Mathematics for Economics 1
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Lecture No. 10
Rules of Differentiation

Hello and welcome to another lecture of the course Mathematics for Economics Part 1. So, the topic that we have been covering is called differentiation. It is quite a long topic, we have covered some parts of it and the particular sub topic within this topic that we were talking about in the last lecture is called limits. So, let us go to that particular sub topic.

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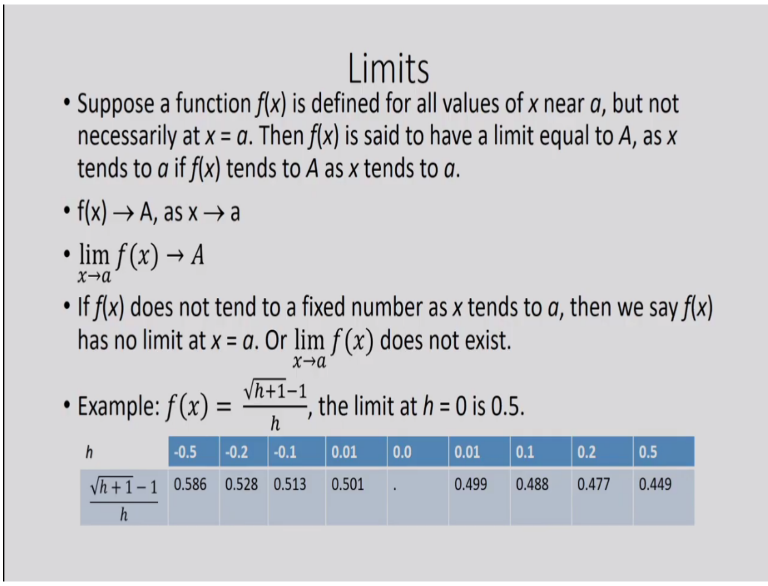

So, here is the definition of the limit. So, suppose the function $f(x)$ is defined for all values of x near small a, but not necessarily at $x = a$ then $f(x)$ is said to have a limit equal to A as x tends to a if $f(x)$ tends to A as small x tends to a. So, basically what one is talking about is that take any function.

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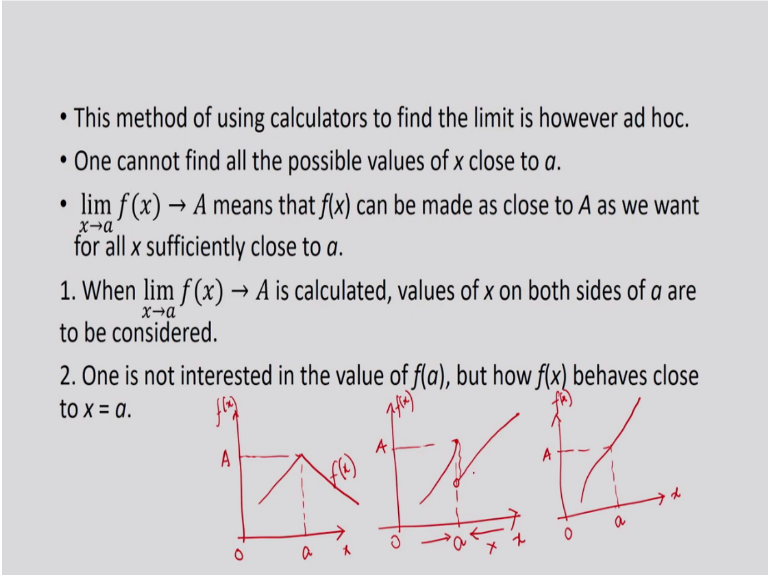

Let me draw a particular diagram to motivate the issue. So, if you have a function something like this and you have a particular value of x which is a, then this function the value of the function approaches this particular value A as x approaches the value a and we have examined one particular example which is like this suppose small $f(x)$ has this particular $f(x) = \frac{\sqrt{h+1}-1}{h}$.

