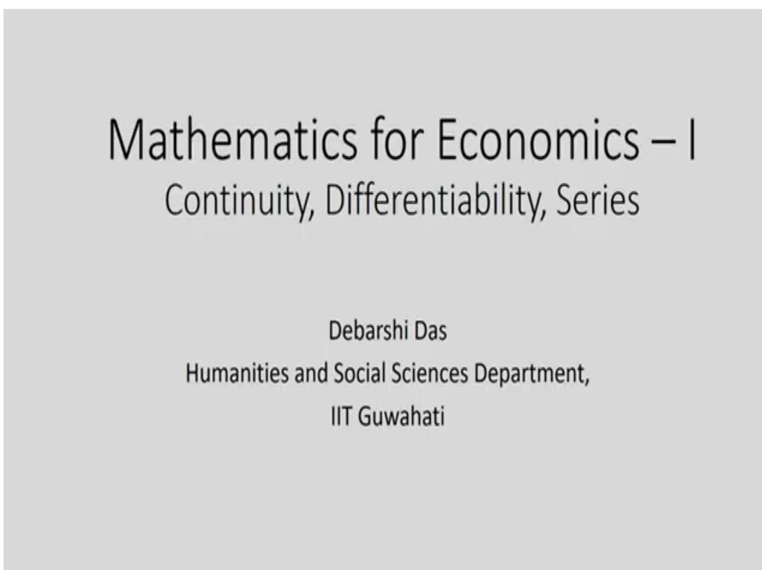
Mathematics for Economics - I
Professor Debarshi Das
Humanities & Social Sciences Department,
Indian Institute of Technology Guwahati
Lecture 14
Differentiability, series, PDV

Welcome to another lecture of this course called Mathematics for Economics Part I. So, so far we have been talking about differentiation and we have looked at different aspects of differentiation. Now, what we are going to do in the next three lectures is to look at the implications of differentiation and we are going to talk about certain special kinds of functions also, exponential functions and logarithmic functions, and we are going to present some rigorous definitions of the concepts that we have introduced before like limits, continuity etc, etc.

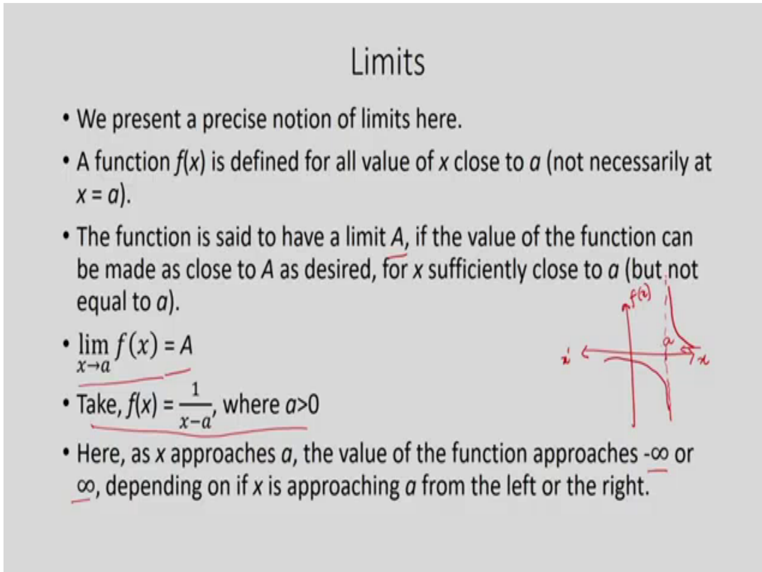
So, this is sort of kind of continuation of the previous theme which was differentiation, but at a greater depth.

(Refer Slide Time: 01:27)



So, as you can see on your screen, the topic that I have chosen for this three lectures is continuity, differentiability and series, but this is just a very short form of what is there in these three lectures. The lectures will contain much more than what is there in the title.

(Refer Slide Time: 02:12)



So, we start with what are limits. We have talked about limits before. Now, we are going to discuss these at a more rigorous level. So, we present a precise notion of limits here. A function $f(x)$ is defined for all values of x close to a, a is a particular value of x, but it is possible that the function is not defined at that particular value of x, not necessarily at $x = a$. This function is said to have a limit A, if the value of the function can be made as close to A as desired, for x sufficiently close to small a, but not equal to a.

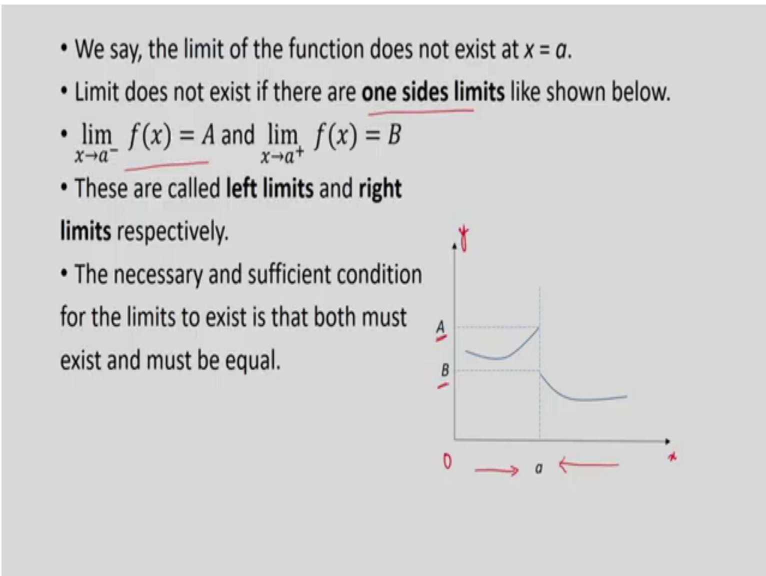
So, the value of the function can be made as close to the limit as possible. What is the limit? Limit is A. If we take the independent variables x to very close to this particular value a, if that is satisfied, this criterion is satisfied, then we can say that this function $f(x)$ has a limit and that limit is A at the value $x = a$. So, this is a short way to write the same thing, $\lim_{x \to a} f(x) = A$.

So, here is an example. So, here is an example where $f(x) = \frac{1}{x-a}$, where $a > 0$. So, what happens here? We will see that here there is no limit at $x = a$. Why, the reason is as x approaches a, the value of the function approaches minus infinity or plus infinity, depending on if x is approaching a from the left or the right. So, how will this look like?

Suppose, you have this four quadrants. So, suppose you are approaching a and x is approaching a and where is a, a is suppose here. Suppose you are approaching a from x greater than a. So, in general, $x - a$ will be positive. So, this $\frac{1}{x-a}$ will be positive. But if x becomes very close to a, then$x - a$ becomes very small and so 1 divided by a very small number is going to approach infinity. So, you are going to have a kind of function like this. So, it is going up and up as you are approaching a from the right.

On the other hand, if you are approaching a from the left, then this value $x - a$ is as such negative, but as you are getting very close to a from the left hand side, then it becomes very close to 0, the denominator becomes close to 0. So, the fraction becomes, it approaches minus infinity. So, you have a shape like this. So, therefore, at $x = a$, you cannot say that there is a limit, because there is no fixed or finite number A to which you approach as x approaches a. So, that is the reason why for this particular function there is no limit of this function at x equal to a.

