Mathematics for Economics 1

Professor. Debarshi Das
Department of Humanities and Social Sciences
Indian Institute of Technology, Guwahati

Lecture No. 11

Partial Differentiation

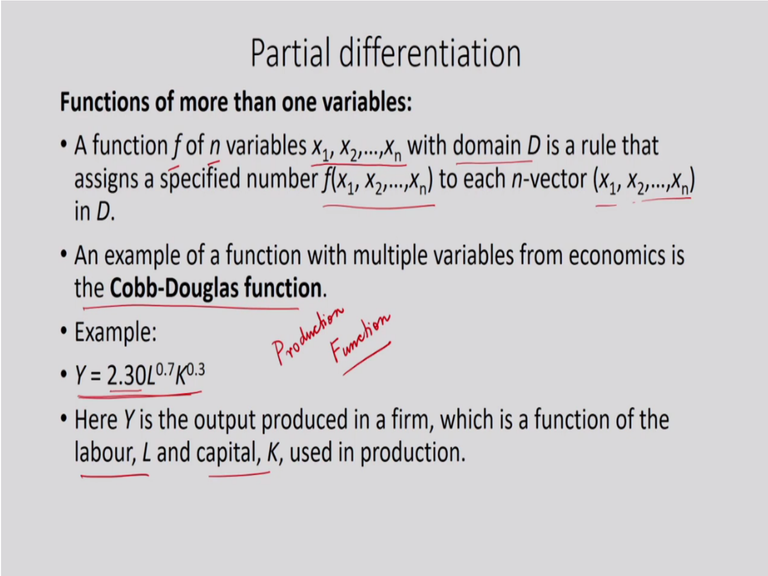
Welcome to another lecture of this course Mathematics for Economics Part 1. So, the topic that we have been covering is called differentiation as you can see on your screen differentiation. We have been talking about the definition of differentiation, we talked about limits and there are different rules of differentiation, we talked about them and today we shall deal with another subtopic within differentiation.

This is called partial differentiation. So, what happens in the partial differentiation remember so far we have been talking about a function of a single variable so $y = f(x)$, so x is the only independent variable or explanatory variable of this variable called the dependent variable which is why, but in economics we often encounter functions which have multiple independent variables.

And if you have multiple independent variables, then each of these independent variables can change and if they change then what happens to the value of the function, how much does the value of the function change. So, that is what we are going to discuss today, which is the topic of partial differentiation. So, this is a technique to find out the instantaneous rate of change of the dependent variable with respect to one of the many independent variables because there are multiple independent variables.

And it is possible that some of them are changing all of them might not be changing may be just one of them is changing or two of them are changing then how does the dependent variable change, what is the instantaneous rate of change of the dependent variable. So, those things are the subject of today's talk. Let us go to the particular portion.

(Refer Slide Time: 2:56)



So, this is what we are going to deal with today: partial differentiation. Functions of more than one variable. So a function f of n variables, n could be any number and these variables let us call them $X_1, X_2, X_3 ..... X_n$, so there are n number of independent variables and they belong to this particular set D remember this is the domain set. So this function f is a rule that assigns a specified number $f(X_1, X_2, X_3 ..... X_n)$ to each n vector $X_1, X_2, X_3 ..... X_n$ in the domain D.

As we know a particular function is a rule which assigns a specified value of the dependent variable for a particular value of the independent variable which belongs to the domain. Here instead of one independent variable you have n number of independent variables so you have a particular vector $(X_1, X_2, X_3 ..... X_n)$. With respect to this particular vector you get a value of the dependent variable which we are denoting as $f(X_1, X_2, X_3 ..... X_n)$.

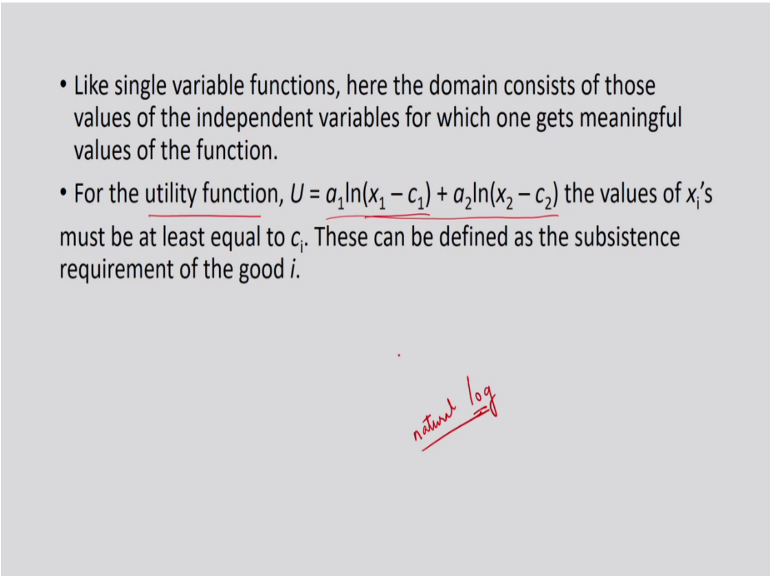
So, to fix our ideas I start with an example an example of a function with multiple variables from economics is the Cobb-Douglas function. So, this is very well known form of functions in economics often used. This is called Cobb-Douglas function and here we are dealing with particular Cobb-Douglas function this is called the production function,$Y = 2.30L^{0.7}K^{0.3}$. We have encountered production functions before.

Here, on the right hand side you have the level of output which is denoted by capital Y it is a function of in this particular case two independent variable L and K. L is the amount of labor and K is the amount of capital. So, these are the usual conventional notations we use K for capital and L for labour used in the process of production and labour and capital are used and combined and we get a certain level of output which is denoted by this function, $Y = 2.30L^{0.7}K^{0.3}$.

And in this particular case look at the form of the function. First we have a constant here it is $Y = 2.30L^{0.7}K^{0.3}$. So, both L and K have their powers and this powers in this particular case both of them are less than 1 and you can see that if L and K change, then Y will go on changing if any of them is equal to 0, then the level of output becomes equal to 0 because of the fact that both of this inputs are indispensable.

You cannot put zero amount of labor and expect to produce any level of output with the help of capital alone so that is not possible. So, that is why this form is taken.