And we have taken values of h which are close to 0, at $h = 0$, the value of this function becomes undefined something divided by 0 cannot be defined, but as you can see as we are taking h very close to 0, 0.01, 0.1, 0.2 etcetera, etcetera the value of the function approaches 0.5 and from the left hand side also as this should be minus as h approaches 0 from the negative side you have minus 0.05 minus 0.2, minus 0.1 minus 0.1 the value of this function is approaching the value of 0.5.

So, therefore by this sort of heuristic demonstration we can say that the limit as h goes to 0 the limit of this function is 0.5. Here for a general case this limit of this function is A and where we can say that the limit does not exist suppose you take a function like this so here it is defined, but suppose if you take x greater than small a, then this function is actually starting from here.

Then we say this limit is not there, the limit does not exist of this function at $x = a$ because from the left hand side there you can make the value of the function to as close to A as possible as x becomes very close to a the value of the function becomes as close to A as possible, but that is from the left hand side, but from the right hand side this is not true because from the right hand side you cannot make the value of the function as close to A as possible because as you can see that there is a gap.

So, we say that the limit does not exist at this particular value of x is equal to a. But suppose you take a kinked sort of shape ,then can we say the limit exist. So, suppose the function is something like this. Actually here you can say that the function has limit as x tends to small a then the value of the function tends to A so the limit of the function exist at $x = a$.

So, if there is a kink like this then there is no problem, but if you have a gap like this then there is a problem of the existence of the limit.

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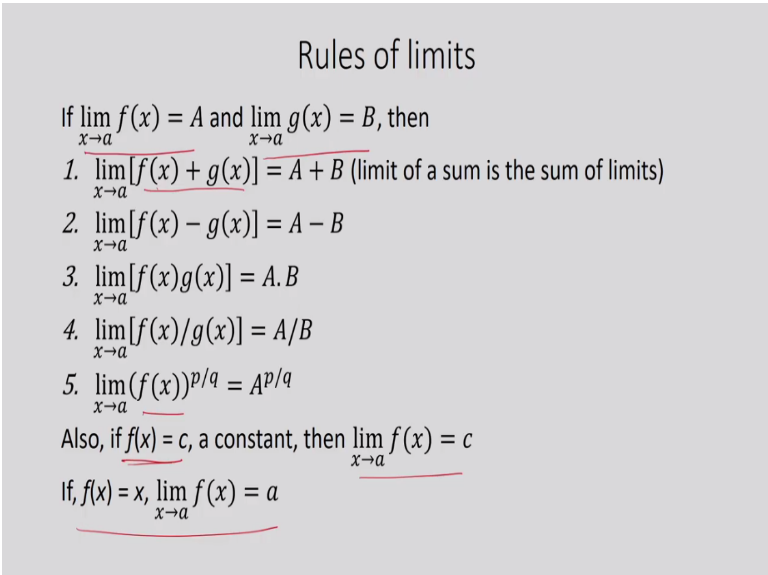

Now there are some rules of the limits so that is what we want to start in this class. So, suppose this is given to us that is $\lim_{x\to a} f(x) = A$and $\lim_{x\to a} g(x) = B$, then what about the sum, what about the sum of $f(x)$ and $g(x)$. As it turns out the limit of the sum is the sum of the limits.

On the left hand side you have the limit of the sum that is $f(x) + g(x)$and on the right hand side you have the sum of the limits which is $A + B$. So, this seems quite intuitive. Similarly, the second rule is if you take the gap that is the subtraction of $f(x)$ and $g(x)$ if you subtract $g(x)$ from $f(x)$ and take the limit of that, then the result is the same as if you take the limit of f and then from that subtract of g in both cases x is tending to a.

Thirdly, if you multiply two function so if you have $f(x)$ and $g(x)$ and take the product of $f(x)$and $g(x)$ and take the limit of that as x tends to a, then the result is same as limit of $f(x)$multiplied by limit of $g(x)$ so that will be A multiplied by B. Fourthly, if you take the ratio of two functions, then what about the limit of that ratio and this is equal to the ratio of the limits because on the right hand side you have A divided by B.

