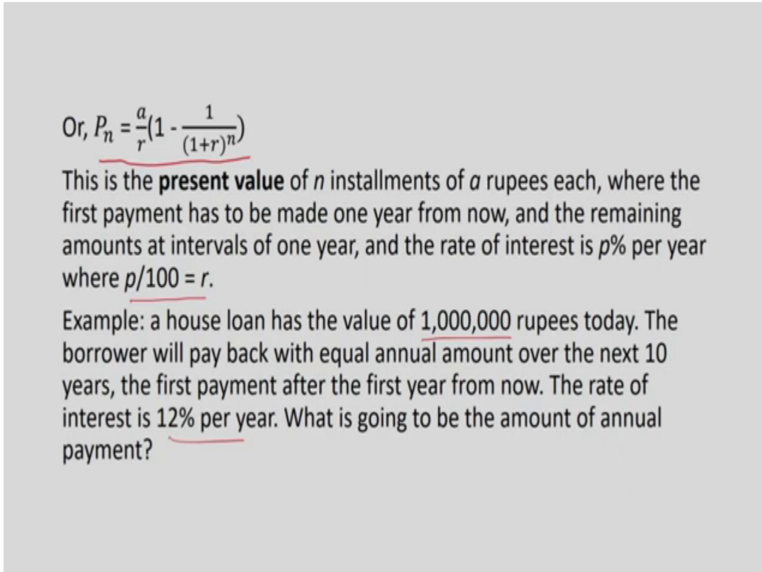

So, we were in the last lecture talking about series and sequences and their applications and the thing that we were discussing is present discounted value or PDV. We have talked about one example of PDV and this is the general form that we have found that you have a limited number of periods in which a person can get certain income or profit, suppose, that is constant that is a, then present discounted value, or in short, present value, of this series of payments in future is given by this expression $P_n = \frac{a}{r}(1 - \frac{1}{(1+r)^n})$, where r is the rate of interest.

It is also called rate of discounting, n is the total number of periods and small a is the payment per period or we can call it the installment per period. So, there are n installments and r $r = p/100$, where p can be thought of as 10 percent or 15 percent. So, if you divide 10 by 100 you get r which is the rate of discounting or it is also sometimes called the rate of interest.

Here is another example of how these ideas could be used, the idea of present discounted value. A house loan has the value of INR1 million today. So, a person has taken a loan from the bank INR1 million that is INR10 lakh today and this borrower will pay back with an equal amount annually over the next 10 years, the first payment after the first year from now. The rate of interest is 12 percent per annum. What is going to be the amount of annual payment? So, we are given certain information. We are given the information about P_n.

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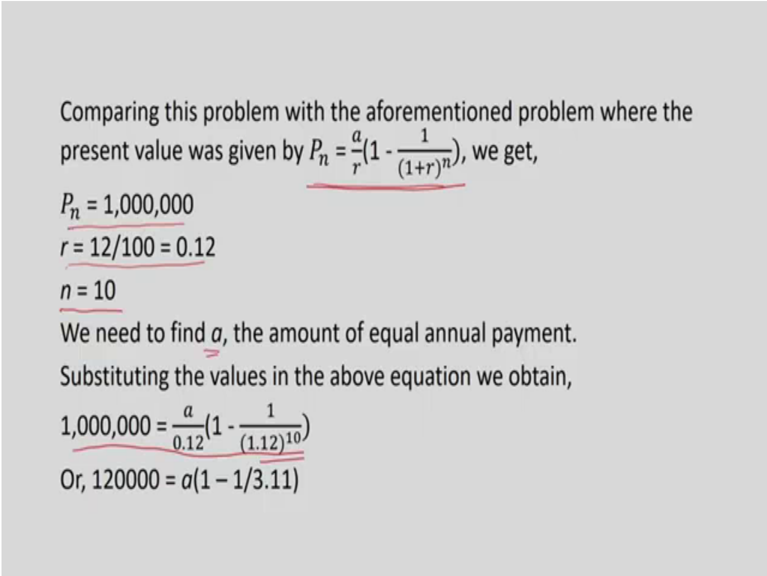

So, this is what we know. $P_n = \frac{a}{r}(1 - \frac{1}{(1+r)^n})$, where P_n is the present value, a is equal payment annually, r is the rate of interest and n is the total number of periods. Now, in this formula what we know is P_n which is INR1 million, we know the rate of interest which is 12 percent that is 0.12, how many periods over which the payments will have to be done it is $n = 10$. So, what we need to find is a, the amount of equal annual payment.

Now, we have to basically substitute all these values into this formula and the only thing that is unknown is a. So, if we do so, we get $1000000 = \frac{a}{0.12}(1 - \frac{1}{(1.12)^{10}})$. And then we basically simplify this relation.

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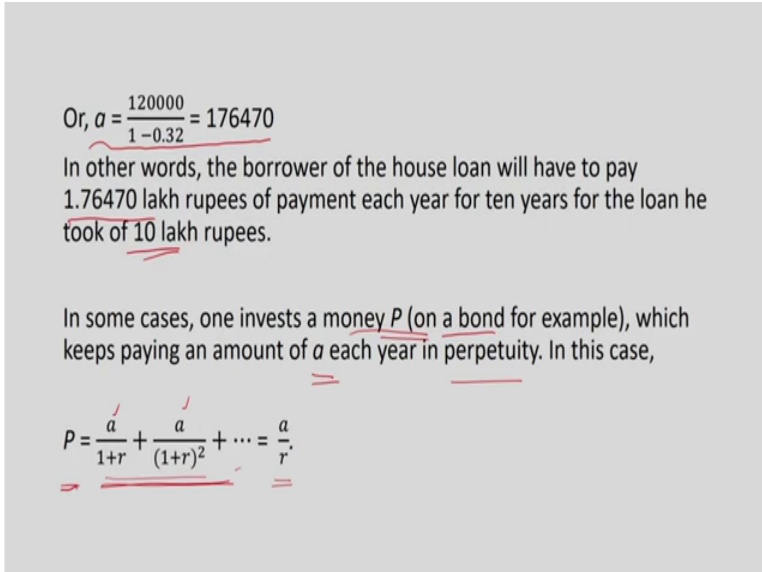

And what we get in the end is a, which is a = 176470. In other words, the borrower of the house loan will have to pay INR1.76470 lakh of payment each year for 10 years for the loan that he took of INR10 lakh. So, this is the answer.

And notice he has taken INR10 lakh loan today and he is going to make 10 payments, but the amount of payment each year is not INR1 lakh, it is INR1.76470 lakh, which is more than INR1 lakh. And this is because the value of money tomorrow is less than the value of money today. The same amount of money will be worth more today than tomorrow. That is why you have to make the payment more if you want to pay it in future to account for the time.

In some cases, one invests a money P suppose, on a bond for example, which keeps paying an amount of small a each year in perpetuity, in perpetuity means forever. In this case, we similarly use the same idea that if you are going to get a amount of money per year from let us say next year, so the present discounted value of this stream will be $P = \frac{a}{1+r} + \frac{a}{(1+r)^2} + ... = \frac{a}{r}$. And this is the present discounted value of these payments that are going to be made to you tomorrow. And obviously that payment that you are going to get tomorrow should be equal to the present discounted value which is the price that you are paying for the bond.