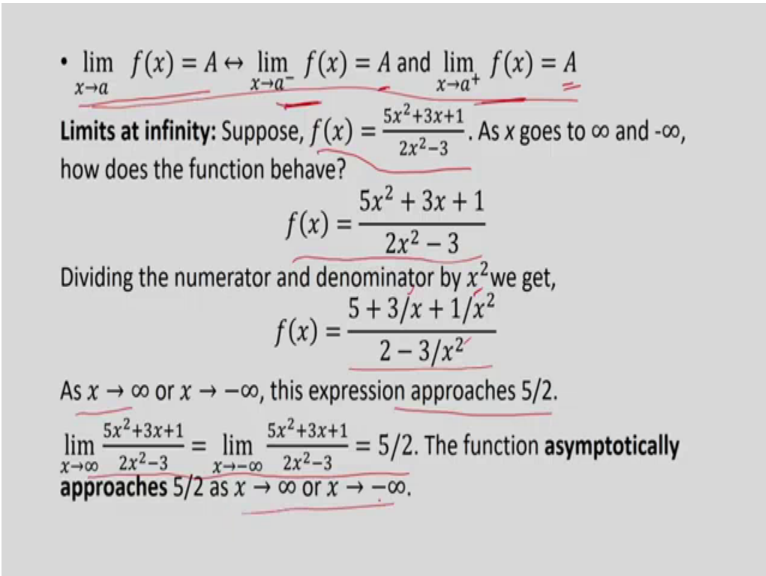
(Refer Slide Time: 06:22)



We say the limit of the function does not exist at $x = a$. Limit does not exist if there are one sided limits like shown below. This is the other case where limit does not exist. So, here I have just drawn the diagram. I have not given you the form of the function. So, suppose the function is $f(x)$ is such that if you are approaching a from the left hand side then the value of the function becomes very close to A. So, there is a limit. As x goes to a minus the $f(x)$ approaches A. This is A.

However, if you approach a from the right, the value of the function does not approach A. It approaches something which is different, it is B. So, these two values A and B are called left limit and right limit of this function at $x = a$. So, here you do not have a limit, because these two values are different. The necessary and sufficient condition for the limits to exist is that both must exist and must be equal, both means left hand and the right hand limits must be there, they should exist and they should be equal. Here they are not equal.

(Refer Slide Time: 08:05)



So, this is what is written here in mathematical terms. The left hand limit, this is the left hand limit, and the right hand limit they should exist and they should be equal. They are A. So, that means, they are equal. And in that case only we can say that the function has a limit at $x = a$ and that is given by A. What happens as x goes to infinity, does the function have a limit? Well, here is an example. Suppose $f(x)$ is given by this. It is a fraction $f(x) = \frac{5x^2+3x+1}{2x^2-3}$.

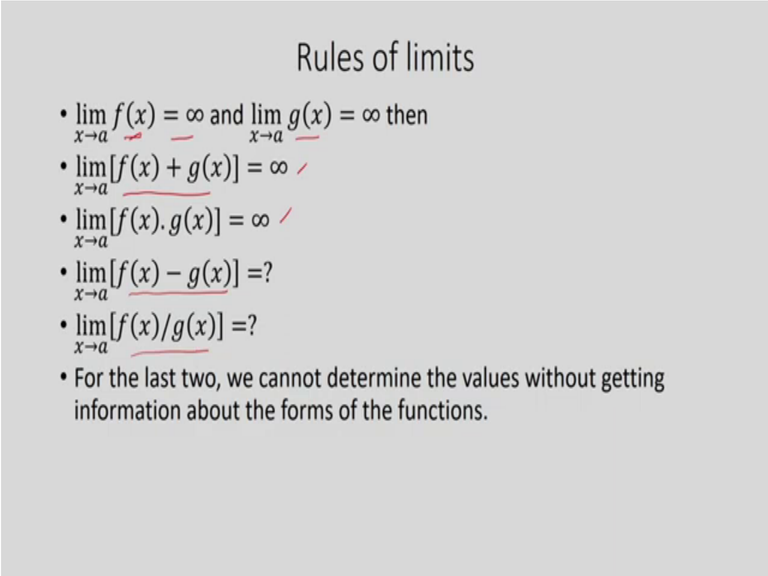
As x goes to plus infinity and minus infinity, how does the function behave? Let us try to see that. So, $f(x) = \frac{5x^2+3x+1}{2x^2-3}$. Next, what we do? We divide the numerator and denominator by x^2.

What is the speciality of x^2? x^2 is the highest power of x, highest power of x is 2. So, x^2 is that term which contains the highest power of x. And we divide both numerator and denominator by x^2 and I get this expression $f(x) = \frac{5+3/x+1/x^2}{2-3/x^2}$.

What happens now is that suppose $x \to \infty$, then this term $f(x)$becomes very close to $5/2$. And the same thing happens if $x \to -\infty$, in that case also $f(x)$approaches $5/2$. And you can see that why it is what I said it is because $3/x$ goes to 0, $1/x^2$ goes to 0, $3/x^2$ goes to 0 as $x \to \infty$ or as $x \to -\infty$. So, this term, this term, this term all these drop out so you are left with $5/2$.

So, this is how we write it that $\lim_{x \to \infty} f(x)$or $\lim_{x \to -\infty} f(x) = 5/2$ and we say that the function asymptotically approaches $5/2$ as $x \to \infty$ or $x \to -\infty$.

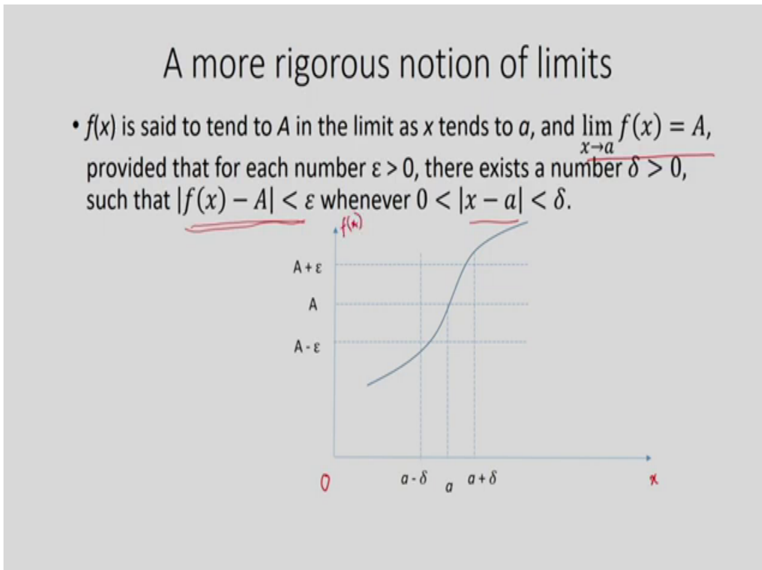
(Refer Slide Time: 10:47)



Rules of limits, so there are certain rules that limit satisfy. Suppose it so happens that you have a function $f(x)$ and the $\lim_{x \to a} f(x) = \infty$ and you have another function $g(x)$ and $\lim_{x \to a} g(x) = \infty$, then we can say certain things about $f(x)$ and $g(x)$. Summation of $f(x)$ and $g(x)$ if you take the limit of that $x \to a$ that also goes to ∞. Similarly, the product of $f(x)$and $g(x)$ if you take $x \to a$ the product also goes to ∞.

What about the difference, well, for difference and quotient that is $f(x) - g(x)$ or $f(x)/g(x)$ we cannot say anything a priori as x goes to a, what happens to these things unless we have some information about their forms. So, if you have some information of the form of $f(x)$ and $g(x)$ then maybe you can say something about $f(x) - g(x)$ or $f(x)/g(x)$, but without knowing the form we cannot say anything a priori.