What is A, A is limit of $f(x)$as x tends to a and B is limit of $g(x)$ as x tends to a. So, limit of the ratio is the ratio of the limits and then you have an interesting property of power. So, you have $f(x)$ raise to the power of p divided by q and then you are trying to take the limit of that and the result is same as you take the limit of $f(x)$ and which is A and raise the power to p divided by q. So, these are the 5 sort of simple rules of limits.

There are some specific functions that one can take, suppose it is a constant function so suppose $f(x)$is the same. It is constant, suppose it is c and you want to find out what happens to the limit as x tends to a and the limit is same as the value of the function because the value of the function is not changing in the limit also it is the same as the value. What about very simple function where $f(x) = x$.

So, this is like the 45 degree line in the first quadrant if I take x only non negative values and this is the 45 degree line, then the limit of this function as x tends to a will be a. The reason being that at $x = a$ the value of the function is a, so as x becomes very close to a in the limit the value of the function becomes very close to a.

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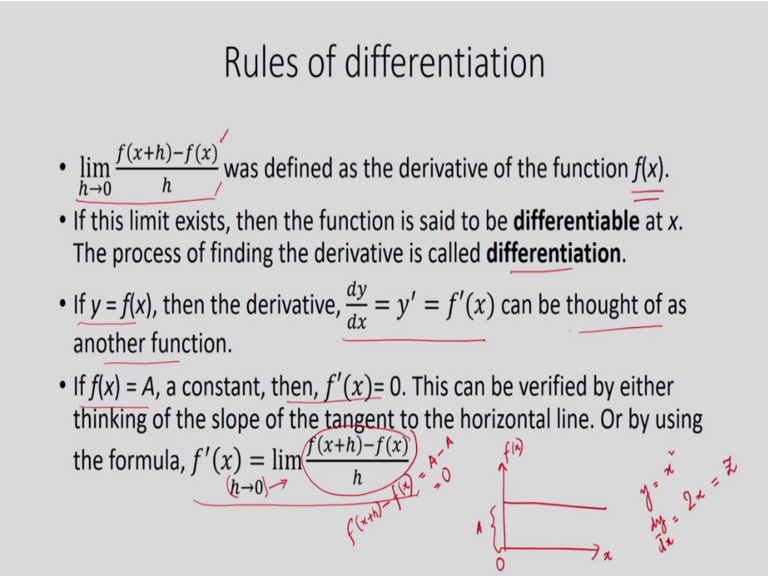

Now, we come to differentiation. We have already talked about the differentiation, but while talking about the differentiation the idea of limit was there. So, that is the reason why we had made this short sort of discussion about limits. Now, we are back to the discussion on differentiation and let us talk about some rules of differentiation. As we know the differentiation of any function can be done and this is called the derivative of this function $f(x)$ at any value x.

So, this is called the derivative $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. If this limit exists, then the function is said to be differentiable and we have seen that the limit may not exist in many cases and if the limit does not exist, then the function is not differentiable. The process of finding the derivative is called differentiation. So, there are two terms that we have defined here.

One is a function, can be differentiable it may not be differentiable at a particular point and this process of finding the derivative is called differentiation. Suppose, $y = f(x)$is the function, then the derivative is denoted by these notations, $\frac{dy}{dx} = y' = f'(x)$. It is sometimes written as dy dx and it is also written as y dash or sometimes it is called y prime and it is sometimes written as f dash of x it is also called as f prime of x.

Now what is being said is that it can be thought of as another function so this can be shown very simply suppose you have $y = x^2$,then by using this definition of derivative what you can say is that $\frac{dy}{dx} = 2x$. Now this can be seen as a different function altogether let us call this as Z, $Z = \frac{dy}{dx}$. Now, one can see that Z itself is a function of x.