So, bond is what? Bond is a certain thing where you are lending to someone the amount of money. In this case, this is P. So, that is the money that you are investing, you are lending. And the person who has borrowed from you, he is going to pay a amount of money each year. So, these two sides must be equal, otherwise someone is making a loss. And if this infinite series is summed up, you are going to get just $\frac{a}{r}$, where a is the payment each year and r is the rate of interest. So, P, which is the price of the bond, $P = \frac{a}{r}$.

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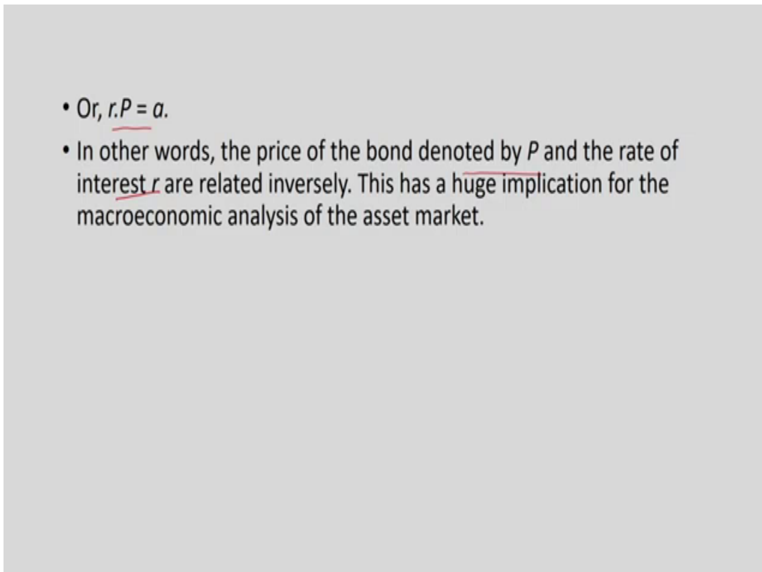

In other words, if I multiply both sides by r, I get $r.\ P = a$. In other words, the price of the bond, which is noted by P, and the rate of interest r are related inversely, because a is constant. So, if r changes, P must change in the opposite direction to keep the product at the same level.

So, that is why I am saying P and r are inversely related. And this relationship, this inverse relationship between P and r has a huge implication for macroeconomic analysis of the asset market. So, this is something which is dealt with in macroeconomics where people analyze how the money market functions, how the bond market functions, et cetera, et cetera.

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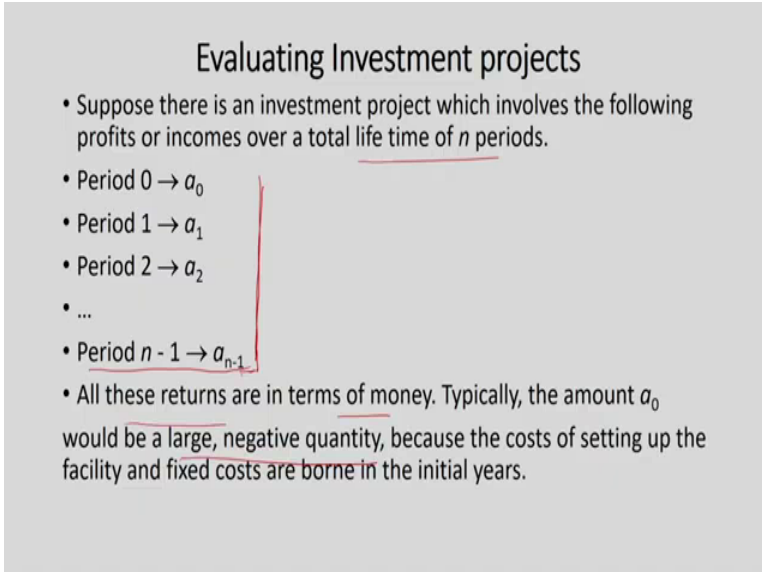

Now, we come to another application of this same topic which is series, evaluating future projects. Suppose there is an investment project which involves the following profits or incomes over a total lifetime of n periods. So, there is an investment project, suppose you are a prospective investor and if you invest in that project then over a period of n periods, these are the profits or the income stream.

So, in the first period that is period 0, you are going to get a_0, period 1, a_1, period 2, a_2, like that. So, there are a total number of n periods. So, the last period will be n-1, because we are starting from 0. So, in the n -1 th period, the payment is a_{n-1}. Now, all these returns are in terms of money. Typically, the amount a_0 which is the payment in period 0 will be a large negative quantity. Why, the reason is the costs are involved. So, this is the project in its totality.

So, there will be some payments that you will have to make as an investor. So, those are like outgoings, outflows and those will be registered as negative and there will be some earnings, which are positive like profits. So, those earnings might be in the future. But definitely in the initial period, there will not be any positive inflow. In the beginning there will be an outflow. So, therefore, a_0 is likely to be a large quantity, because fixed costs are involved and it is likely to be negative.

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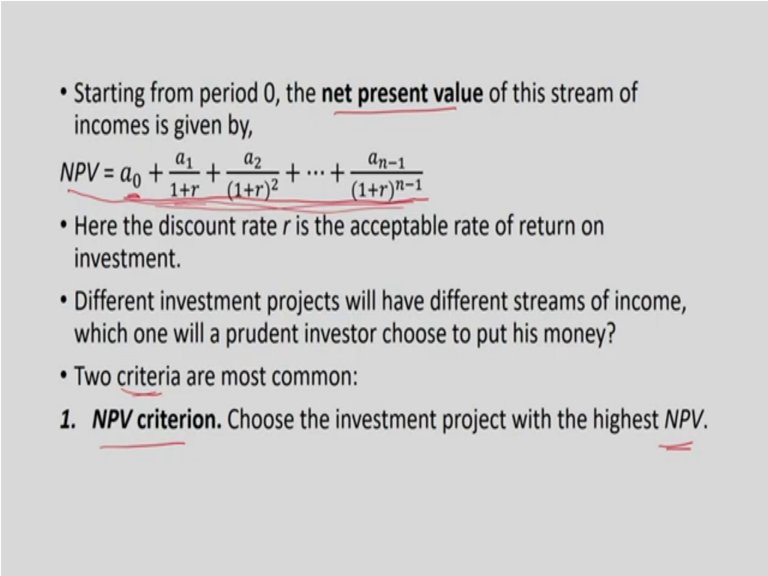

Now, starting from period 0, the net present value of this stream of income is given by NPV, this quantity, this expression, $NPV = a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_{n-1}}{(1+r)^{n-1}}$. And this is similar to what we have seen before present discounted value. You are standing in period 0. So, you are not discounting a_0 which is in this period itself, but from the next period onwards the incomes are discounted and r is the rate of interest. So, this is how it is going to look like. The last period's payment is a n-1. So, its present discounted value is $\frac{a_{n-1}}{(1+r)^{n-1}}$.

Here the discount rate r is the acceptable rate of return on investment. So, the investor when he is investing in this particular project, he will try to find out what is the acceptable rate of return on investment and suppose r is the acceptable rate of return, then r is going to be that rate at which the stream is going to be discounted. Now, different investment projects will have different streams of income. a_1, a_2, all these things could be different from one project to another.