(Refer Slide Time: 7:14)



Like single variable functions here the domain consists of those values of the independent variables for which one gets meaningful values of the function. So, remember in case of a single variable function we said what is the domain? Now domain if it is not specified otherwise it will consist of those values of the independent variable which give you meaningful value of the dependent variable or which makes sense.

In this particular case for example L or K cannot be negative, they could be positive, they could be 0 that is fine, but unless it specified L and K we can assume they cannot be negative and they should be minimum 0, but can go on increasing as far as possible depending on the ability of the producer or the availability of resources. So, that is how the domain is defined.

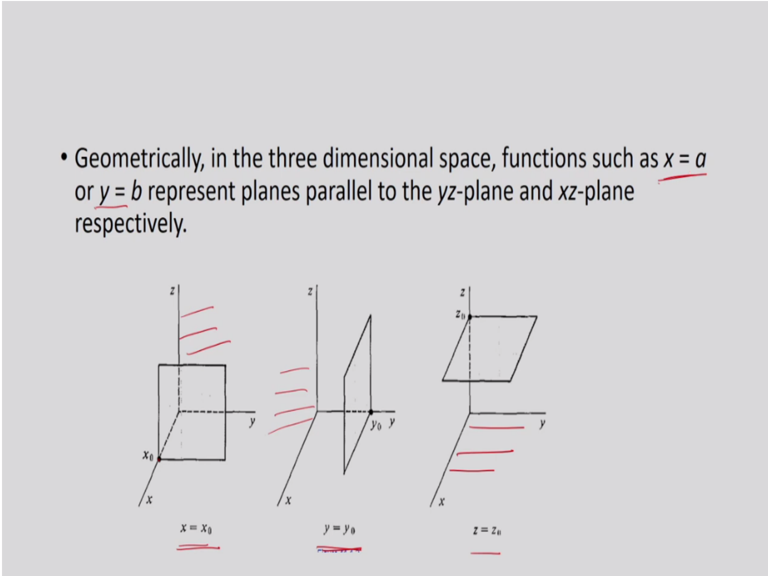
Here is another example of what is known as the utility function, $U = a_1 ln(x_1 - c_1) + a_2 ln(x_2 - c_2)$. Here U is denoted on the left hand side which represents the utility a person gets and you can see on the right hand side you have a particular expression a1 ln I have not discussed this log, this particular term before which we shall deal with later on, but this is called natural log ln. For the time being it will suffice for us to note that log or natural log is defined only for positive values, in the minimum they can be 0 and they cannot be negative.

So, log of a negative number is undefined. So, in this case you have $a_1 ln(x_1 - c_1) + a_2 ln(x_2 - c_2)$. The values of $x_i's$ must be at least equal to c_i because you know if c_i is less than x_i only then the value inside becomes positive. At the most c_i can be equal to x_i which will give us a 0 value log of 0 is somewhat problematic, but obviously ci cannot be more than x_i.

So, therefore here this function is defined only if x_i is greater than c_i or in the minimum x_i can be equal to c_i. So, what are $x_i's$are the values of the goods that a particular consumer consumes. So, x_i is the amount of good x_i which the particular consumer consumes. So, in this case x_i has to be greater than or equal to c_i that basically can be interpreted as c_i is the subsistence requirement of consumption of a particular good that basically means that suppose you are an individual.

You will need certain basic amount of food for example to survive. So, your food consumption cannot go below a certain level otherwise, you will not be able to leave. Therefore, you have this kind of function which basically captures the fact that consumption cannot be below a certain level.

(Refer Slide Time: 11:20)



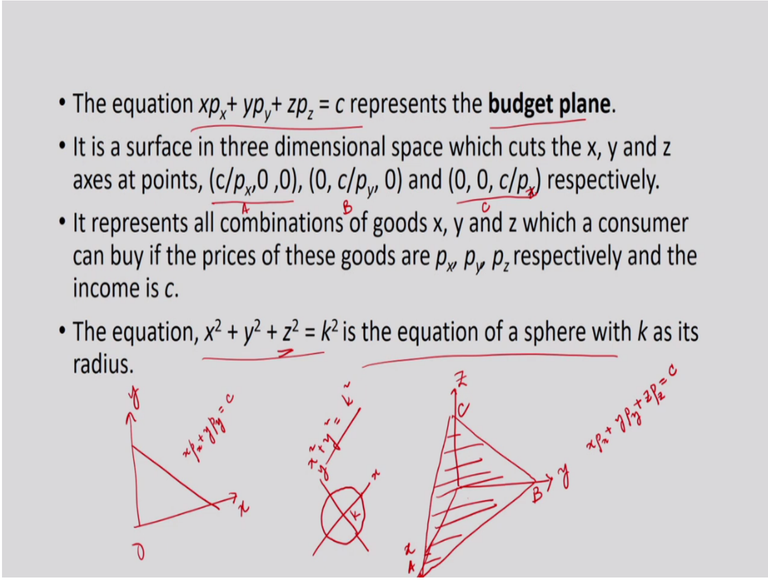
Now we are talking about three variables here. Remember at least three variables because two variables are independent and one variable could be dependent, the number of independent variables can become more than two also, but in the least they can be two. Geometrically here you have a three dimensional space. In three dimensional space function such as $x = a \; or \; y = b$. They represent planes parallel to the yz plane and xz plane respectively.

So, here is an example so instead of a I have taken suppose $x = x_0$ how does it look like, what does it represent. So, in this three dimensional space suppose this is the value of x_0 and x should always be x_0 which means you have a plane sort of wall kind of structure at $x = x_0$. So, $x = x_0$ represents any point on this wall because on this wall the value of x is fixed.

Similarly, here you have value of y fixed and remember this $y = y_0$ it is also a wall and it is parallel to what plane? It is parallel to this yz plane. Similarly, your $x = x_0$ is parallel to this yz plane and this is parallel to xz plane and if you take $z = z_0$ so the z coordinate is fixed, then you have another plane which is like a kind of roof and along the roof you have z which is always fixed and this roof is parallel to xy plane.

So, this sort of equation such as x is a fixed number, y is equal to fixed number, z is equal to fixed number they are actually planes parallel to the zy plane, parallel to the xz plane and parallel to the xy plane respectively.