So, that is the simple point that is being made that if you take the derivative of a function then the derivative itself is a function. So, now we can talk about another very simple function suppose $f(x) = A$which is a constant then what about the derivative. The derivative is equal to actually 0 $f'(x) = 0$. What is the proof this? This can be verified by two ways. You can think about the slope of the tangent to any horizontal line.

So, here $f(x)$ is what, $f(x) = A$,this value is A. Now, as we know the interpretation the geometric interpretation of derivative of a function is that it is the slope of the tangent to the graph of the function. Now, if you have a horizontal line, then the line itself is horizontal so its slope is 0. So, therefore the derivative is also going to be 0 so that is one way to think about it.

Another way to understand why the derivative should be equal to 0 is to go by the definition $f^{'}(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, then we know $f(x) = A$. So, $f(x + h) - f(x) = A - A = 0$.

So, the numerator is 0 so this is called if you remember the Newton ratio. The Newton ratio becomes 0 therefore it does not matter what limit I take the $f^{'}(x) = 0$.

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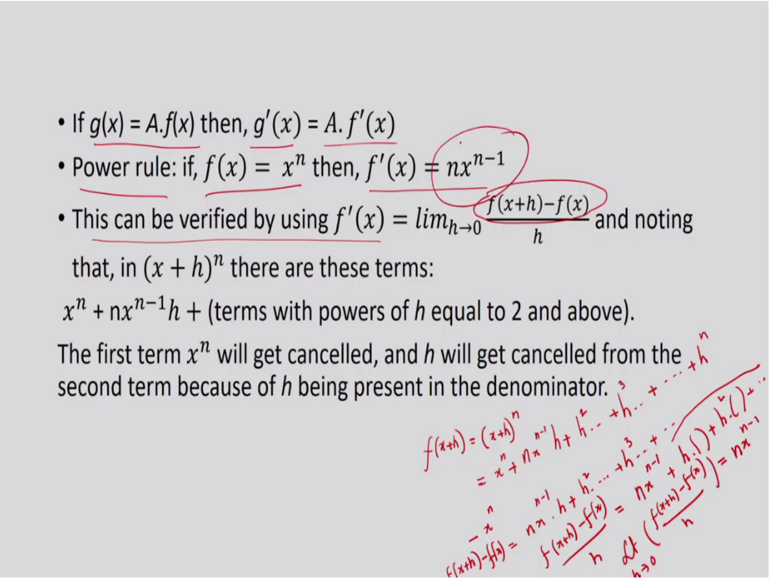

Now another function let us take which is a constant multiplied by a $f(x)$. So, $g(x) = Af(x)$ $f(x)$ is a function of x obviously then what should be the formula for the derivative of $g(x)$. Here, the derivative of $g(x)$ which you are calling as $g^{'}(x)$. $g^{'}(x) = A.f^{'}(x)$.

So, any function multiplied by a constant if I take that to be a new function then the derivative of this new function is constant multiplied by the derivative of that function. Now another important rule which is called the power rule so suppose you have $f(x) = x^{n}$then what about . $f^{'}(x)$ and this is given by n$x^{n-1}$and what is the proof of this.

We can verify this by again using the definition of derivative which is $f^{'}(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. Now in this case what is $f(x + h)$? It should be this because

$f(x) = x^n$ so $f(x + h) = (x + h)^n$. Now, this term can be decomposed as $x^n + nx^{n-1}h + n(n - 2)/2$......

But whatever these terms are the power of h should be 2 multiplied by something then h to the power 3 multiplied by something plus etcetera, etcetera and the last term will be h to the power n. Now from this I am going to deduct x^n that will be the numerator. $f(x + h) - f(x)$. So, from here I have to deduct $f(x) = x^n$.

So, this term becomes $f(x + h) - f(x) = nx^{n-1}.h + h^2.multiplied\ by\ something + h^3.multiplied\ by\ something +$...... and then I have to divide this by h. If I divide it by h then I have $\frac{f(x+h)-f(x)}{h} = nx^{n-1} + h.() + h^2() +$..