Question is which one will a prudent investor choose to put his money? Now, there are two criteria which are generally used and they are most common. One is the NPV criteria, NPV criterion, net present value. And what does this criterion say? It says a very simple thing. Choose the investment project with the highest NPV. So, that is why it is called NPV criterion.

So, for any project, I look at this quantity NPV and I find out the NPVs of all the projects I am considering and I can obviously arrange them in a descending order and I will choose that project which has the highest NPV. Suppose I have to invest only on one project then I will choose that project which has the greatest NPV. So, this is called the NPV criterion.

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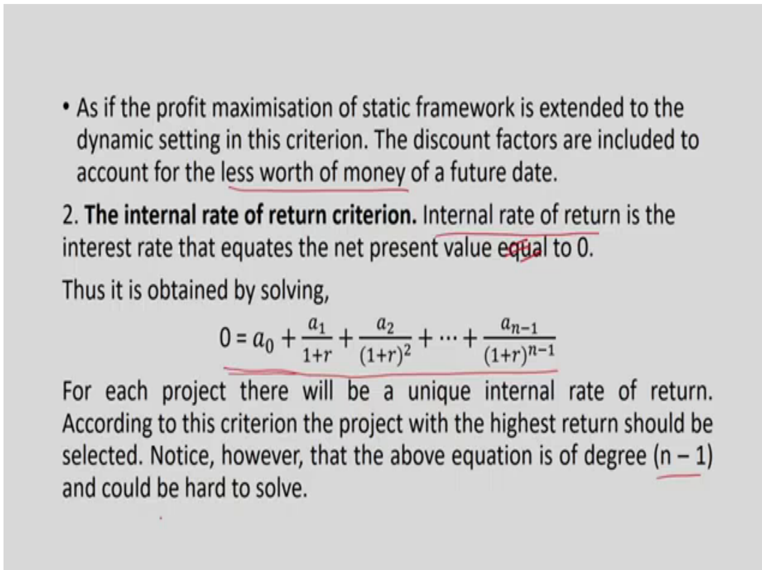

How do we interpret this criterion? As if the profit maximization of static framework is extended to the dynamic setting in this criterion. What is profit maximization? Remember, we talked about this earlier that a capitalist always likes to maximize the profit in a particular period. So, that is a static framework. There are no two or three periods involved. There is just a single period and he wants to maximize the profit that is it. But here since there are many periods involved, we are extending the same idea of maximizing profit over many periods.

The additional thing that is coming here is the discount factors. The discount factors are included to account for the less worth of money of a future date. So, this is something we have discussed before. So, you have to include the discount factor, if you are talking about the future. This was the first criterion. And the second criterion is the internal rate of return criterion.

What is the internal rate of return? The internal rate of return is the interest rate which equates the net present value to 0. So, it is obtained by solving this equation,

$0 = a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_{n-1}}{(1+r)^{n-1}}$. Remember, this is the expression for net present value on the right hand side. So, the NPV has to be equal to 0. And if we have this condition then there will be one rate of interest that is r, which is going to solve this equation, that rate of interest or rate of return is called the internal rate of return.

So, for each project there will be a unique internal rate of return and according to this criterion, the project with the highest return should be selected. So, if you have say two or three projects to evaluate and you want to pick up only one project, then for each project you solve this equation, because for each project this a_0, a_1, a_2, these things will be different.

And so the r which solves the equation of a particular project will be different from the r of another project. Notice, however, that the above equation is of degree n-1. The power of r is what? The highest power is n-1. So, this equation might be quite difficult to handle. It will be very difficult to solve.

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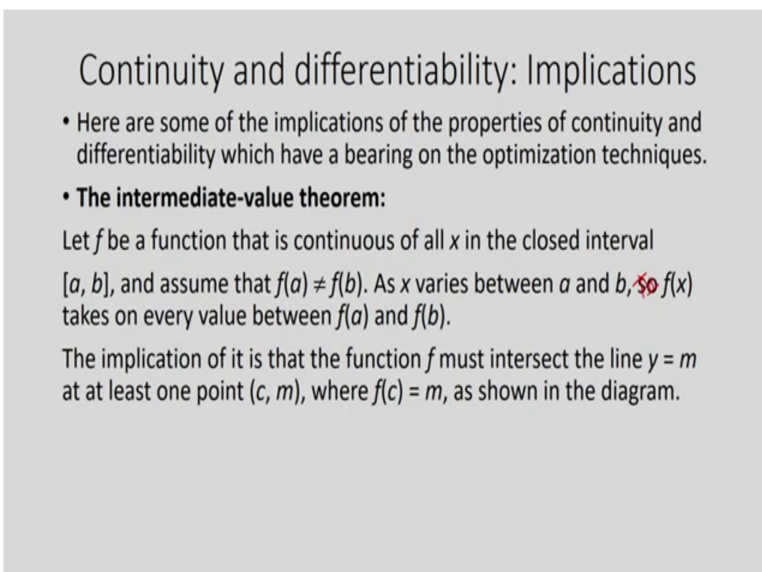

Now, we look at the implications of continuity and differentiability from a theoretical point of view. Here are some implications of the properties of continuity and differentiability which have a bearing on the optimization techniques. So, after this we are going to look at how to optimize a particular function, so how to maximize a function, for example, or minimize a function. Now, if we want to do that, then certain theoretical results are important.

And those results are derived from the properties of continuity and differentiability. So, here we are going to talk about those important results and theorems. Now, this is the first one, the intermediate value theorem. So, what does it say? Let f be a function that is continuous of all x in the $[a, b]$ and assume that $f(a) \neq f(b)$. As x varies between a and b, $f(x)$ takes on every value between $f(a)\ and\ f(b)$.

So, this is what is the, what is known as the intermediate value theorem. Remember, here it is important that f is a continuous function and interval is a closed interval and $f(a) \neq f(b)$. So, they have different values. The implication of this theorem is that, that the function must intersect the line $y = m$ at least one point (c, m), where $f(c) = m$as shown in the diagram.

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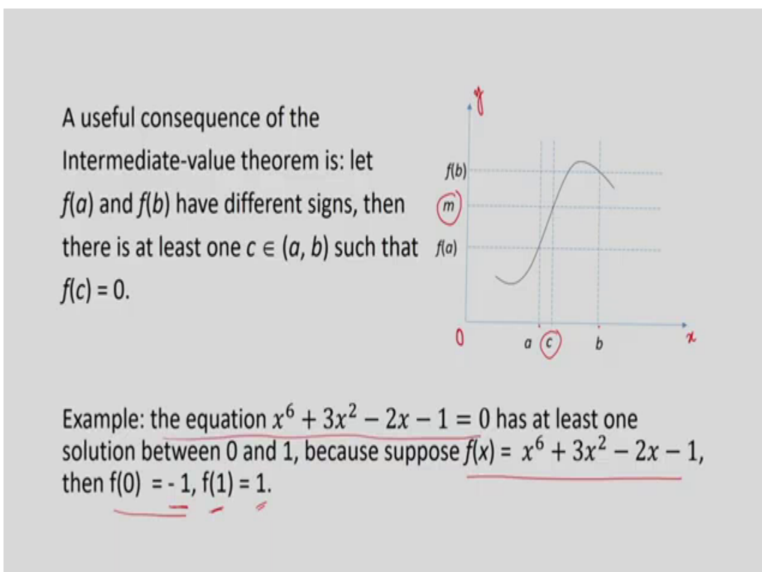