(Refer Slide Time: 13:57)



And now let us talk about a more complicated sort of plane. Let us talk about this $XP_x + YP_y + ZP_z = c$. This is what we are calling as the budget plane why we are calling it the budget plane. Remember when we were talking about only two goods. Suppose x and y were the two goods then we talked about what is called the budget set and the budget line.

Remember this was x and you have $XP_x + YP_y = c$, c is the income. So, this is two dimension, but if you have three dimensions so three goods are being purchased then you have another term so you have $XP_x + YP_y + ZP_z = c$. So, your total income is getting subdivided among three goods and you are spending your total income then geometrically how do we represent this particular equation.

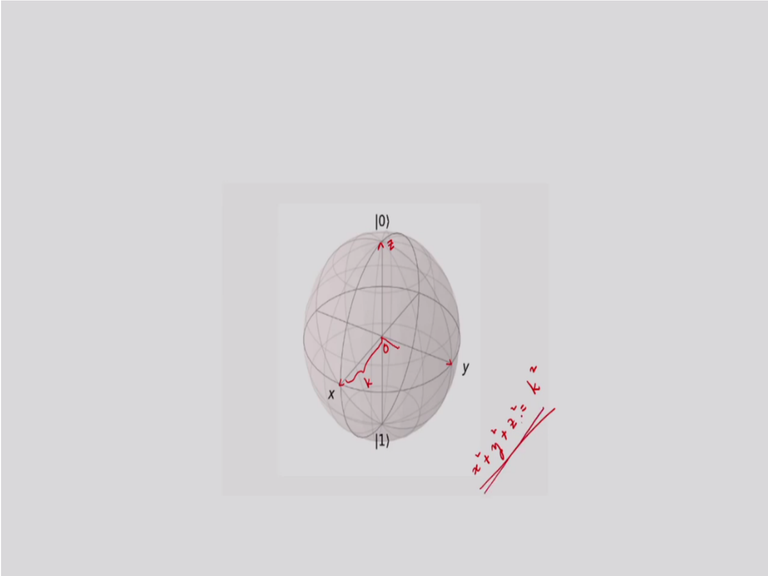
Well here it is not a straight line, it is basically a plane in the three plane so this is the plane that we are talking about. It is a triangular plane which is having some point of intersection with a three axis and these are the points of intersection (c/P_x, 0, 0) so this is that point suppose this is A, this is B, this is C. So, A represents this point, B represents this point and C represents this particular point (0, 0, c/P_z).

And we know that this is obvious because suppose you are talking about point A here the person is not consuming any y and z the entire money income is being spent on x only. So, how much x can he buy? He can buy c divided by px so that is why the coordinate of point A is (c/P_x, 0, 0) likewise for the other two points and these are the corner points, but inside if you look at the interior points they are having positive values of all the three goods.

So, it represents all combinations of goods x, y and z which a consumer can buy if the prices of these goods are P_x, P_y, P_z and the income is c. So, this is one equation now we talk about a more complicated equation. Here you do not have power to be one, but here now we have squares, that is, the power is 2. So, suppose you take this particular equation $x^2 + y^2 + z^2 = k^2$.

And we are saying this is the equation of a sphere with k as its radius. So, remember again if you had $x^2 + y^2 = k^2$, then we have seen this is the equation of a circle I am sorry so this is k the radius is k. So, this is your x axis, this is your y axis and this is the circle, but now you have a third element z^2 and we are claiming that from a flat circle we go to a three dimensional sphere.

(Refer Slide Time: 18:31)

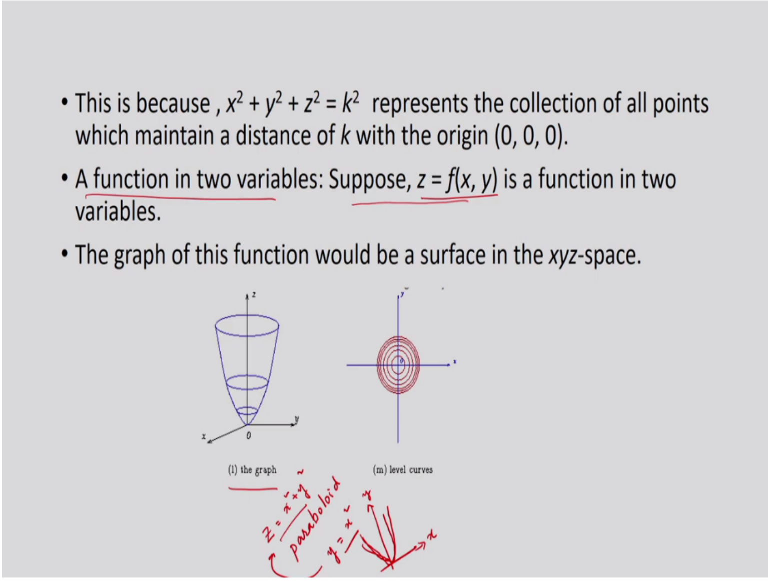


So, this is how it is going to look like you have x axis along this x and you have y axis and here along the vertical direction you have the z axis and you are saying that this is k. So, here the equation is $x^2 + y^2 + z^2 = k^2$. Now what is the argument why we are claiming that if you have this equation then the corresponding figure or graph in the three dimensional plane will be a sphere.

This is because $x^2 + y^2 + z^2 = k^2$ represents the collection of all points which maintain a distance of k with the origin 0, 0, 0. So, this is easy to see that you take any point on the surface of the sphere and you think about the distance of that point from the origin, origin is 0, 0, 0, then one can show that this distance is equal to $\sqrt{x^2 + y^2 + z^2}$.

And what is this distance after all it is the radius so if you take the squares of both sides so therefore you have radius square is equal to $x^2 + y^2 + z^2$ so therefore you have this equation. So, that is the demonstration.