Now limit of this will be the other terms will vanish all this terms will go to 0. So you are left with only nx^{n-1}. So that is the final result that if you take the derivative of x^n, then what you get is this nx^{n-1}.

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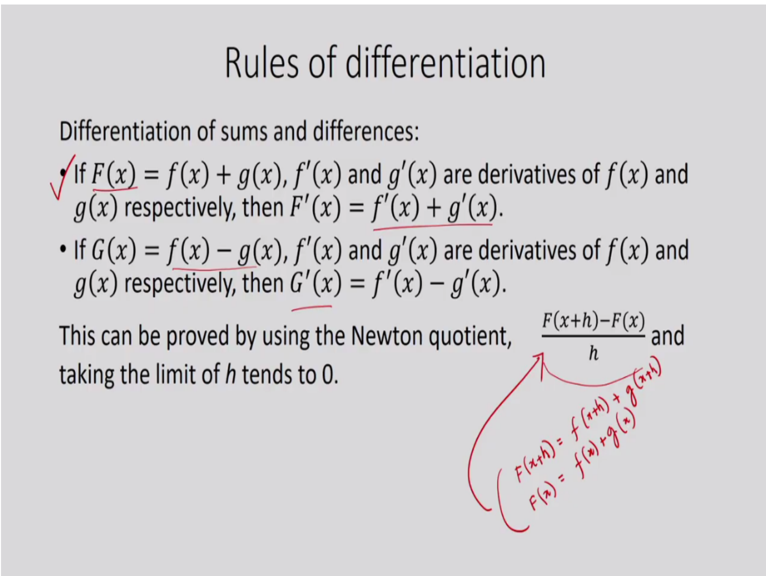

Now there are certain rules of differentiation. So, first we are talking about the summation so suppose you have two functions f and g and we know what are the derivatives are $f'(x)$ and $g'(x)$, then if I take the summation of them and call the summation as capital $F(x)$, then the derivative of $F(x)$ is equal to the summation of the derivatives that is $F^{'}(x) = f^{'}(x) + g^{'}(x)$.

So, basically derivative of summation is equal to the summation of the derivatives. Similarly, if I take two functions and deduct one function from another so you have f one function and g another function, $f(x) - g(x) = F(x)$ suppose and like before we know $f^{'}(x)$ and $f^{'}(x)$ are the derivatives of $f(x)$ and $g(x)$, then $G^{'}(x) = f^{'}(x) - g^{'}(x)$.

So, both these results are intuitive and how can we proof that we can actually start from here this is the Newton quotient and we know if we take the limit of h goes to 0 of the limit of the Newton quotient, then we get the derivative. So, if I use this and then I use the fact that $F(x + h) = f(x + h) + g(x + h)$ for the first result.

And similarly $F(x) = f(x) + g(x)$. And then we substitute these things back here and that will give us this result by simplifying the expression similarly for the second result.

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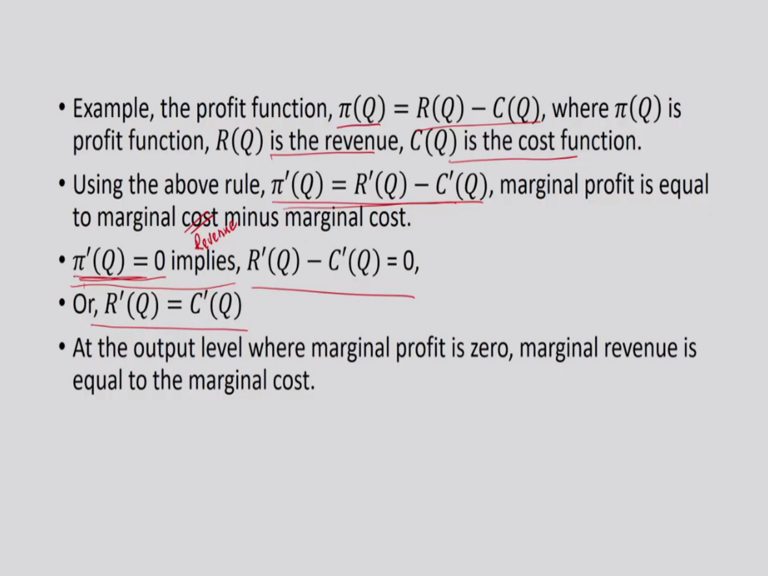