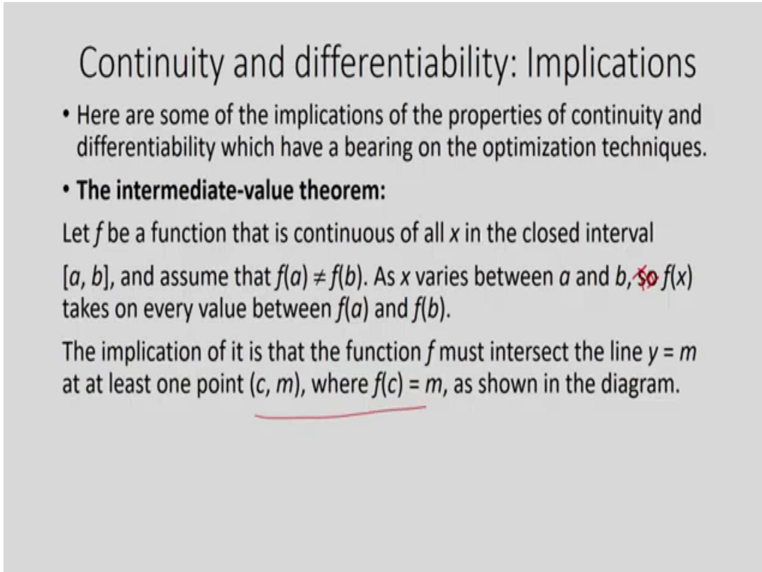

So, here is a particular function that I have just drawn. Here is a and here is b, and the function is continuous in this particular interval. Now, if I pick up any particular m which lies between $f(a)$ and $f(b)$, then what this result is telling me that there is at least one c. With respect to m, I will get at least one c such that $f(c) = m$. So, that is what this theory is guaranteeing.

So, obviously, here geometrically $y = m$ is a straight line. It is a horizontal line. Then this line will intersect this function at least once. In our case it is intersecting only once and that point is having the value of $x = c$. Now, a useful consequence of the intermediate value theorem is, let $f(a)$ and $f(b)$ have different signs, then there is at least one c between a and b such that $f(c) = 0$. So, here is an example of how it can be used.

Suppose you have this equation $x^6 + 3x^2 - 2x - 1 = 0$. Now, what is the proof that this equation has a solution? So, what is the guarantee that there is one x at which this equation is satisfied? Now, we can show that there is one, at least one x which satisfies this equation by looking at $f(0)$. $f(0)$ if you take $x = 0$, where $f(x)$ is given by this function $x^6 + 3x^2 - 2x - 1$ and put $x = 0$ then it turns out to be minus 1, $f(0) = -1$. And if you take $f(1)$ and then it becomes 1.

So, the value of the function from minus 1 it goes to plus 1. Therefore, there must be at least one solution between 0 and 1. So, because this comes from the intermediate value theorem itself, because 0 lies between minus 1 and plus 1.

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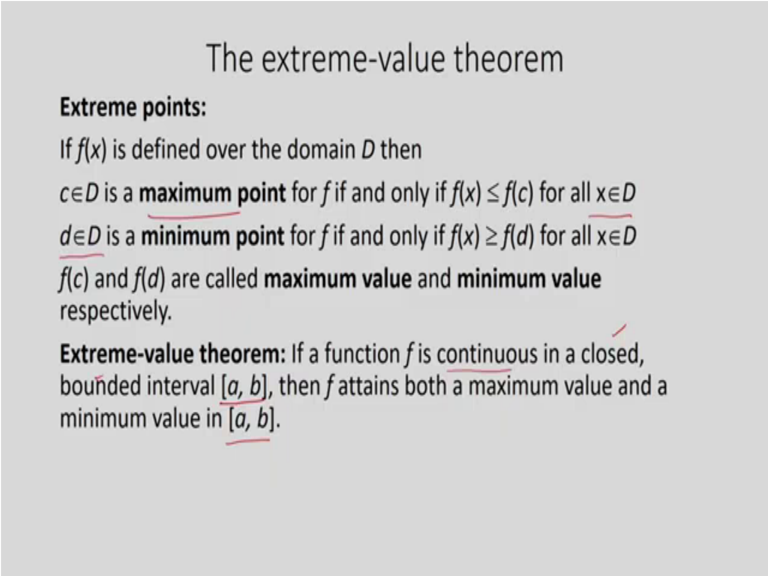

Now, we come to another theorem which is called the extreme value theorem. Now, for that I have to define what are known as extreme points. If $f(x)$ is defined over the domain capital D then $c \in D$ is a maximum point for f if and only if $f(x) \leq f(c)$ for all $x \in D$. That means, at c the value of the function is never less than the value of the function at other values belonging to the domain and then it is called a maximum point.

Similarly, $d \in D$ is a minimum point for f if and only if $f(x) \geq f(d)$ for all $x \in D$. $f(c)$ and $f(d)$ are called maximum value and minimum value, respectively, so, maximum point, maximum value, minimum point minimum value.

Now, we come to the extreme value theorem. If a function f is continuous in a closed bounded interval [a,b] then f attains both the maximum value and the minimum value in the same interval [a,b]. So, this is very important and look at the conditions, f has to be continuous. It is a continuous function and it is defined over a closed bounded interval [a,b] then we have both maximum and minimum values in the interval. So, this is called the extreme value theorem.

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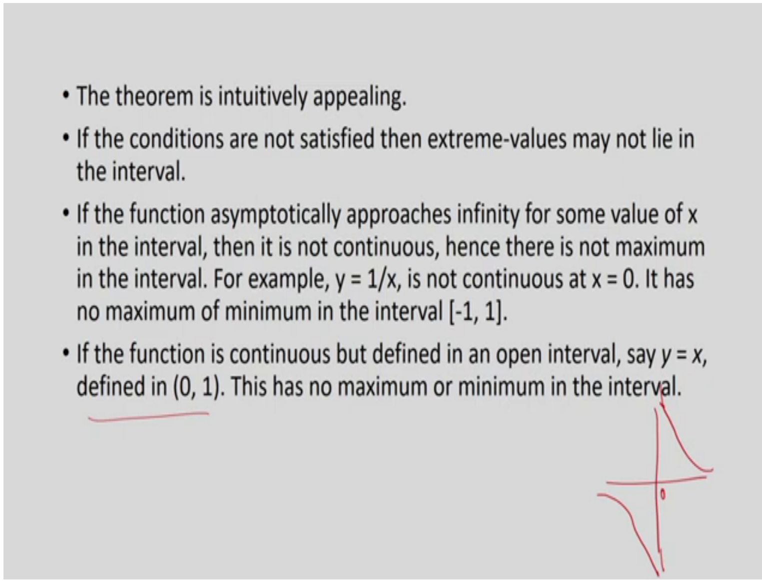