(Refer Slide Time: 12:08)

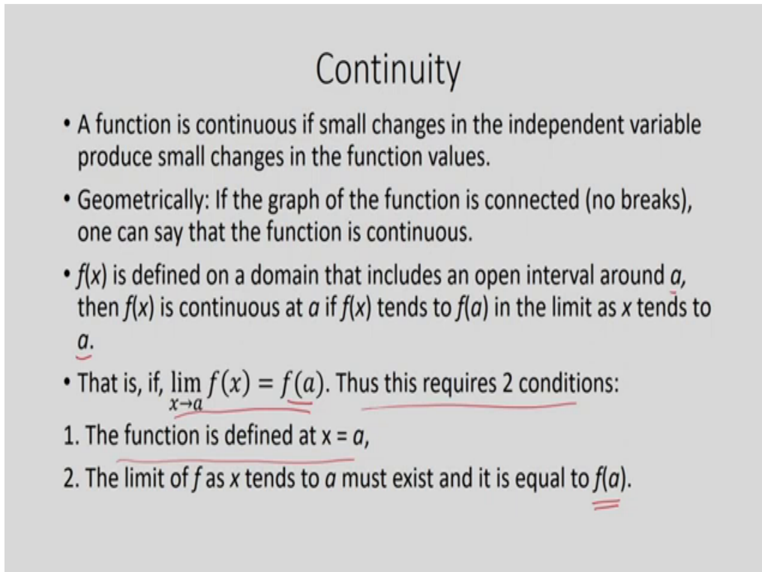


Here is a more, even more rigorous notion of limits. $f(x)$ is said to tend to A in the limit as x tends to a and we say $\lim_{x \to a} f(x) = A$ provided that for each epsilon which is greater than 0, $\epsilon > 0$, there exists a number delta which is also greater than 0, $\delta > 0$,such that $|f(x) - A| < \epsilon$ whenever $0 < |x - a| < \delta$. So, it is a little bit complicated statement.

So, what we are saying is that you can make the value of the function as close to A as possible and that is what is meant by this $|f(x) - A| < \epsilon$. So, you can take any arbitrary very small epsilon and still the difference between A and the value of the function will be less than that.

And when does this $|f(x) - A| < \epsilon$, when you take x to be very close to a, whenever $|x - a| < \delta$. So, give me any epsilon very small and you want to make the value of the function very close to A less than epsilon and I will be able to give you a delta, delta means you are very close to the x, sorry, the value of x is very close to a that means $|x - a| < \delta$ and correspondingly the value of the function will be very close to A. So, these are the axes.

(Refer Slide Time: 14:24)



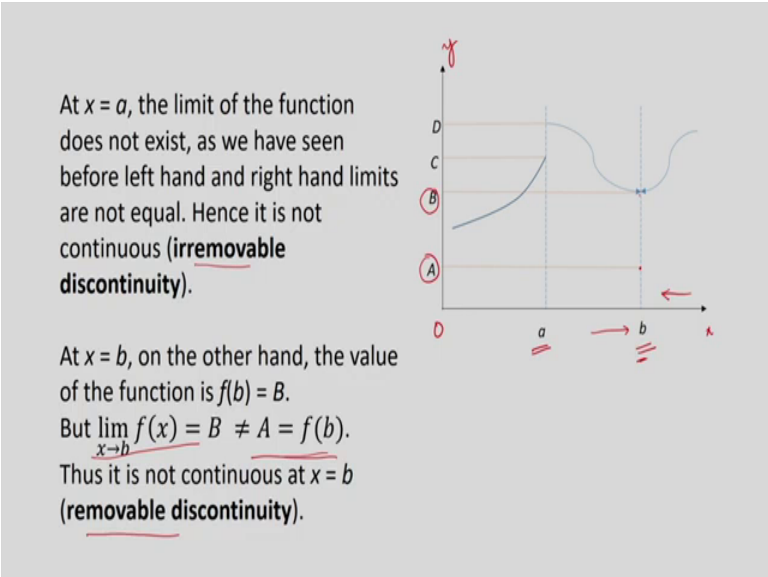
Now, we come to something known as continuity. Now, this idea of continuity we use often in our general language in a commonsensical manner, but in mathematics how we understand continuity. A function is continuous if small changes in the independent variable produce small changes in the function values. Geometrically, if the graph of the function is connected, that is there are no breaks, one can say that the function is continuous.

So, what is meant by there are no breaks is that, if you are drawing the graph of the function, then you do not have to lift your pen from the paper. You can draw the function or the graph of the function at the same stroke of your pen. In that case, we say that the function is continuous. $f(x)$ is defined on a domain that includes an open interval around a, then $f(x)$ is continuous at a if $f(x)$ tends to $f(a)$ in the limit as x tends to a.

So, if you have $\lim_{x \to a} f(x) = f(a)$ then we can say that the function is continuous at $x = a$. So, basically we need the function to have a limit. And secondly, the value of the function at that particular value should be equal to $f(a)$. Thus it requires two conditions to be satisfied that the function is defined at $x = a$. Remember, this was not required when we define limits. The function was not necessarily defined at $x = a$. But for continuity we need that. The function has to be defined at $x = a$. And secondly, the limit of f as x tends to small a must exist and it is equal to $f(a)$.

So, continuity is basically more stronger requirement than the existence of limit at $x = a$. For the function to be continuous add a particular value we need stronger conditions to be satisfied than only the function having a limit at $x = a$.

(Refer Slide Time: 17:18)

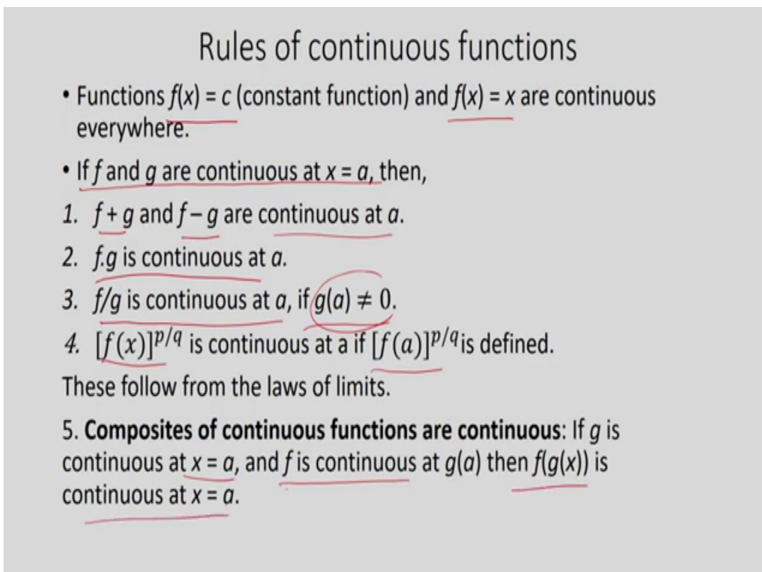


At $x = a$, the limit of the function does not exist as we have seen from the left hand side and the right hand side the limits are not equal. So, in this particular example, you can see straight away from the diagram, I have taken two values of x, one is a and b. Suppose, we are talking about a. Does the limit exist at a, does not, because the left hand limit and the right hand limit they are not equal. Hence, it is not continuous. If the function does not have a limit, it is not continuous. And this kind of discontinuity is called irremovable discontinuity. This discontinuity cannot be removed.

And secondly, there is another kind of discontinuity which is called removable discontinuity. And the example is given here. You have $x = b$. At $x = b$, the function is discontinuous and it is irremovable discontinuous. What is meant by that is that at $x = b$, the value of the function is $f(b) = A$. This is the value of the function. This value.

On the other hand, if you look at the graph, what is the limit of the function at $x = b$. At $x = b$, the limit of the function is actually B, this, $\lim_{x \to b} f(x) = B$. You can see that. As you approach b the value of the function approaches this value, both from the left and the right. So, the limit is at B. Whereas, the value of the function defined at b is A and A and B are not equal and this kind of discontinuity is called removable discontinuity.