So, here is an example of economics. So, you have suppose what is called the profit function which is usually denoted by $\pi(Q)$, Q is the output level. So, profit function is expressed as $\pi(Q) = R(Q) - C(Q)$ where R is the revenue and C is the cost. So, $R(Q)$is the revenue function, $C(Q)$ is the cost function. Once the cost is deduced from the revenue, then we get profit very simple. Now, suppose we want to want to find out what is $\pi^{'}(Q)$?

Then we use this rule here and if we apply this rule here then $\pi'(Q) = R'(Q) - C'(Q)$. So, how can we interpret this the interpretation is that marginal profit is equal to marginal revenue this should be revenue, marginal revenue minus marginal cost. So, $R'(Q)$ is called marginal revenue and $C'(Q)$ is called marginal cost. Now one might be interested to find at what output level the profit is getting maximized.

The producers could be thought of as maximizing profits. So, what output level is the profit getting maximized that is a practical question. So, at the output level where profit is getting maximized one can show we can shall see it later on that level of output can be found out by setting $\pi'(Q) = 0$. So, if $\pi'(Q) = 0$ then the profit is getting maximized at that particular output level.

So, this is a necessary condition so if this is not satisfied, then the profit is not getting maximized as we shall see later on. So, if you apply this rule that is $\pi'(Q) = 0$, then from this above we get $R'(Q) - C'(Q) = 0$ that means $R'(Q) = C'(Q)$. So, this is a necessary condition of profit maximization. At the output level where marginal profit is 0, marginal revenue is equal to the marginal cost.

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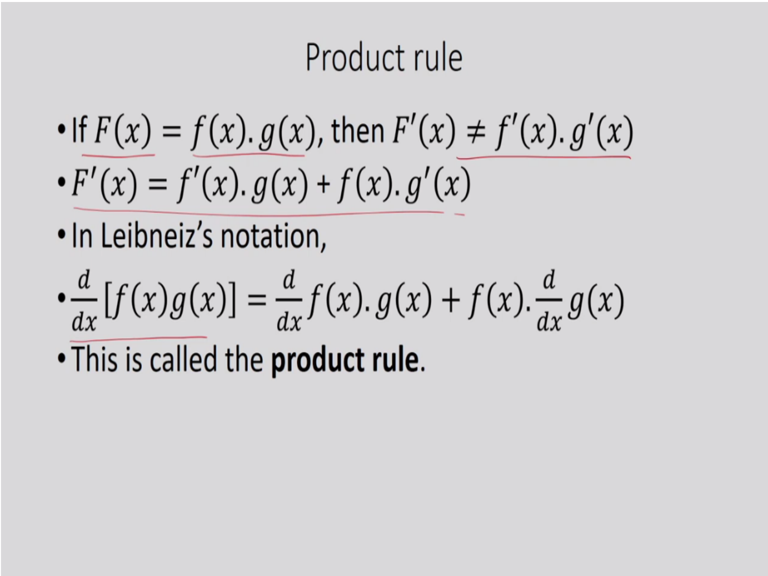

We come to another rule which is called the product rule of differentiation. So, here if you take the production of two functions f and g and suppose the product is called F. So, capital

$F(x) = f(x).g(x)$, then interestingly unlike what our intuition would tell us $F^{'}(x) \neq f^{'}(x).g^{'}(x)$.

It is actually equal to this. So, $F^{'}(x) = f^{'}(x).g(x) + f(x).g^{'}(x)$. So, if we use the Leibniz notation then this is how it looks like $\frac{d}{dx}[f(x).g(x)] = \frac{d}{dx}f(x).g(x) + f(x)\frac{d}{dx}g(x)$. This is called the product rule.