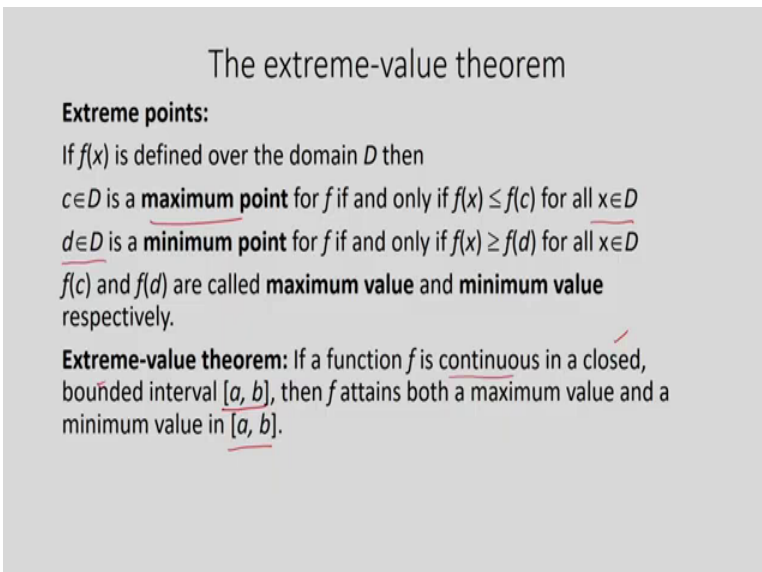

The theorem is intuitively appealing. If the conditions are not satisfied, then extreme values may not lie in the interval. So, what are the conditions, conditions are, it has to be continuous. The interval has to be closed and bounded. If the function asymptotically approaches infinity for some value of x in the interval then it is not continuous. Hence, there is no maximum in the interval. For example, you take $y = 1/x$, then this function is not continuous at x equal to 0.

We have seen that before that this function is not continuous. It has no maximum or minimum in the interval minus 1 to plus 1. So, this is how it looks like. The function will be something like this. So, it is discontinuous at 0, and therefore, I cannot say that it has any maximum or minimum. In fact, it does not have. In this particular case, it has no maximum or minimum, because these lines are going asymptotically to infinity and minus infinity.

If the function is continuous but defined in an open interval, say $y = x$ is defined over (0 , 1) then also the maximum or minimum is not guaranteed. This has no maximum or minimum in this particular interval, because as you approach $x = 1$, then the value of the function goes on rising. And since it is an open interval, there is no maximum here.

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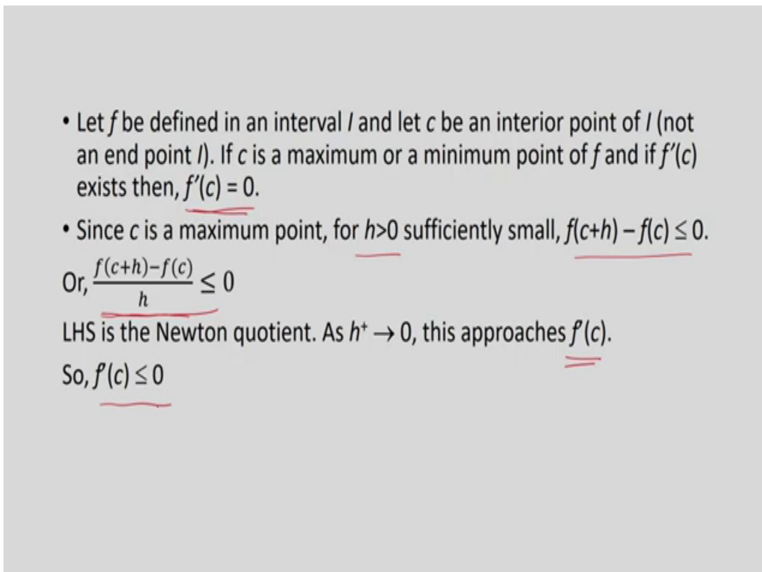

Let f be defined in an interval I and let c be an interior point of I . An interior point is, what it is and it is not an end point in that interval I. If c is a maximum or minimum point of f and if $f'(c)$ then, $f'(c) = 0$. So, this is a very important property. So, you have a function which is defined over an interval and suppose there is a point in that interval which is let us suppose c and it is an internal point and suppose there is a maximum or minimum at c then it must be the case that $f'(c) = 0$ if $f'(c)$ exists.

So, what is the demonstration of that? Suppose c is a maximum point. So, since c is the maximum point, then if we take $h > 0$ sufficiently small then this has to be satisfied, $f(c + h) - f(c) \leq 0$ because of the very fact that at $x = c$ the function has a maximum. And if I manipulate this, then it becomes like this, $\frac{f(c+h)-f(c)}{h} \leq 0$. This has to be less than or equal to 0. Now, the left hand side is the Newton quotient. As h plus goes to 0 that is we are approaching 0 from the positive side that is the right hand side then this left hand LHS here, the Newton quotient approaches $f^{'}(c)$. So, this is the definition of derivative. So, therefore, $f^{'}(c) \leq 0$.

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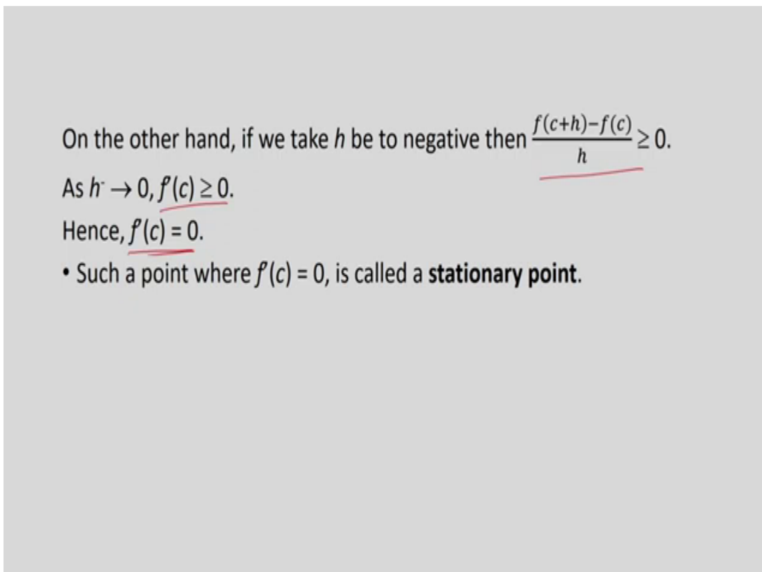

Similarly, I can take h to be negative in that case this Newton quotient, $\frac{f(c+h)-f(c)}{h} \geq 0$. And similarly, we see that $f^{'}(c) \geq 0$. And both of them are satisfied $f^{'}(c) \geq 0$ and $f^{'}(c) \leq 0$, which means that $f^{'}(c) = 0$. So, our proof is done. That if we have a maximum or minimum of a function at a particular point and if $f^{'}$, that is the derivative is defined, then $f^{'}(c) = 0$, that is the derivative is equal to 0, such a point is called a stationary point.