(Refer Slide Time: 19:32)

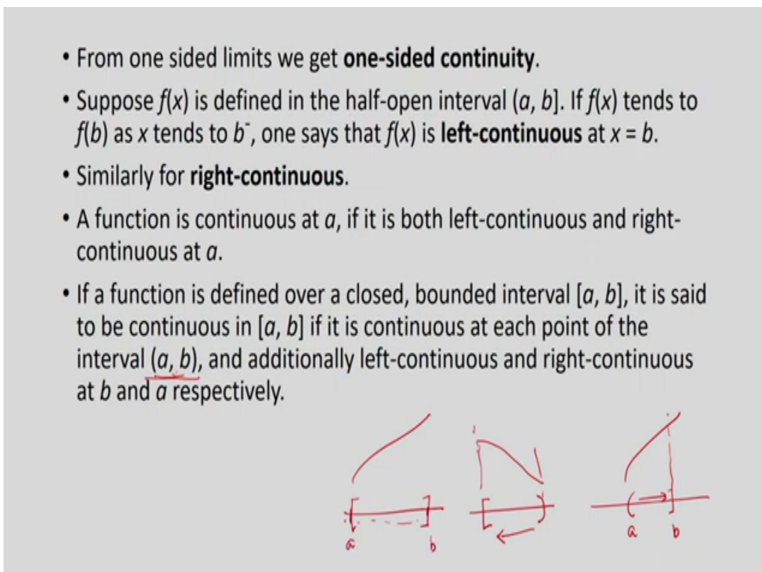


The rules of continuous function, there are certain rules. Functions of this form $f(x) = c$, which is a constant function or functions of this form $f(x) = x$, so this is the 45 degree line, these are continuous everywhere. So, at every point in its domain the functions are continuous. And there are certain other general rules. Suppose the f and g are continuous at $x = a$, then $f + g$ and $f - g$ are also continuous at a. That is the summation and the difference of two continuous functions is a continuous function, continuity defined at a particular point.

Similarly, $f.g$ that is the product of two functions which are continuous at a particular point is also continuous. Similarly, the ratio f/g is also continuous. Also, we have to mention that it should be continuous if you do not have $g(a) = 0$. So, this has to be satisfied. The denominator cannot be equal to 0. And if you take the power, so you take $[f(x)]^{p/q}$ then this will also be continuous at a if f is continuous at a, and obviously, we need that $[f(a)]^{p/q}$ be defined. If it is not defined, we cannot talk about continuity. And these properties actually follow from the laws of limits.

And finally, you have composite functions. Composites of continuous functions are also continuous. So, suppose you have if g is continuous at $x = a$ and f is continuous at $g(a)$, that is $f(g(x))$. g is continuous at $x = a$ and f is continuous at g where $x = a$, then we say $f(g(x))$ is continuous at $x = a$. So, composite functions of two continuous functions is also continuous.

(Refer Slide Time: 22:12)



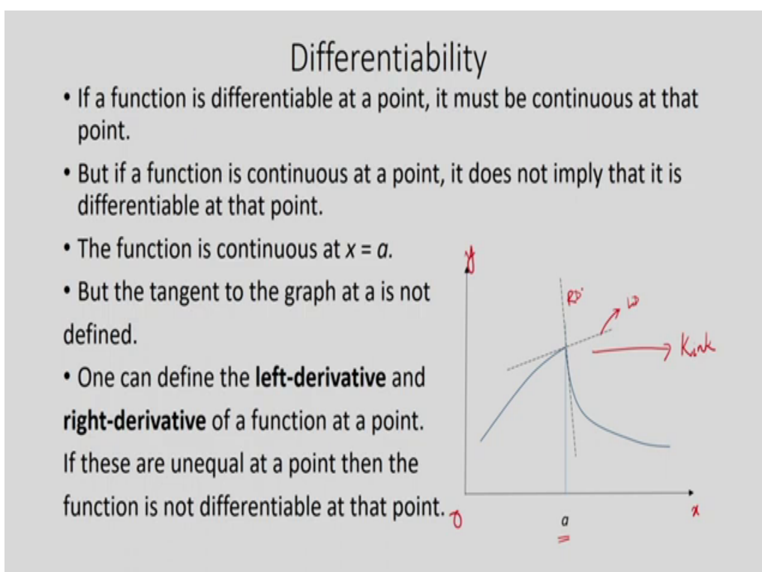
From one-sided limits, we get one-sided continuity. Remember we talked about one sided limits, now, we are talking about one-sided continuity. Suppose $f(x)$ is defined in the half open interval (a, b] that is, a is not included, but b is included in this interval. So, $f(x)$ is defined over that half interval, half open interval. If $f(x)$ tends to $f(b)$ as x tends to b^-, b^- because x is coming from the left hand side so that is why b^-, one says that $f(x)$ is left-continuous at $x = b$.

So, think about the geometry you have a and you have b and you have a function like this, so you are approaching this value x is approaching b and remember this is a closed interval at b and then we say that if the value of the function approaches $f(b)$, we can say that the function is left-continuous at $x = b$.

Similarly, one can talk about right-continuity. So, here the diagram will just be the opposite of this. So, here you are coming from the right. And suppose the function is like this. So, this will be the case of right-continuity. A function is continuous at a, if it is both left-continuous and right-continuous at a. It cannot be just continuous at one side. And if it is continuous only at one side then we cannot say that the function is continuous at all.

If a function is defined over a closed bounded interval [a, b], it is said to be continuous in [a, b] if it is continuous at each point of the open interval (a, b). a and b are not included here. This is an open interval. And additionally left-continuous and right-continuous at b and a, respectively. So, here we are talking about continuity in an interval. So, there is a function like this. So, this function is said to be continuous in this bounded interval [a, b] if it is continuous at each point here. Additionally, each point I mean in the open interval. Additionally it has to be left-continuous, because we are talking about b and right-continuous at a.

(Refer Slide Time: 25:14)



And then we come to what is known as differentiability. So, you see there are three things that we are talking about one after another. First there was the idea of limits, then we talked about continuity and we saw that continuity has stricter conditions, set of conditions and differentiability as we shall see it has even more stricter conditions. If a function is differentiable at a point it must be continuous at that point.

So, differentiability implies continuity. But if a function is continuous at a point it does not imply that it is differentiable at that point. So, continuity does not imply differentiability. So, basically, continuity is a necessary condition for differentiability and differentiability is a sufficient condition for continuity.

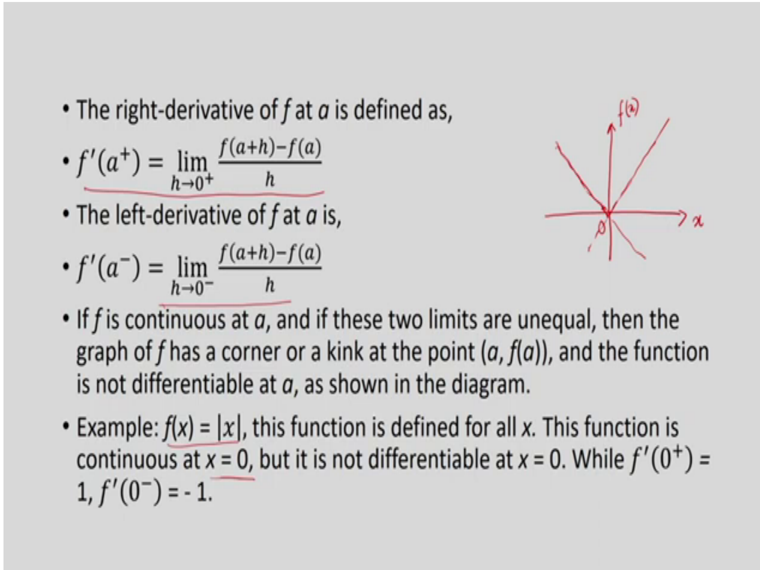
So, let us take this example: you have y on the vertical axis, x on the horizontal axis, and you have this function which is in blue color and you can see that at $x = a$ I have met the function having a sort of point which is an angular point. This point is called a kink point. It is not smooth at this particular point. At $x = a$, the function is not smooth. And this kind of point is called a kink point.

However, we can verify that this function is continuous at $x = a$. You do not have to lift your pen from the paper to draw this graph at, if you want to draw the graph you do not have to lift your pen at $x = a$. But what happens is that tangent to the graph at a is not defined. Since the function is having this angular shape at this a, you cannot draw a particular tangent here. It is not properly defined. You can in fact draw many lines which go through this point, multiple lines are there, therefore, the tangent is not defined.