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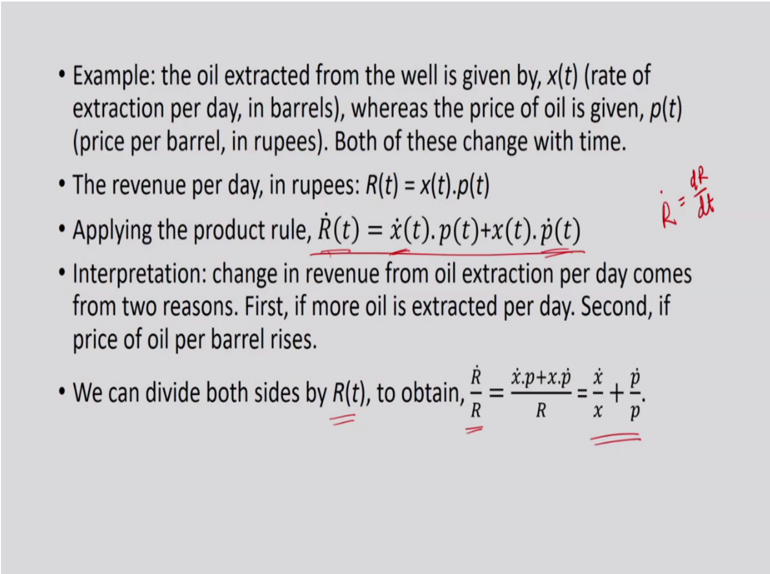

So, here is an example of how we can use the product rule. Oil is extracted from well and suppose it is given by $x(t)$ is the rate of extraction per day in barrels. So $x(t)$ amount of oil is being extracted whereas the price of oil is given by $p(t)$ which is price per barrel in rupees. Both of this that is x and p they change with time that is why they are functions of t. Now the revenue per day in rupees will be what? Capital $R(t) = x(t).p(t)$.

The amount of oil that is being extracted multiply that by the price of oil you get the revenue per day. Now we can now use the product rule. So, we are calling it as $\dot{R}$. So, $\dot{R} = \frac{dR}{dt}$. So, this is the notation that we generally use when time is involved. Here t is time. So, applying the product rule what we should get is $\dot{R}(t) = \dot{x}(t).p(t) + x(t).\dot{p}(t)$.

So, what is the interpretation of this? On the left hand side you have $\dot{R}(t)$ which is change in revenue from oil extraction per day. As time changes how much the revenue is changing. So, it comes from two reasons. First, if more oil is extracted per day that is x is changing $\dot{x}$.

Second, if price of oil per barrel rises. So, if the price itself is rising even if the production is remaining constant revenue will rise.

So therefore, you have this $\dot{p}$ term and from this we can actually divide both sides by $R(t)$ which is the revenue and we get this expression $\frac{\dot{R}}{R} = \frac{\dot{x}.p+x.\dot{p}}{R} = \frac{\dot{x}}{x} + \frac{\dot{p}}{p}$. So, $\frac{\dot{R}}{R} = \frac{\dot{x}}{x} + \frac{\dot{p}}{p}$.

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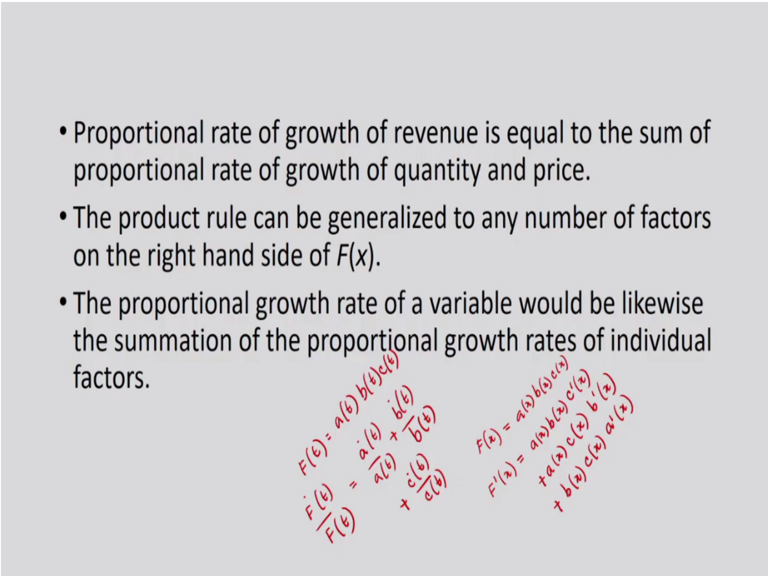