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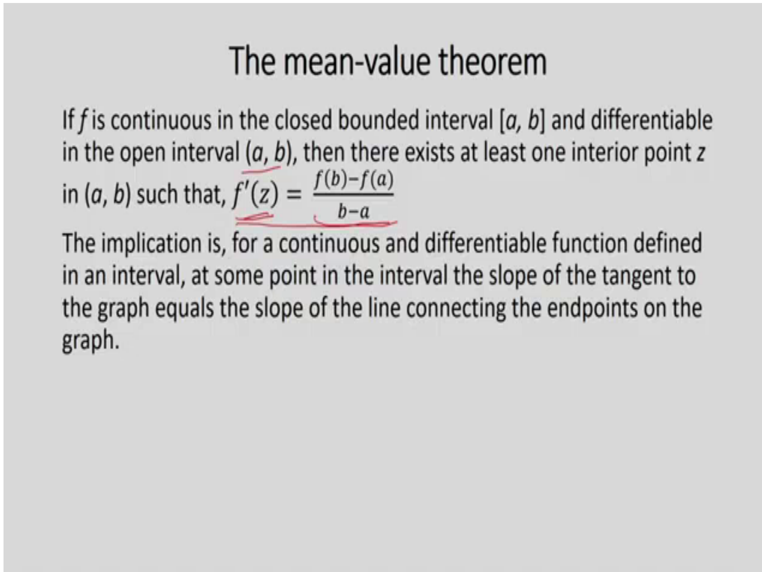

Now, we come to another theorem, which is called the mean value theorem. If f is continuous in the closed bounded interval [a, b] and differentiable in the open interval (a.,b) then there exists at least one interior point z in (a.,b) such that $f^{'}(z) = \frac{f(b)-f(a)}{b-a}$. What is the implication of this?

The application is for a continuous and differentiable function defined over an interval, at some point in the interval the slope of the tangent to the graph, so this is that point z, the slope of the tangent to the graph is $f^{'}(z)$ equals the slope of the line connecting the endpoints on the graph. So, this is the slope of the line connecting the endpoints on the graph.

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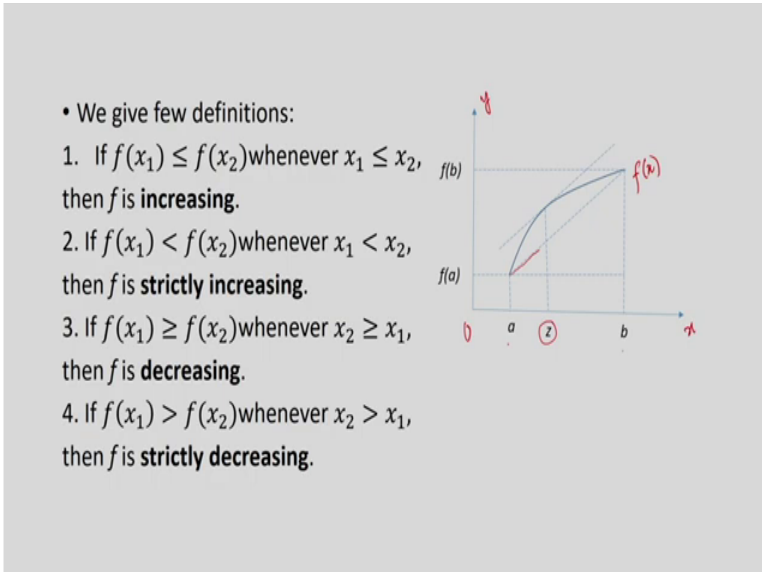

So, here is the geometric way to see this. So, suppose you have this function $f(x)$ and you have two points a and b which belong to the domain. And so the slope of this line ab is given by this. You can see it is, this is the slope of the line. Now, we can find at least one z which is in that interval, such that the derivative of the function at z is equal to the slope of that chord that is the chord connecting a and b or $f(a)$ and $f(b)$.

We give a few definitions now. Number one, if $f(x_1) \leq f(x_2)$, whenever $x_1 \leq x_2$ then f is an increasing function. So, x is rising in that case f is also rising in a weak sense then f is called an increasing function. And if this holds in a strict sense that $f(x_1) < f(x_2)$, whenever $x_1 < x_2$ then f is a strictly increasing function.

Similarly, if $f(x_1) \geq f(x_2)$ whenever $x_2 \geq x_1$ then f is a decreasing function. If $f(x_1) > f(x_2$ whenever $x_2 > x_1$ then f is a strictly decreasing function.

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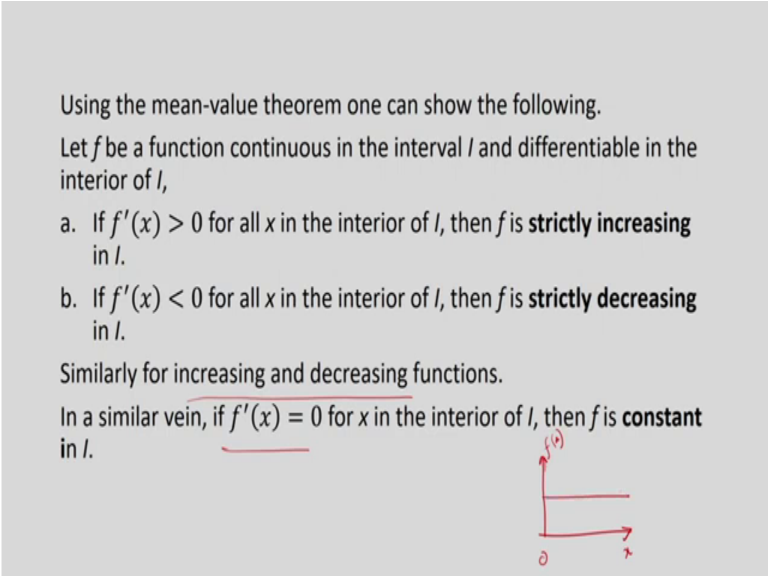

Using the mean value theorem, one can show the following. Let f be a function continuous in the interval I and differentiable in the interior of I, then we can show the following. Number one, $f'(x) > 0$ for all x in the interior of I means that f is strictly increasing in I. Similarly, if $f'(x) < 0$ for all x in the interior of I then f is strictly decreasing in I. These, both these properties are quite intuitive actually.

And these can be proved by invoking the mean value theorem. Similarly, for increasing and decreasing function, so this is true not only for strictly increasing and decreasing, they are also true for weakly increasing and decreasing function. In a similar vein, if $f'(x) = 0$ for x in the interior of I, then f is constant in I. So, this is like this. It is a horizontal line.

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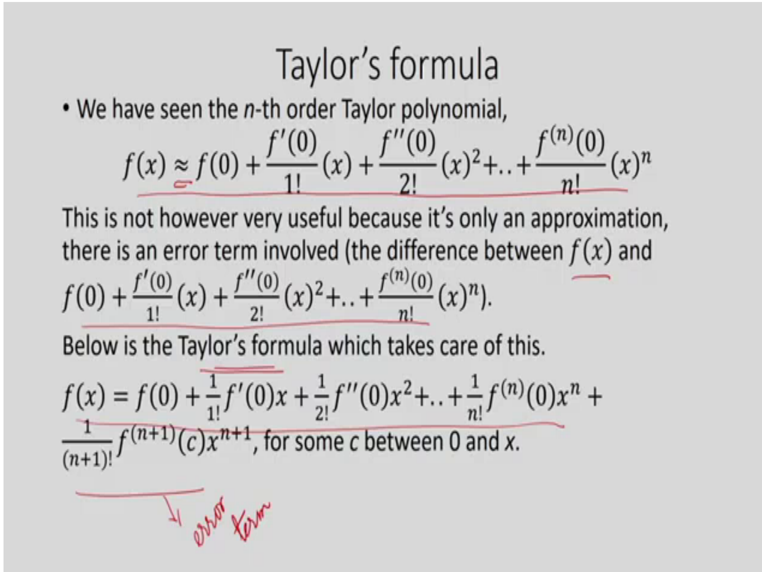

Now, there is something called Taylor's formula. We have seen before the n-th order Taylor polynomial had this form $f(x) \approx f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \dots\dots + \frac{f^{(n)}(0)}{n!}(x)^n$.