One can define the left-derivative and the right-derivative of a function at this point, at point a. If these are unequal at a point then the function is not differentiable at that point. So, this is a general idea that at any point on the graph of the function you can define what is a left-derivative and what is the right-derivative. And if this left and right derivatives are not equal, then we say that the function is not differentiable. But what is the definition of left-derivative and right-derivative?

Well, you can see the diagrammatically how I have done that. Left-derivative will be something like the slope of this line, left derivative and the right-derivative will be the slope of this sort of more vertical line which is a negatively sloped. But here the left-derivative is positive. The right-derivative, in this case, will be negative.

(Refer Slide Time: 29:00)



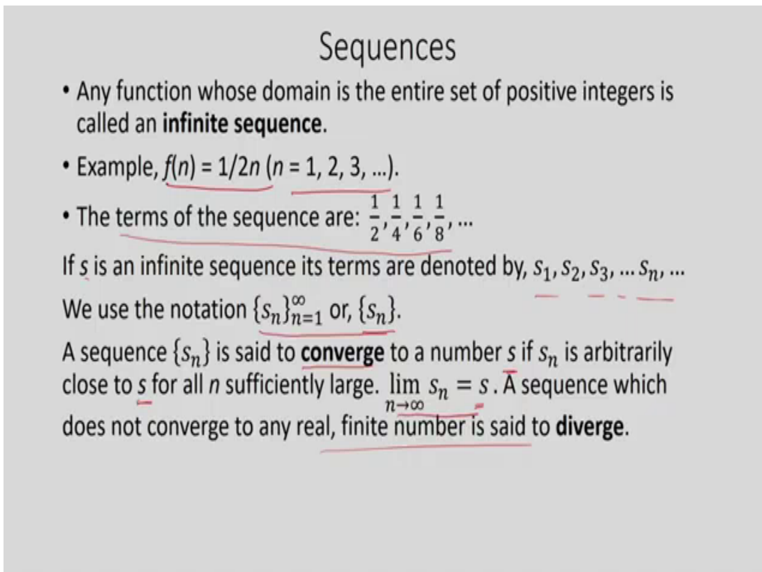
Here they are defined. The right-derivative of f at any point a is defined like this $f^{'}(a^{+}) = \lim_{h \to 0^{+}} \frac{f(a+h)-f(a)}{h}$. That is you are coming from the right hand side. This will be like this coming from the right hand side. Then what is the value of the derivative. On the other hand, the left-derivative of the function at a is this, $f^{'}(a^{-}) = \lim_{h \to 0^{-}} \frac{f(a+h)-f(a)}{h}$. Here h is going to 0 but from a negative value. So, you have this value.

If f is continuous at a and if these two limits are unequal, then the graph of f has a corner or a kink at this point $(a, f(a))$ and the function is not differentiable at a, as shown in the diagram. I have explained that. Here is another example where the form of the function is also given. Example $f(x) = |x|$, this function is defined for all x. This function is continuous at $x = 0$, but it is not differentiable at $x = 0$.

Let us look at the graph. So, here is x and here is suppose $f(x)$. So, $|x|$, so if $x = 0$ or $x > 0$, it is $f(x) = x$. So, it is the 45 degree line. And if you take x to be negative, then it is going to be negative of x. So, you get positive x, $f(x) = x$. Minus of x if x is negative, then you have $f(x)$ like this. So, the basically the functions graph is always in this first and the second quadrant.

And as you can see, here, the left-derivative is the slope of this line, which is this line. And what is the slope of this line, it is minus 1. Whereas, if you take this 45 degree line, obviously, we know that the slope is plus 1. So, here the left-derivative and the right-derivatives are not equal. Therefore, the function is not differentiable. And geometrically also that is true that at this point there is a kink. There is a sort of 90 degree angle in the graph.

(Refer Slide Time: 31:48)



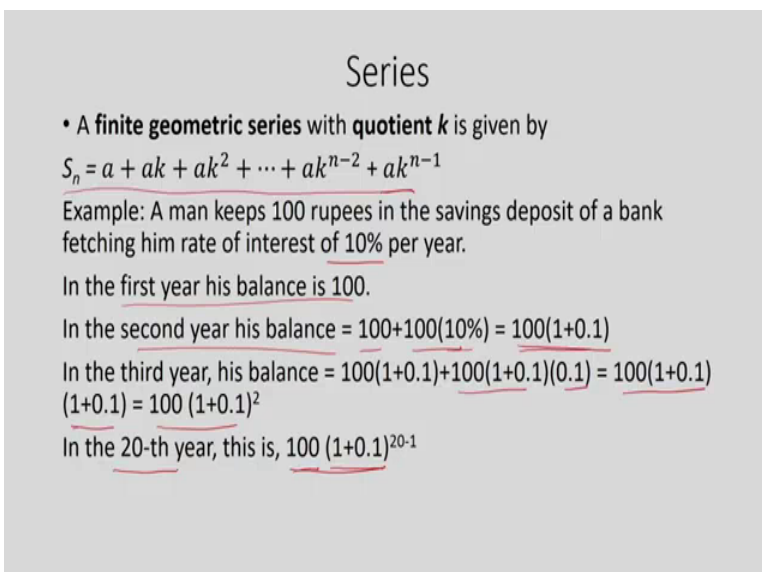
Now, we come to a sort of different topic within this larger theme that we are covering and these are called sequences. These are also related to functions. Any function whose domain is the entire set of positive integers is called an infinite sequence. So, domain is the set of positive integers that means 1, 2, 3 etc, etc, it goes to infinity and you look at the value of the functions and those values will give you the sequence.

For example, suppose $f(n)$ that is the function is defined as, $f(n) = 1/2n$ and as we know n can take all the positive integers. It can take values like 1, 2, 3 and it goes on like that. So, what will be the terms of the sequence? It will be $1/2,\ 1/4,\ 1/6,\ 1/8$…. So, all the even numbers will appear in the denominator, positive even numbers and it will go on like that. So, this is an example of a sequence.

So, sequences are generally denoted by these $\{s_n\}_{n=1}^{\infty}$, or in more shorter form like this $\{s_n\}$ and we covered that with the second bracket. If s is an infinite sequence, its terms are denoted by s_1, s_2, s_3, s_4 ...... s_n.... So, the general term is called small s_n, n is appearing as a subscript. The sequence s_n is said to converge, so you have this notion of convergence, is said to converge to a number small s, if s_n is arbitrarily close to s for all n sufficiently large.

So, as you go on increasing the number of terms if the value of the function that is s_n, it becomes closer and closer to a particular value which is s, then we can say that the function converges. So, this is basically very close to the idea of limit. So, we say that the limit of s_n, n goes to infinity, is s. A sequence which does not converge to any real finite number is said to diverge. So, it is not necessary that all the sequences will converge. Some sequences might diverge also if the limit does not exist as n goes to infinity.

(Refer Slide Time: 34:50)



A related idea is the idea of series. So, we start with an example. In finite geometric series with quotient small k is given by this, S_n. This is the general form of the series. $S_n = a + ak + ak^2 + ... + ak^{n-2} + ak^{n-1}$. This is the n-th term. So, if this is the n-th term, what is the second last term? This will be ak^{n-2}. So, you are summing up all these terms up to the n-th term and that is called S_n and it is called the series.