(Refer Slide Time: 20:15)



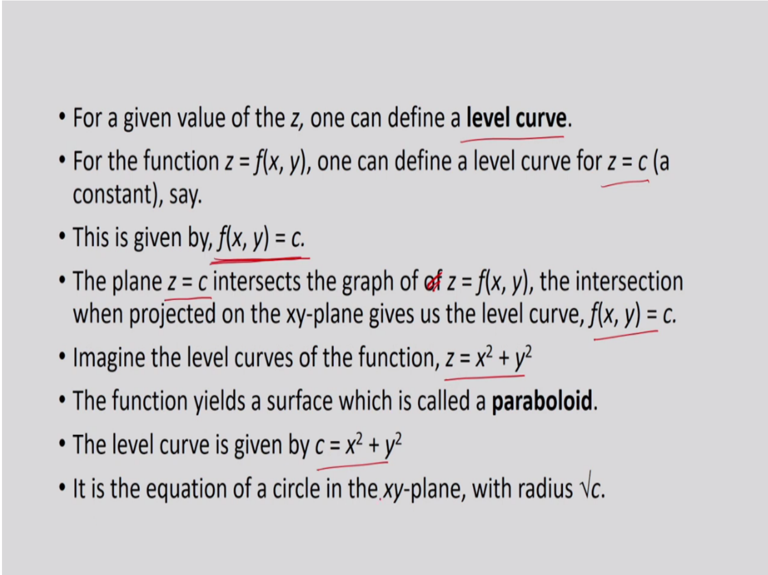
Now from the equations in three variables which are represented by different shapes we now come to what is called a function. Remember there is a mild difference between equation and a function. Now we are talking about a function, suppose. So, you have a function z which is a function of two variables $z = f(x, y)$and the graph of the function would be a surface in the xyz space.

So, suppose you talk about this particular, which is represented on the left hand side, the graph here. Here you have a surface it looks like a funnel kind of thing and if you talk about the equation of this kind of funnel it could be like $z = x^2 + y^2$ and you can add some constant with coefficient of x square, coefficient of y square, but this could be a general shape and this shape is called paraboloid from the word parabola because if you recall the shape is this y and x.

So, this is two dimensions so instead of this you have now this. So, $z = x^2 + y^2$ and you have this sort of surface no longer a curve. So, this generally is called a paraboloid from the word parabola. For the time being ignore the figure on the right hand side.

So, what we are saying is that if you have $z = f(x, y)$ so you have the dependent variable to be a function of two variable, then one can represent this function in the 3 dimensional space I have taken a particular form of function and this form of function will give us this kind of shape and it is called a paraboloid.

(Refer Slide Time: 22:50)



Now for a given value of z we can define what is known as a level curve. Level curve means what? For the function $z = f(x, y)$ one can define a level curve for suppose $z = c$. So the level curve will have this kind of equation $f(x, y) = c$. Note this equation is an equation which involves only two variables only x and y it does not involve z. So, the plane $z = c$ intersects the graph of $z = f(x, y)$.

The intersection when projected on the xy plane gives us the level curve $f(x, y) = c$. That is the full explanation of this. So, first you start with this function $z = f(x, y)$ so correspondingly we will get this kind of surface a paraboloid. Then you take $z = c$, c is a constant. So, here is perhaps one example: suppose this is c this height is c. So, this $z = c$ is actually a plane as we have seen it is a plane parallel to the xy plane.

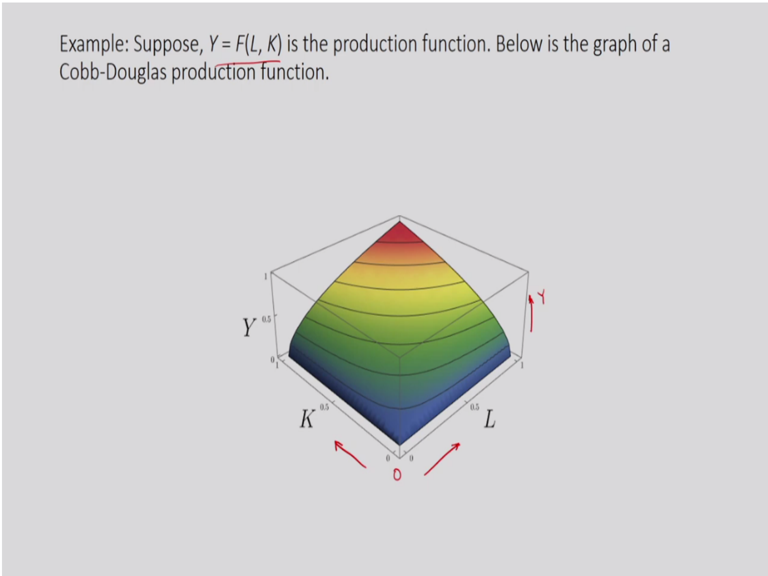
So, this $z = c$ plane will intersect this paraboloid and from this intersection which in this case looks like a circle. We basically project this on the xy plane so this is how the projection is being done and this projection when it is done on the xy plane we get this level curve. So, on the right hand side you have this level curve.

So, for different values of c you will get different level curves for example here $z = c$, but if you could take a different value of c and the projection will also be different. So, there will be a different curve so that is why you have on the right hand side different circles. So, here the level curves are all circles in this particular example of a paraboloid all the level curves are circles on the xy plane and they cover all the four quadrants, you can see that.

So, that is what I have written: the plane $z = c$ intersects the graph of $z = f(x, y)$. The intersection when projected on the xy plane gives us the level curve whose equation is given by $f(x, y) = c$. So, here I have just written what I have just shown you. Imagine the level curve of the function $z = x^2 + y^2$. The function yields a surface which is called a paraboloid.

Here on the left hand side you have the paraboloid the level curve is given by $c = x^2 + y^2$ and as we know $x^2 + y^2 = c$.. It is the equation of a circle in the xy plane and what is the radius? The radius is $\sqrt{c}$, as c is rising you have different sorts of level curves.