However, this expression is not very useful, because it is only an approximation, there is an error term involved which is the difference between $f(x)$ and the term on the right hand side, because it is not equal to, it is approximately equal to. So, there is a gap. And that gap is called the error.

The Taylor formula actually takes care of this error. So, here we write this up till this, but then there is an error term and this is the error term. The error term is given by $\frac{1}{(n+1)!} f^{(n+1)}(c)\, x^{n+1}$. And what is c. c is some value between 0 and x.

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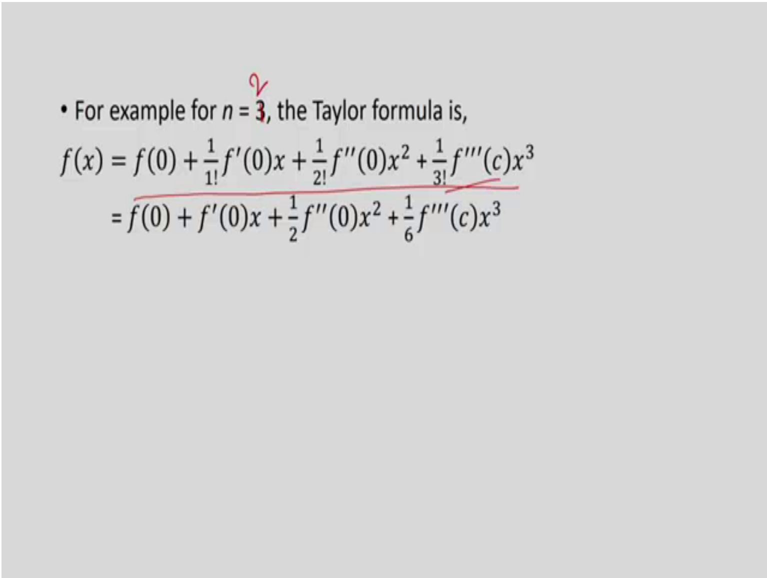

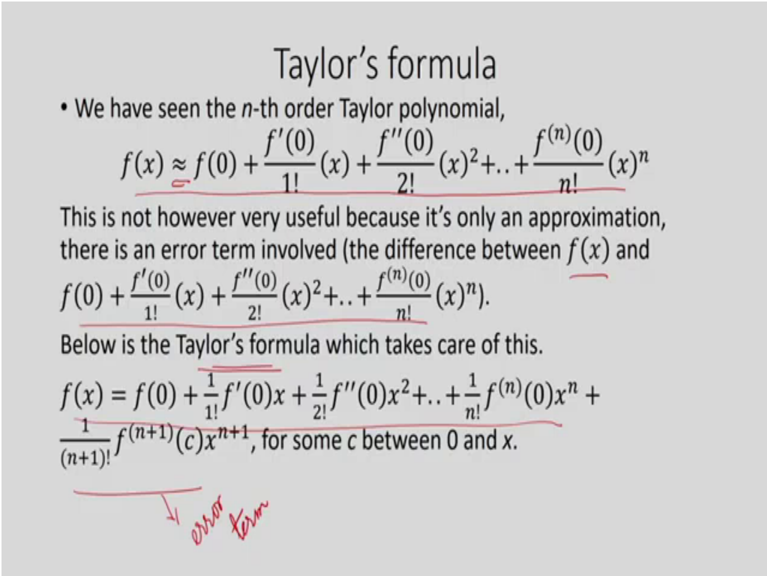

So, for example, if you take any general function $f(x)$ and if you take n = 3, so in that case the formula will be like this. This will be actually for n = 2. So, after the second term you have the error term. So, that is why it is n = 2.

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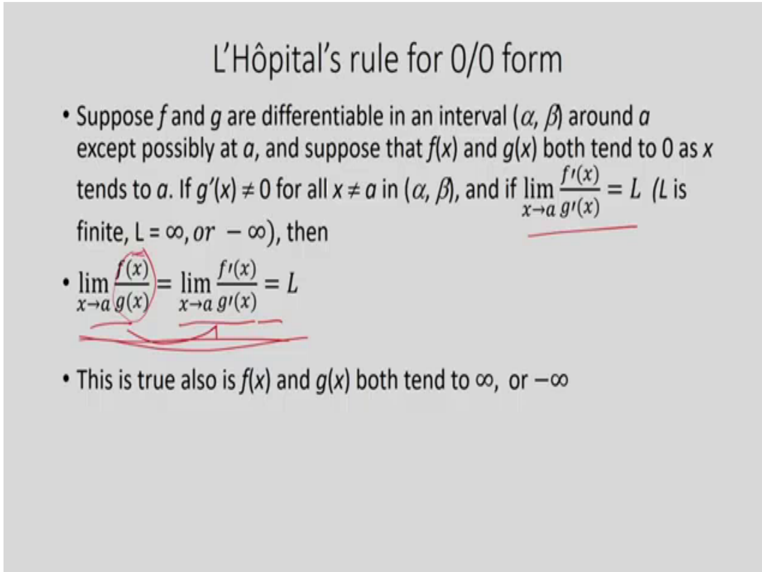

Now, these are different tools and instruments that we have which follow from the properties of limits and differentiation. So, this is called L'Hopital's rule for 0/0 form. Suppose f and g are differentiable in the interval (α, β)around a except possibly at a and suppose that f(x) and $g(x)$ both tend to 0 as x tends to a. If $g^{'}(x) \neq 0$ for all $x \neq a$in (α, β) and if, this is important, $\lim_{x \to a} \frac{f^{'}(x)}{g^{'}(x)} = L$ (L is finite, it could be infinity or minus infinity), then this expression $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{'}(x)}{g^{'}(x)} = L$ and this thing as we know is equal to L.

So, you have $\frac{f(x)}{g(x)}$ and if you take the limit x goes to a suppose this quotient becomes 0 divided by 0 and that is not possible to handle. So, but no worries, you can actually use the L'Hopital's rule which tells me that this expression is the same as this expression. So, basically you differentiate f and g, the numerator and denominator with respect to x and then take the limit. And this is true also if f(x) and $g(x)$ tend to infinity or minus infinity. It is not only 0 by 0, but also, let us say, infinity divided by infinity.