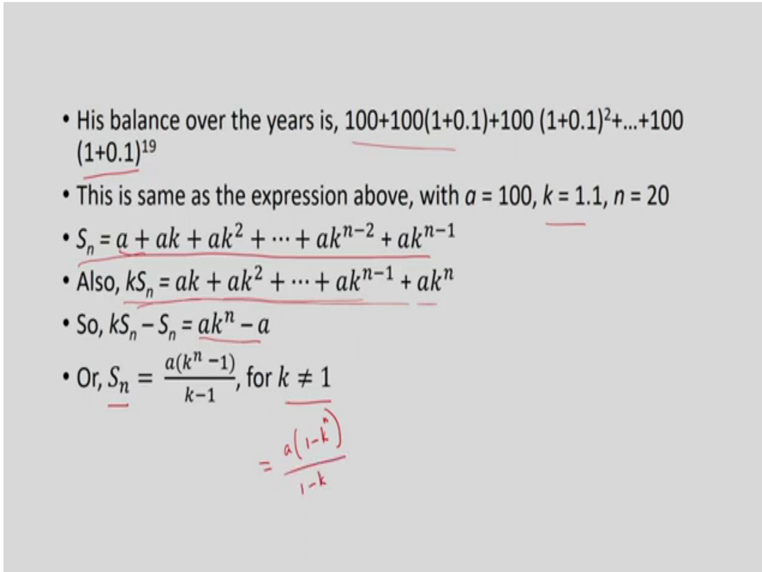
And in this particular series the quotient is there, the quotient is k. What is the role of the quotient? It is the quotient with which each term is getting multiplied and you are getting the next term and this is an example of a finite geometric series. Example, real life example, a man keeps INR100 in the savings deposit of a bank fetching him the rate of interest of 10% per year. So, you have kept INR100 in your bank and when you generally keep your money in the bank in a deposit, the bank pays you some rate of interest. And in this case, suppose 10% is the rate of interest that the bank pays to you for keeping your deposit in the bank.

Now, in the first year, his balance is INR100. In the second year, what is his balance? It will be INR100 plus there is interest that he has earned on that INR100. So, that will be 100+(100)(10%). This is the interest payment and this is the original deposit. So, it becomes 100(1+0.1), 10% is what, it can be written as 0.1. 10 divided by 100 is 0.1. So, this is 100(1+0.1).

Similarly, in the third year his balance is how much? The second year's deposit plus the interest that you will earn on that, so 100(1+0.1)+100(1+0.1)(0.1). And if you take common 100(1+0.1) you can take common and then you will get (1+0.1) in the brackets and this will simplify as $100(1 + 0.1)^2$. So, this is the balance in the third year.

And you can now see there is a general pattern in the 20th year, what will be the balance. The balance will be $100(1 + 0.1)^{20-1}$, because the pattern is that you keep the 100 constant and there is this factor $(1 + 0.1)$ and the power is important. The power is the number of years minus 1. So, this is 20 minus 1, because in the third year it was 2, in the second year it was 1. So, in the 20th year it will be 20 minus 1.

(Refer Slide Time: 38:24)

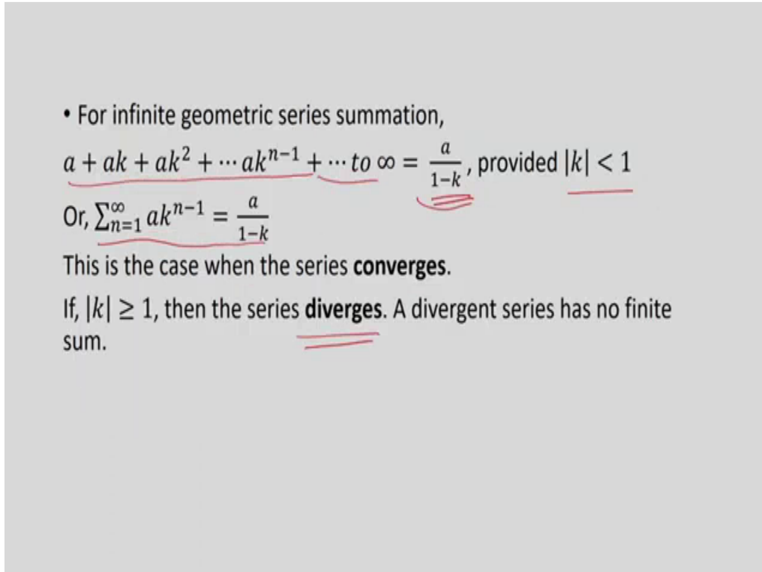


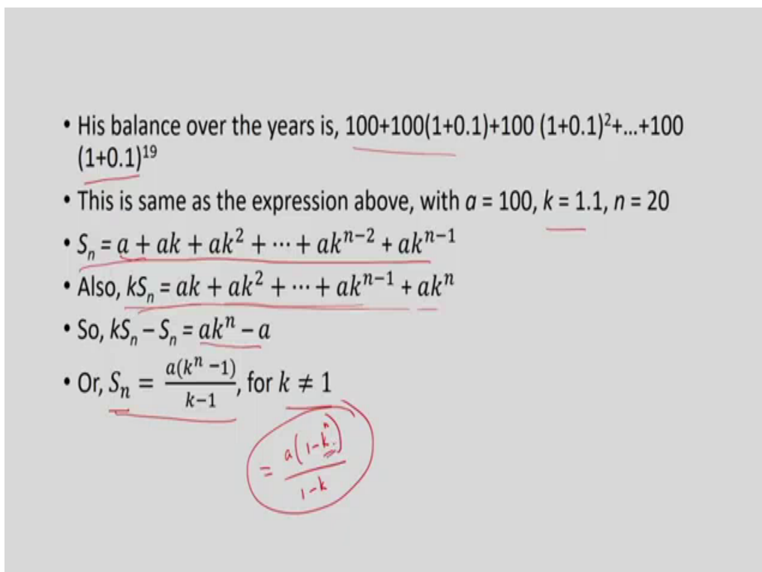
So, if I find out what has been his total balance over the years, so it is a summation of all the balances he had over all this 20 years, then this will be that $100 + 100(1 + 0.1) + 100(1 + 0.1)^2 + \ldots\ldots 100(1 + 0.1)^{19}$. So, this is like the expression above. This is the expression. And in the sense that the a here is 100, what is k, k the thing that is getting multiplied with it is $(1 + 0.1)$ which is 1.1 and n which is the last term, the last term if we look at that, it will be 20, because n minus 1 is 19, so n is equal to 20.

So, you have this series $S_n = a + ak + ak^2 + \ldots + ak^{n-2} + ak^{n-1}$. Now, if I want to find out what is the value of this series, then I do the following manipulation. I multiply both sides by k, then I get this series and then I take the difference of these two. So, on the right hand side I get $ak^n - a$ and then so $S_n = \frac{a(k^n-1)}{k-1}$ and this should be true if you have $k \neq 1$, because if you have $k = 1$ then this will be something divided by 0, which is undefined.

So, you have the expression for S_n. Sometimes it is also written as $S_n = \frac{a(1-k^n)}{1-k}$. Just have to multiply both numerator and numerator by minus 1, you will get the same expression.

(Refer Slide Time: 40:38)



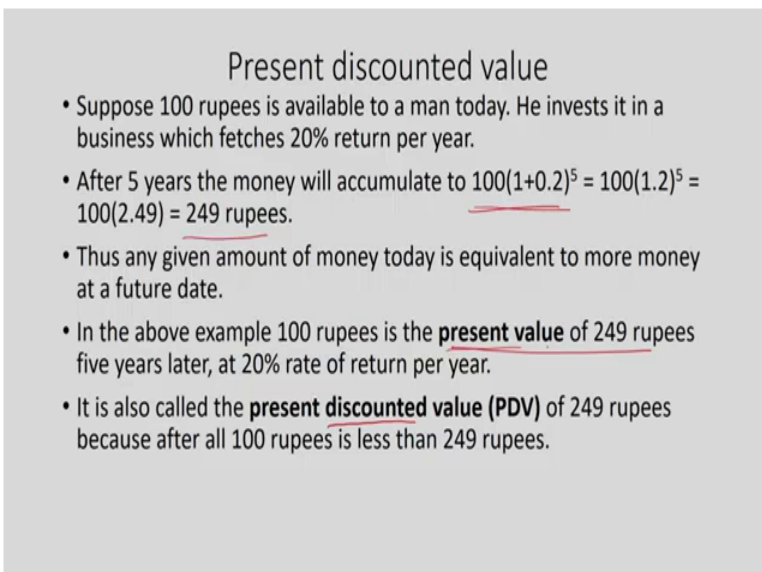


Now, for infinite geometric series, so this was the finite geometric series. Remember, this was a finite geometric series with quotient k and there I have this summation. But if you have an infinite geometric series, but that can also be found out, what is the summation, but that can be done if you have $|k| < 1$, because what is happening is that if you look at the series, this is how it is going to look like. This goes on like that. It is going to infinity. Now, if $k > 1\ or\ k <- 1$, then these terms will become, it will go to plus infinity or minus infinity. So, that will be not possible to sum.