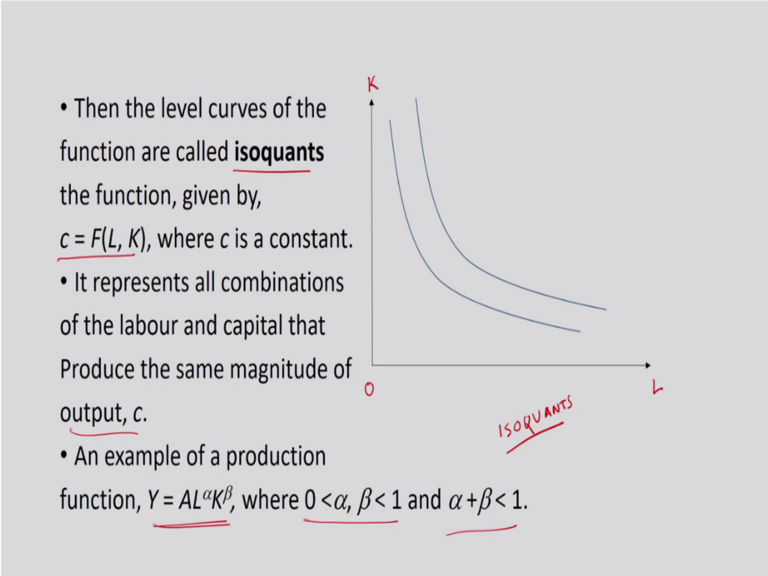
(Refer Slide Time: 26:46)



This was a general discussion about level curves. Now we come to particular discussion in economics. So, here you have $Y = F(L, K)$, remember this is the general form of a production function of two independent variables labor and capital. Below is the graph of a Cobb-Douglas production function. So, you have sort of starting from 0 this is the origin vertically you are going up that is the output level and it is color coded which means that if you are going along the same color the output level is remaining the same.

So, you can see this lines here on the surface of the curve and on the surface you have this lines and this lines are level curves. As you are going up the color is changing from blue it is changing to red which means that the output level is rising. So, greater and greater level of output as you are going to the shades of red.

(Refer Slide Time: 27:58)



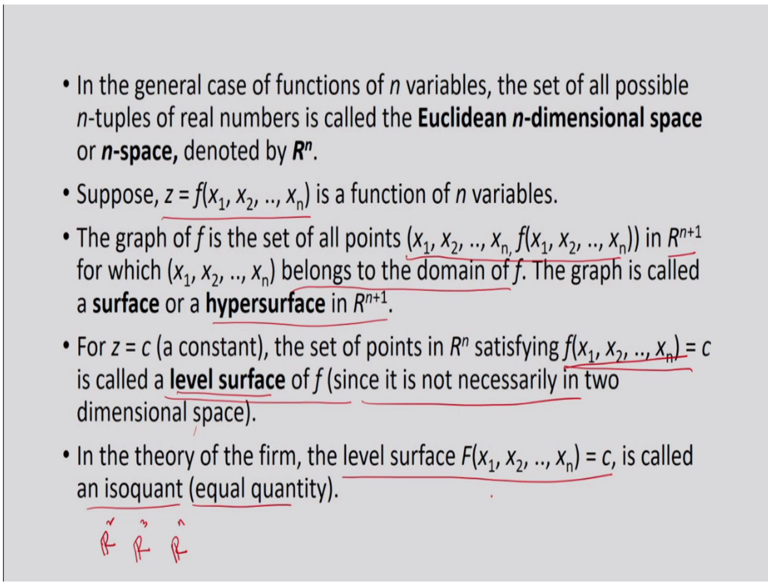
The level curves of this function are called isoquants. So, isoquants are given by this equation $c = F(L, K)$ where c is a constant. So, on the two dimensions so if you draw that because we know the level curves are projections on the two dimension. So, in this case the projection will be on the L, K plane so labor and capital plane. So, you will have this sort of downward sloping functions which are convex to the origin.

So, these functions are parallel, all of them are convex to the origin; these are called isoquants. It represents all combinations of labor and capital that produce the same magnitude of output, c. So that is the interpretation of this. It is not difficult to understand because that is what we are getting here. So, along the level curve the output level is the same but different combinations of labor and capital are there.

So, isoquants therefore represent different combinations of labor and capital which produce the same level of output. So, an example of a production function so you take this particular Cobb-Douglas production function $Y = AL^{\alpha}K^{\beta}$and alpha and beta both lie between 0 and 1 not taking the value of 0 and 1 and alpha plus beta is less than 1, $0 < \alpha, \beta < 1$ *and* $\alpha + \beta < 1$. This could be a particular production function.

From this if we draw the isoquants then the isoquants will look like these downward sloping lines and they are moving away from the origin as they are going down which basically means that they are convex to the origin.

(Refer Slide Time: 30:03)



Now from two dimensions we come to n dimensions, the general case. In the general case of functions of n variables the set of all possible n tuples of real numbers is called the Euclidean n dimensional space or n space. This is denoted by R^n. So, just imagine that we talked about two dimensional space which was R^2. We talked about three dimensional space R^3, but suppose instead of 2 and 3 I take a general n.

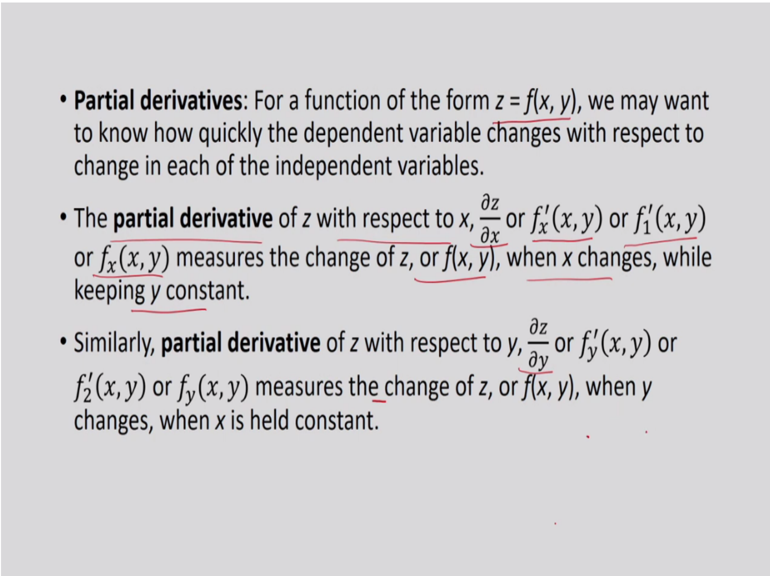
So, that will also be an n dimensional space. So, this is called Euclidean n dimensional space or n space and this is denoted by R^n. Suppose, you have a function which is $z = f(x_1, x_2, x_3, x_4,.... x_n)$ is a function of n variables. The graph of f is the set of all values $(x_1, x_2, x_3, x_4,.... x_n, f((x_1, x_2, x_3, x_4,.... x_n))$ and this will be in what space? It will be R^{n+1} space for which $(x_1, x_2, x_3, x_4,.... x_n)$ belongs to the domain of f obviously, if we are talking about the function, then the independent variable has to belong to the domain.