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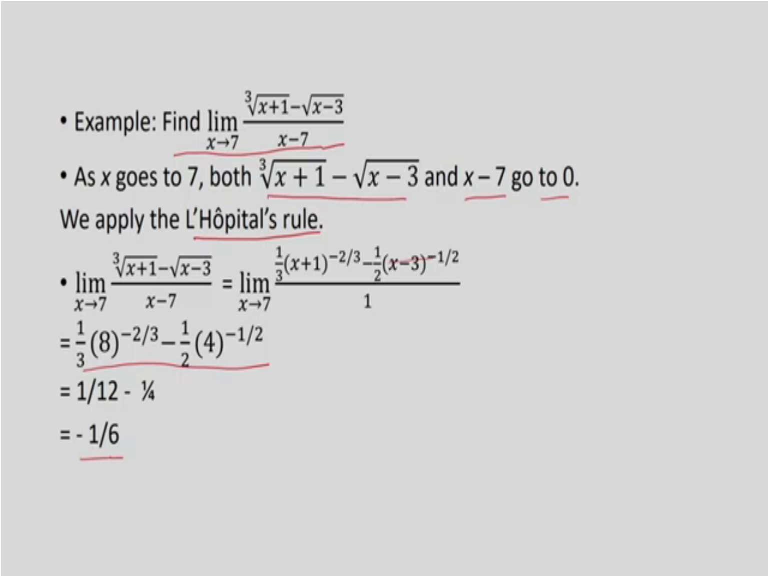

So, here are certain examples. Suppose, we have to find out $\lim_{x \to 7} \frac{3\sqrt{x+1}-\sqrt{x-3}}{x-7}$. Now, as you take the limit that is x going to 7, both the numerator and the denominator they go to 0. So, it becomes 0 by 0 form. So, what you do? You apply the L'Hopital's rules.

And if you do that, what you do is that basically take the derivative of the numerator, so this is the derivative of the numerator, and if you take the derivative of the denominator it is just 1 and then you take the limit that is x goes to 7. If you do that, it boils down to this formula, this expression. And then you simplify, it becomes simply -1/6.

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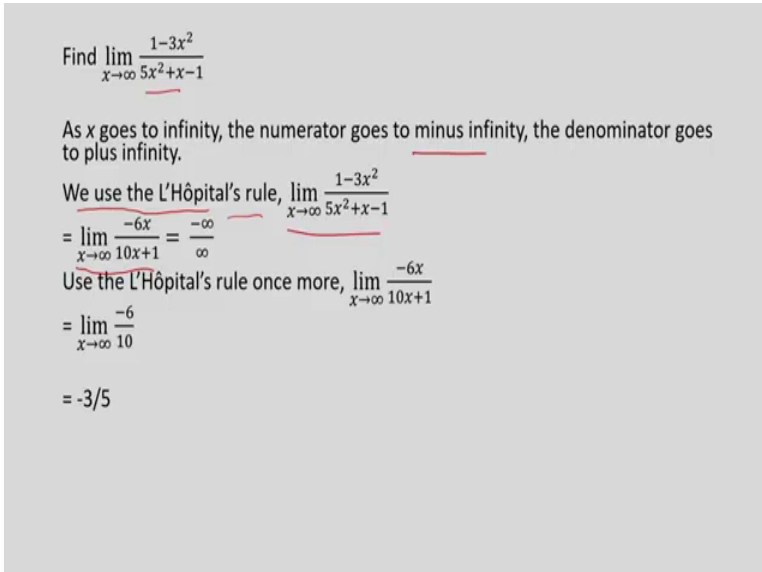

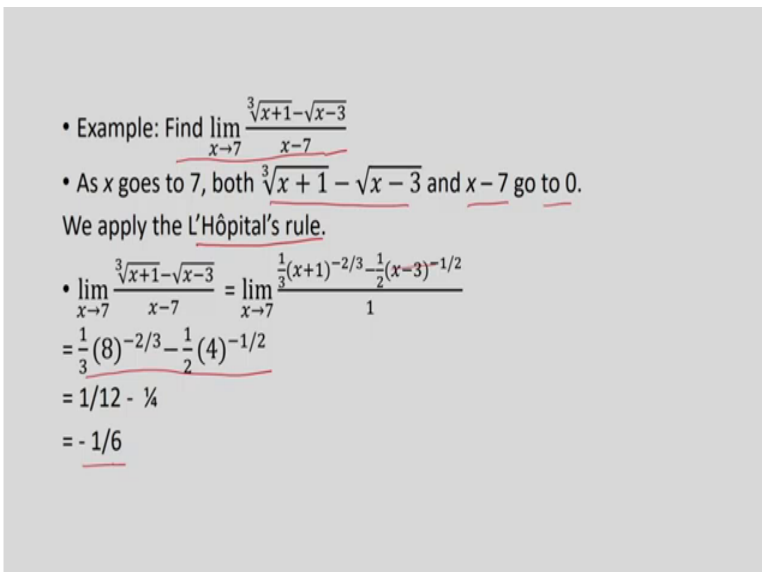

Here is another example. Here you instead of 0 by 0 form you have another kind of form coming up if you take x goes to a. So, you have $\lim_{x\to\infty} \frac{1-3x^2}{5x^2+x-1}$. Now, you can just look at it and you can verify that as x goes to infinity the numerator goes to minus infinity, because you have $1 - 3x^2$. So, as x goes to infinity it goes to minus infinity.

What about the denominator, the denominator goes to plus infinity, because you have $5x^2 + x - 1$. So, it is, as x goes to infinity, the first two terms will go to infinity. Therefore, you have form like minus infinity divided by infinity. So, how to deal with that? And, what we do is that we use L'Hopital's rule, which basically tells me to take the derivative of the numerator.

Now, if you do that, you get -6x and in the denominator you take the derivative of that, it becomes $10x + 1$. Now, you still cannot, you cannot get easy form here, because still if you take x goes to infinity the numerator will go to minus infinity and the generator will go to infinity. So, the same problem remains. Therefore, you apply the L'Hopital's rule once more.

And if you do that, so -6x if you take the derivative it becomes just -6, and $10x + 1$, take the derivative, it becomes 10. So, now it is easy to handle. It just becomes-3/5. So, this is how we use L'Hopital's rule to find the limit of functions which are a little bit difficult to handle and so L'Hopital's rule is very useful in that sense.

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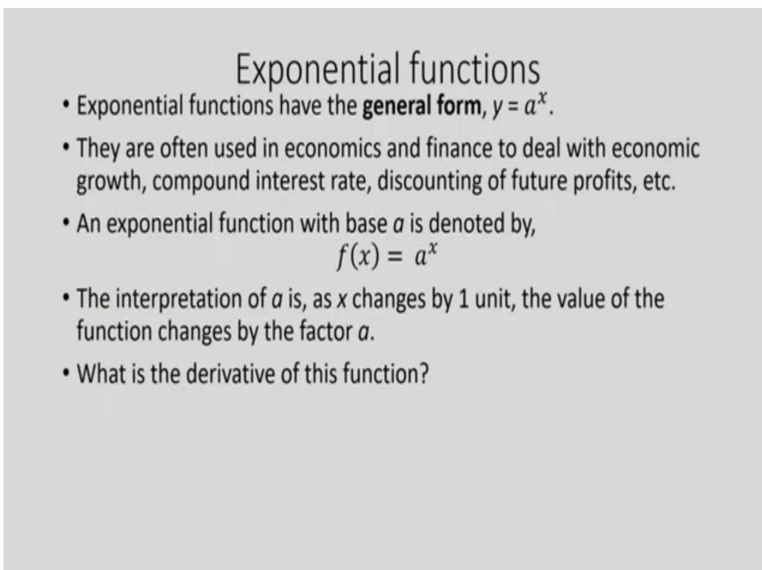

Now, we are going to start with the new topic which is exponential function and logarithmic function. But I guess that will be better if we start them in the next lecture, because it will take some time to cover these two topics in detail, and obviously, there are many applications of exponential functions and logarithmic functions in economics and other fields as well so that we shall take up in the next lecture. So, for the time being, I think I shall call it a day and I will see you in the next lecture. Thank you for attending.