So, therefore, the summation is possible only when you have $|k| < 1$. Even if it is equal to 1 then we cannot find the summation. It will go to infinity or it might fluctuate if it is minus 1. If you have $|k| < 1$, then I can sum it up and the summation is equal to $\frac{a}{1-k}$ and so we can write it like this, summation of this, $\sum_{n=1}^{\infty} ak^{n-1} = \frac{a}{1-k}$. So, this sigma denotes the summation as we know. And this is the case where the series converges.

And as we have just discussed, if $|k| \geq 1$, then the series does not converge. It might diverge. And then there is no finite sum. So, how do I know that this is the form that we are going to get if $|k| < 1$, the reason is this, take this expression $S_n = \frac{a(1-k^n)}{1-k}$, as n goes to infinity and if k is less than 1, then this term will go to 0. So, therefore, this term drops out and you will get $S_n = \frac{a}{1-k}$, as n goes to infinity.

(Refer Slide Time: 43:09)

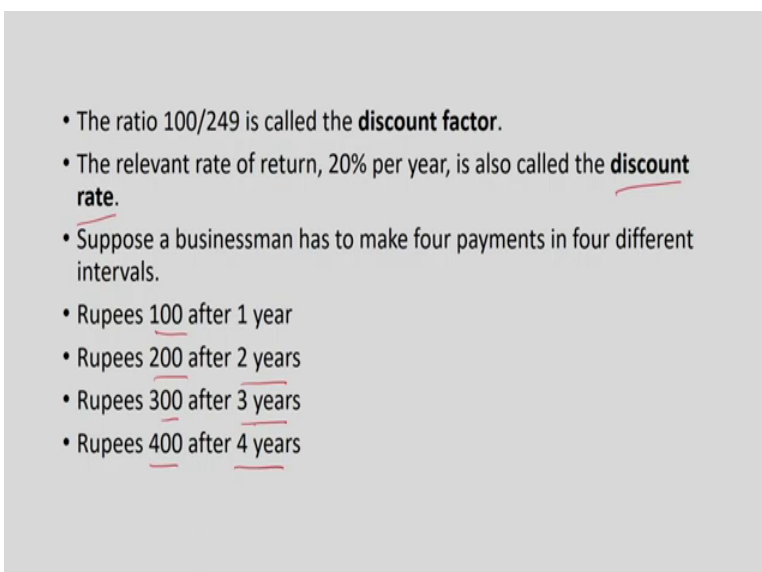


Now, we are going to look at certain applications and applications of these series and sequences. And one very important application of these is the present discounted value. Let's start with an example to motivate. Suppose INR100 is available to a man today and he invests it in a business which fetches 20% return per year. So, each year he is going to get 20% on the money that he is going to invest in the business. Now, after five years what is the amount of the money that he will get?

After five years the money will accumulate to this amount $100(1 + 0.2)^5$, because after one year it becomes 100(1+0.2), after two year it becomes $100(1 + 0.2)^2$ and like that it will grow and so after five years it becomes this amount $100(1 + 0.2)^5$. So, this is nothing but, if you simplify, it becomes INR249. Thus, any given amount of money today is equivalent to more money at a future date, because today you have INR100, the same money grows into INR249 after five year at 20% rate of return per year.

In the above example INR100 is the present value of INR249 five years later at 20% rate of return per year. So, this is the idea of present value. It is also called the present discounted value or PDV of INR249 because after all INR100 is less than INR249. Discounted means you are basically reducing the value. So, that is correct here, because INR249 is what you will get after five years, which is equivalent to INR100 which you have now. INR100 is less than INR249. So, therefore, we say INR100 is the PDV or the present discounted value of INR249.

(Refer Slide Time: 45:47)

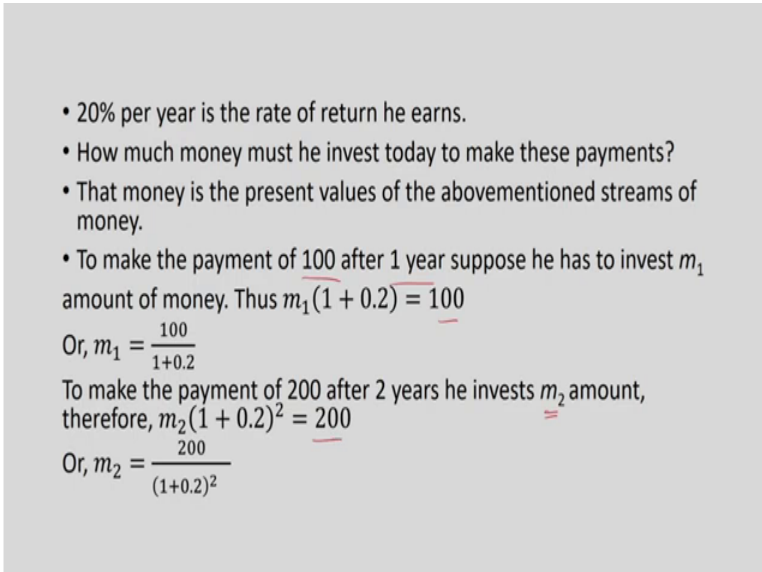


And this ratio 100/249 is called the discount factor in this particular example. This is called the discount factor where you are taking the present value and discount it by the future value and this is called the discount factor after five years. This is the case where you have five years at 20% rate of interest. The relevant rate of return 20% per year is called the discount rate.

You can call it the rate of interest if you are talking about lending the money or you can say that this is the rate of return when you are not lending as such, but maybe you are investing it in a business then this is the 20% rate of return. But whatever it is this is called the discount rate.

Here is another example. Suppose a businessman has to make four payments in four different intervals. So, he has to make INR100 payment after one year, INR200 after two years, INR300 after three years and INR400 after four years. I have constructed the example so that there is a symmetry between time and the money. It is easier to remember that way.

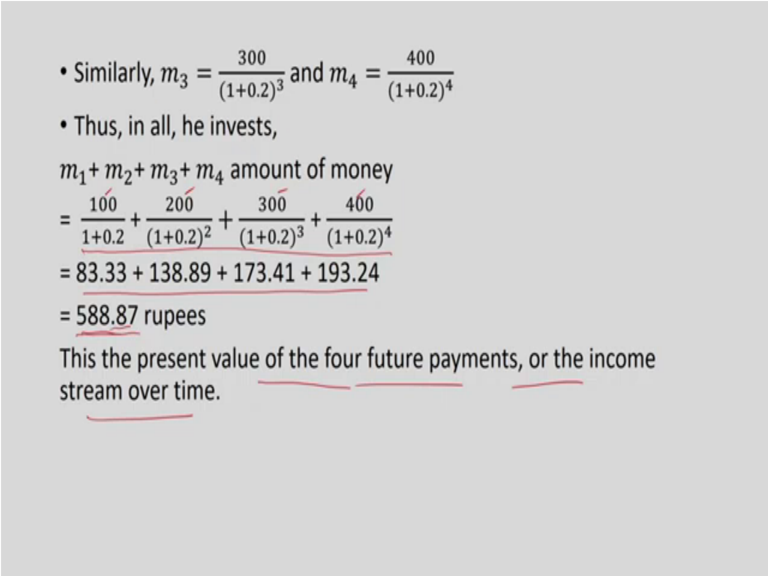
(Refer Slide Time: 47:13)



And suppose 20% per year is the rate of return that he earns. How much money must he invest today to make these payments? So this is the question. So, he has to make certain future payments. So, what is the amount of money that he must keep maybe in a bank which gives him 20% rate of interest or rate of return. And that money is the present value of the above mentioned streams of money. Here you do not have a single thing like five years, but you have many payments to be made over a period of time. So, we can say that this is a stream of payments.