The graph is called a surface or a hypersurface in R^{n+1}. So, this is just a parallel of what we have seen before. So, here you had a paraboloid which was also surface and here you have a sort of dome kind of shape for a Cobb-Douglas production function, but that dome is itself a surface like a tent kind of surface and now we are saying that for a general case this is called a hypersurface or a surface in general.

For $z = c$ a constant, the set of points in R^n satisfying this $f((x_1, x_2, x_3, x_4, ... x_n) = c)$ is called a level surface of f. We are not saying that it is a level curve. If it is a curve then we are saying that it in two dimensions, but it could be more than two dimensions therefore we are saying that this is a level surface of f. So, this is the general form of a level surface. In the theory of the firm the level surface this $F(x_1, x_2, x_3, x_4, ... x_n) = c$ is called an isoquant.

So, for the isoquant the name does not change if you have two inputs it is called as isoquant in that case it is just a curve in a two dimensional space, but if you have multiple more than two inputs, then also it is called an isoquant, iso means equal, quant means quantity, isoquant means equal quantity because the output remains the same along the isoquant.

(Refer Slide Time: 33:31)

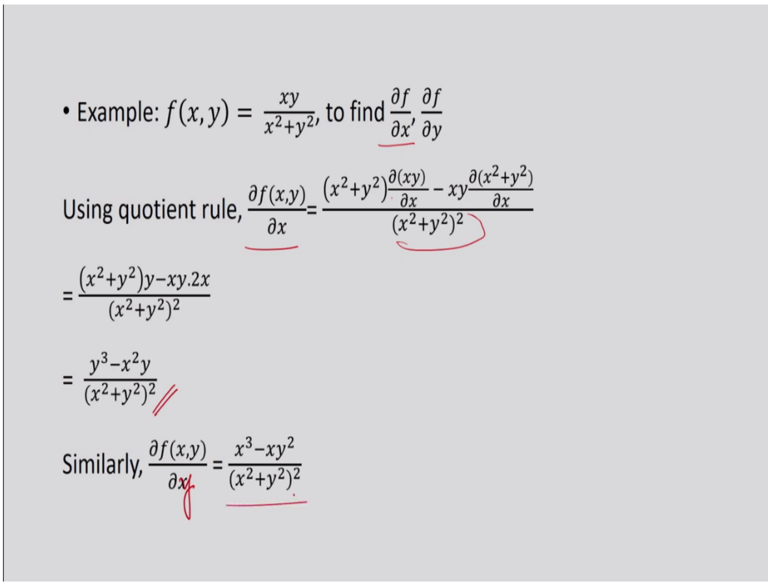


Now we come to the actual discussion of partial derivatives. So far we were just building the blocks, we were gathering the building blocks of partial derivatives. Partial derivatives for a function of the form $z = f(x, y)$. So, you have just two independent variables, we may want to know how quickly the dependent variable changes with respect to change in each of the independent variables.

So, x might change without any change in y, but if x is changing then z will change. So, we want to find out what is the rate of change of z. The partial derivative of z with respect to x is denoted by this $\frac{\delta z}{\delta x}$ $or\, f'_x(x, y)$ this is another way to denote the same thing or $f'_1(x, y)$, why 1, because x is the first variable or more explicitly $f_x(x, y)$ these are all denoting the same thing which is the partial derivative of z with respect to x. It measures the change of z or $f(x, y)$ when x changes while keeping y constant.

So, this is the interesting thing to note that when we are taking the partial derivative of a function with respect to a particular independent variable we are assuming that the other independent variables do not change. So, this was partial derivative with respect to x. Similarly, partial derivative of z with respect to y will be denoted by this $\frac{\delta z}{\delta y}$ *or* $f_y^{'}(x, y)$ *or* $f_2^{'}(x, y)$ *or* $f_y(x, y)$. It measures the change of z or $f(x, y)$ when y changes when the first variable that is x is held constant.

(Refer Slide Time: 35:50)

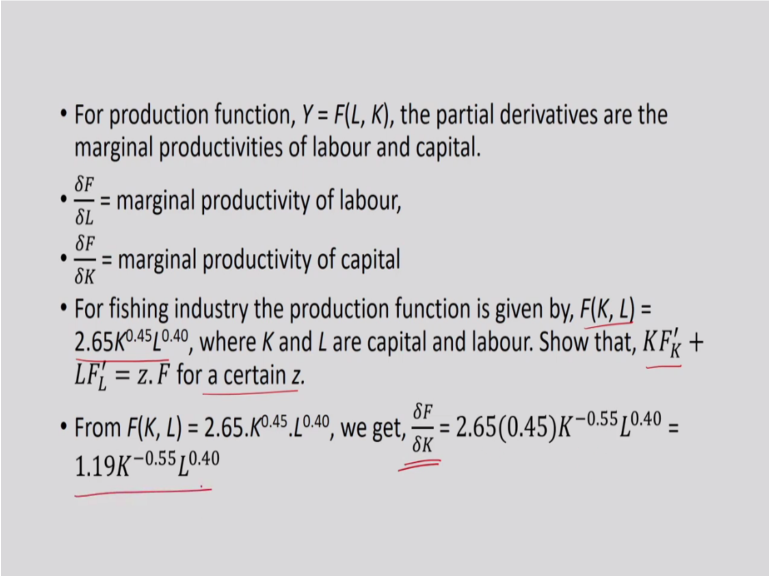


So, here is an example, so you have been given a function $f(x, y) = \frac{xy}{x^2+y^2}$ we have to find out what is $\frac{\delta f}{\delta x}$ *and* $\frac{\delta f}{\delta y}$. So, to find $\frac{\delta f}{\delta x}$ I used the quotient rule because you have a quotient here $\frac{xy}{x^2+y^2}$. So, we know how to do that. I take the square of the denominator in the denominator and then the denominator multiplied by the partial derivative.

Here is the important thing you have to take the partial derivative of the numerator with respect to x minus the numerator multiplied by the partial derivative of the denominator with respect to x and the rest is just simplifying this particular expression and if we do that I will arrive at this expression which is $\frac{y^3 - x^2y}{(x^2+y^2)^2}$.

Similarly, the function is symmetric so if you have $\frac{\delta f}{\delta y}$, then this should be the corresponding expression, $\frac{\delta f(x,y)}{\delta y} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$.

(Refer Slide Time: 37:14)

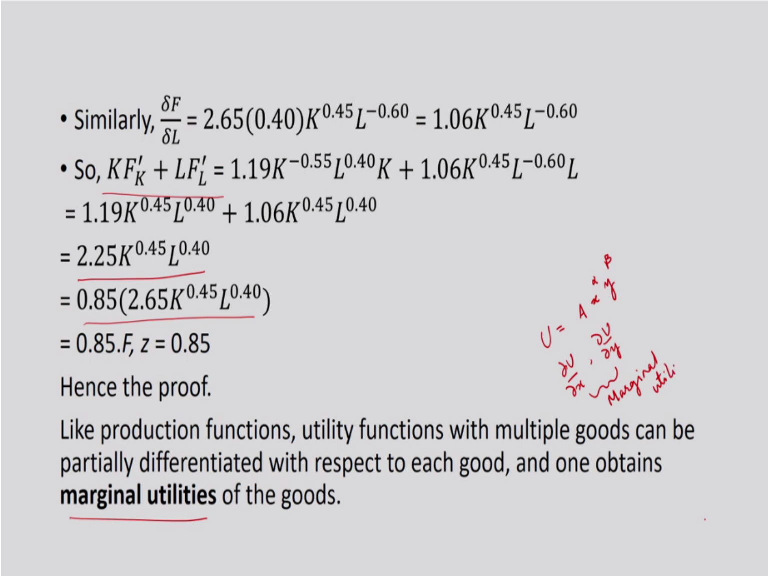


For production function $Y = F(L, K)$, the partial derivatives are the marginal productivity of labour and capital. So, this is like the marginal productivity of labour when the production function was taken as a function of single input. We have done that before, but now suppose both the inputs can change then you have marginal productivity of labour is equal to the partial derivative of F with respect to L, $\frac{\delta F}{\delta L}$ and marginal productivity of capital is partial derivative of F with respect to K, $\frac{\delta F}{\delta K}$.

So, here is an example for the fishing industry: the production function is given by $F(K, L) = 2.65K^{0.45}.L^{0.40}$. So, this is a typical Cobb-Douglas production function where K and L are capital and labour. **Show** that $K.F_K^{'} + L.F_L^{'} = z.F$ for a certain z, certain value of z. Here $F_K^{'}$ and $.F_L^{'}$ are obviously partial derivatives of F with respect to capital and labour.

So, let us first find out the left hand side, that is I have to find out the partial derivative of the production function with respect to capital. So, I simplify this and I get this expression: $\frac{\delta F}{\delta K} = 1.19K^{-0.55}L^{0.40}$.

(Refer Slide Time: 39:04)

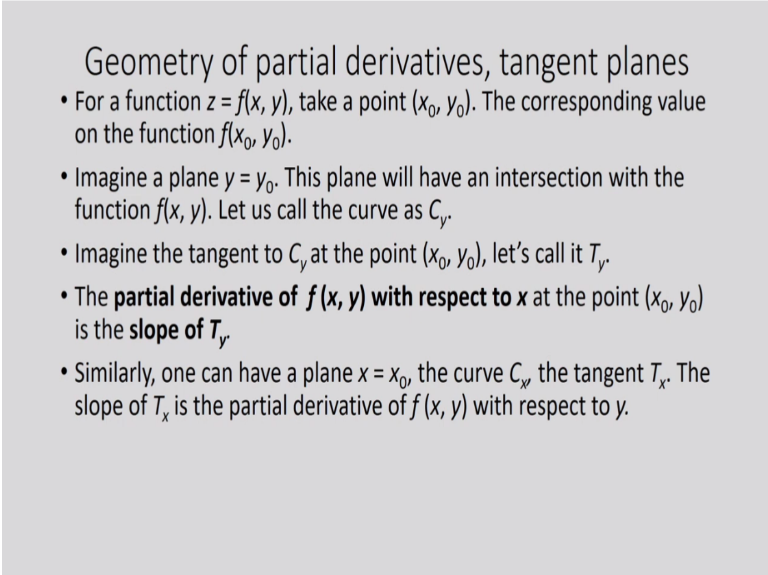


Similarly, the partial derivative of the production function with respect to labour will be $\frac{\delta F}{\delta L} = 1.06\, K^{0.45}L^{-0.60}$ and then I tried to find out what is the value of the left hand side, left hand side is $K.F_K^{'} + L.F_L^{'}$. So, I substitute these values that we have found into this expression.

And I simplify and I get this $2.25K^{0.45}L^{0.40}$ and I can express this, as this 0.85 multiplied by the production function actually, $0.85(2.65K^{0.45}L^{0.40})$. So, we have basically proof the thing for $z = 0.85$. So, the proposition that was supposed to be proof is correct if $z = 0.85$.

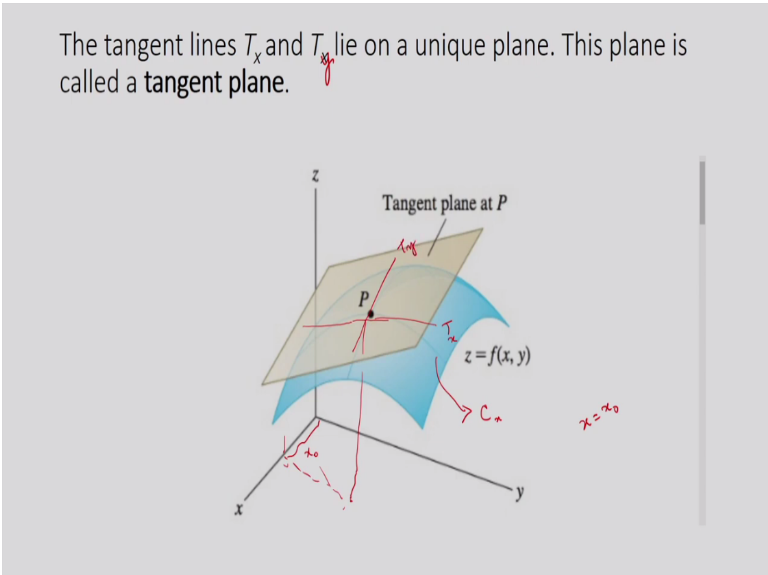
Like production functions, utility functions with multiple goods can be partially differentiated with respect to each goods, and one obtains what are known as marginal utilities of goods. So, you have suppose $Y = Ax^{\alpha}y^{\beta}$, A is constant, where x and y are the goods or the quantities of the goods the consumer is consuming then you can find out what is $\frac{\delta U}{\delta x}$, $\frac{\delta U}{\delta y}$. These will be marginal utilities.