To make the payment of 100 after one year suppose he has to invest $m_1$amount of money. Therefore, m_1(1+0.2)=100, because 0.2 is 20%. He will get a 20% rate of interest so the total amount of money will become INR100. So, that is the idea. So, therefore, $m_1 = \frac{100}{1+0.2}$. Similarly, to make the payment of INR200 after two years, suppose he invest m_2 amount. Therefore, $m_2(1 + 0.2)^2 = 200$. And therefore, $m_2 = \frac{200}{(1+0.2)^2}$.

(Refer Slide Time: 48:44)

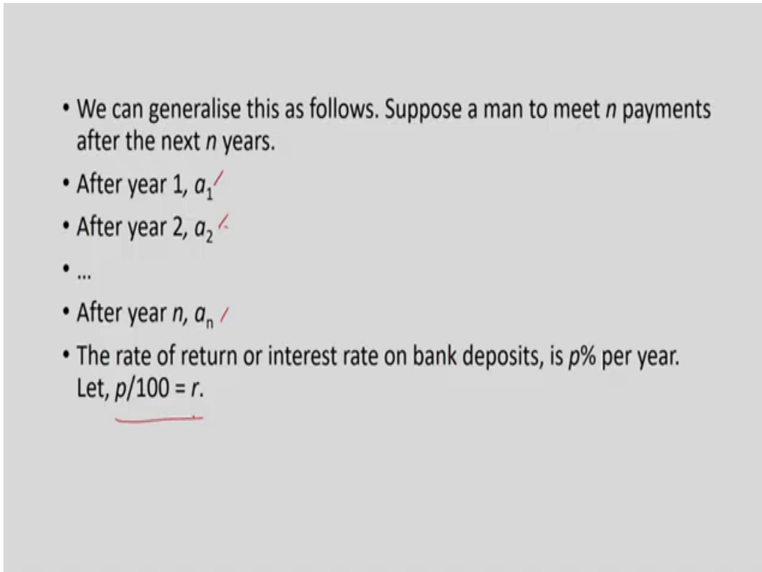


Similarly, $m_3 = \frac{300}{(1+0.2)^3}$ and $m_4 = \frac{400}{(1+0.2)^4}$. Thus, in all, he has to invest $m_1 + m_2 + m_3 + m_4$ amount of money the total amount of money to make all these four payments. And so you just add up these values and you are going to get this $= \frac{100}{1+0.2} + \frac{200}{(1+0.2)^2} + \frac{300}{(1+0.2)^3} + \frac{400}{(1+0.2)^4}$.

And if you simplify this, you will get four terms. I have surrounded it off up to two decimal places and I get INR 588.87. So, INR 588.87 is the present value of four future payments or the income stream over time. Now, if you notice, the amount of money that he is paying in these four year intervals is what, it is INR100 plus 200 plus 300 plus 400. So, in all, he is going to make payments of INR1000 if I just sum up these amounts.

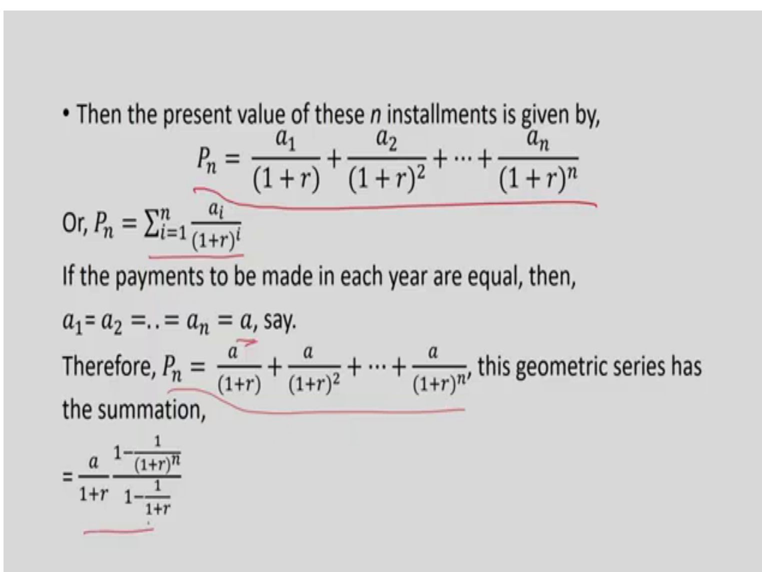
Whereas, today he is not investing or keeping in bank INR100, he is keeping in bank a much more less amount of money, nearly half of that, because that amount of money is going to earn some rate of interest and through those rate of interests and the original amount he is going to make all these payments.

(Refer Slide Time: 50:30)



We can generalize this example as follows. Suppose a man has to meet n payments after the next n years, so after year one he is going to make a payment of a_1, after year two, a_2, etc. etc. and after year n he is going to make a payment of a_n. And suppose the rate of return or rate of interest on the bank deposit is p percent per year. And p/100 = r .

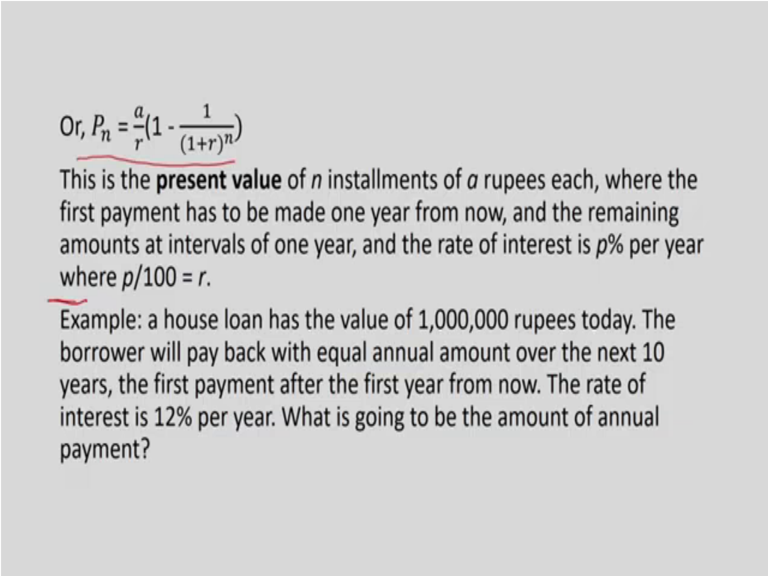
(Refer Slide Time: 51:07)



Therefore, the present value of these n installments is given by this P_n. P_n is the present value. So, I have just generalized that example and I have got this particular expression. $P_n = \frac{a_1}{(1+r)} + \frac{a_2}{(1+r)^2} + ... + \frac{a_n}{(1+r)^n}$. And I can, on the right hand side, just sum it up and write it as $P_n = \sum_{i=1}^{n} \frac{a_i}{(1+r)^i}$.

If the payments to be made in each year are equal, so suppose the case that $a_1 = a_2 = ... = a_n = a$, then the expression becomes much more easier to see, $P_n = \frac{a}{(1+r)} + \frac{a}{(1+r)^2} + ... + \frac{a}{(1+r)^n}$. And this is a geometric series and I can just add it up and simplify this and I get this.

(Refer Slide Time: 52:14)



So, $P_n = \frac{a}{r}(1 - \frac{1}{(1+r)^n})$. This is the present value of n installments of a rupees each, where the first payment has to be made one year from now and the remaining amounts at intervals of one year and the rate of interest is p percent per year where p/100 = r.

Let me stop here. In the next lecture, I am going to start from another example of this present value and its calculation and see you there. Thank you.