(Refer Slide Time: 40:57)



Finally we talk about the geometry of partial derivatives and we talk about what are known as tangent planes. For a function of this form $z = f(x, y)$, take a particular point (x_0, y_0). The corresponding value on the function will be $f(x_0, y_0)$. So, if you take $z = f(x, y)$, you basically have a surface.

(Refer Slide Time: 41:31)



So, here take this for the time being, ignore the plane which is the tangent plane concentrate on this particular blue surface that you have got. So, you have basically from this point, particular value of let us say x_0 and this is let us say y_0. So, you have (x_0, y_0) at this value and at this point on the xy plane and $z = f(x, y)$ will give you a particular value of z and you are reaching here. So, this is the value of $z = f(x_0, y_0)$ so that will be equal to z_0.

Imagine a plane $y = y_0$. This plane we have an intersection with the function $f(x, y)$. Let us call the curve as C_y. So, suppose here y is equal to 0 and $y = y_0$, we know it is a plane. So, you have basically a kind of wall kind of structure here and this wall is going to intersect with the curve or with the surface that is $z = f(x, y)$. And that intersection is called C_y.

So, here is basically the C_y this curve is called C_y $y = y_0$, this plane will have an intersection with the function $f(x, y)$. Let us call the curve as C_y. Imagine the tangent to C_y at the point $f(x_0, y_0)$ Let us call it as T_y. So, you have this C_y and imagine the tangent there will be tangent to this curve at point P and this tangent let us call that as T_y.

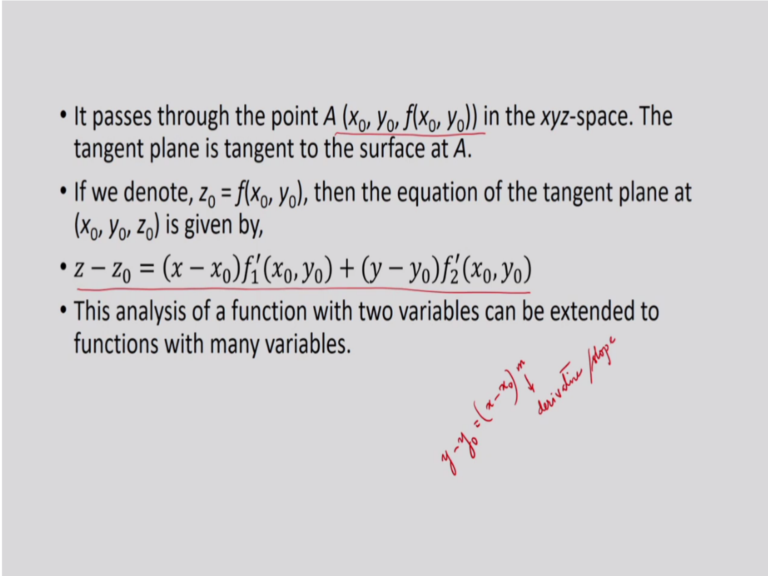
The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is the slope of T_y. So, you have got this line T_y. The slope of this T_y is the partial derivative. So, this is f_x evaluated at (x_0, y_0), so the slope of T_y is equal to this. So, this is the geometry of the partial derivative. What does it mean when you say partial derivative geometrically this is what it means. Now this is partial derivative with respect to x.

What about partial derivative respect to y? For that what we need to do, we need to do a similar thing with $x = x_0$. So, you can have a plane $x = x_0$ and similarly a curve C_x and similarly a tangent T_x. The slope of T_x is the partial derivative of $f(x, y)$ with respect to y. So, if you take $x = x_0$ so suppose this is x_0 Then here you can imagine similarly another plane $x = x_0$ and that plane will have an intersection with this surface and that curve will be C_x.

And similarly, we will have a tangent which is T_x. Now slope of this T_x is like before, slope of this T_x will be the partial derivative of this function with respect to y evaluated at (x_0, y_0). Now these two tangent lines T_x and T_y. So, this will be T_x and T_y lie in a unique plane. This plane is called the tangent plane. So, this is T_y and similarly here is T_y so both of them lie on a particular plane and that plane is called the tangent plane.

So, this is visually shown here this plane that you are looking at is the tangent plane at a particular point that particular point is P.

(Refer Slide Time: 46:50)



So, this tangent plane passes through the point A $(x_0, y_0, f(x_0, y_0))$ in the xyz space that we have seen it passes through this point P which we are calling at as A in our text. The tangent plane is tangent to the surface at this point A if we denote $z_0 = f(x_0, y_0))$, then the equation of the tangent at (x_0, y_0, z_0) is given by this. So, this is the equation of a tangent plane.

$z - z_0 = (x - x_0)f_1'(x_0, y_0) + (y - y_0)f_2'(x_0, y_0)$. So f_1' and f_2' are the partial derivatives of the function with respect to x and y respectively. Just think about it a bit and compare this with the two dimensional case there also you had the equation of a straight line where you knew the slope of this line derivative.

And you knew that it passes through (x_0, y_0) and there, what was the equation it was $y - y_0 = (x - x_0)m$. This is the derivative or the slope at this point (x_0, y_0). So, this is just a generalization of the case in case of three variables and it could be generalized further, if you have n number of variables, then it could be extended to that case as well. So, here I think I shall stop the discussion of partial derivatives. In the next lecture I shall start with something new. Thank you and have a nice day.