And in words what does it mean? It means the proportional change, proportional rate of growth of revenue is equal to the sum of proportional rate of growth of quantity and price. Remember the derivative of R with respect to time if I divide that by R itself that is the expression for proportional growth of the revenue. So, therefore proportional growth of revenue is equal to the sum of proportional rate of growth of quantity and price.

The product rule can be generalized to any number of factors on the right hand side of $F(x)$. What is meant by this is that suppose $F(x)$ is instead of just small f and small g I take this as $F(x) = a(x)b(x)c(x)$. Suppose, only three functions are involved so you do not have two function on the right hand side, but three functions of x, then what happens to the product rule.

Here the product rule will be the following you take $F'(x) = a(x)b(x)c'(x) + a(x)b'(x)c(x) + a'(x)b(x)c(x)$. So, you are taking one derivative at a time. Out of the three functions you are taking derivative of one function at a time and keeping the others as function itself they are not differentiated. So, you are going to get three terms on the right hand side. The proportional growth rate of a variable would be likewise be the summation of proportional growth rate of individual factors.

So, if you have something like this that $F(t) = a(t)b(t)c(t)$and suppose you want to find out this, $\frac{\dot{F}(t)}{F(t)}$The proportional rate of growth of F, then what is being said is this will be equal to this, $\frac{\dot{F}(t)}{F(t)} = \frac{\dot{a}(t)}{a(t)} + \frac{\dot{b}(t)}{b(t)} + \frac{\dot{c}(t)}{c(t)}$. So, it is basically summation of three terms each of these terms denote the proportional rate of growth of each of this functions, three functions on the right hand side.

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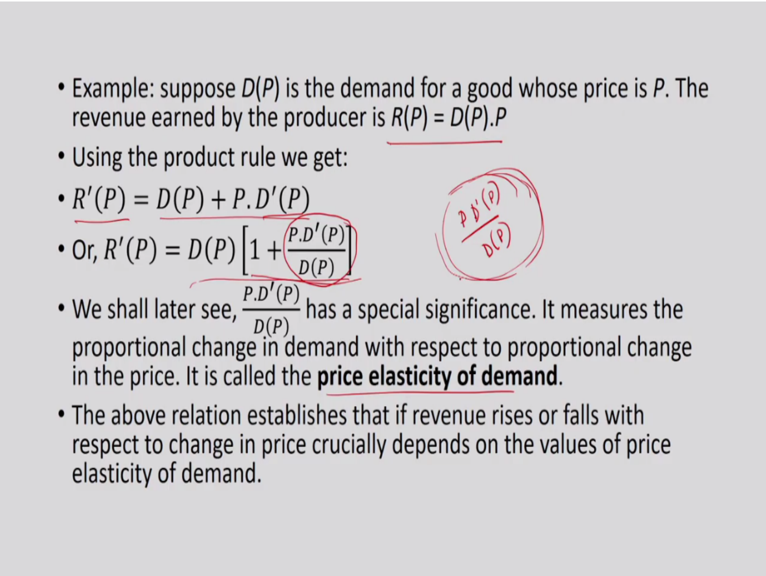

Here is another example where we can use these rules of the product rule. Suppose, $D(P)$ which is the demand for a good whose price is given by capital P. So D is demand, P is price D is a function of price. The revenue earned by the producer will be $R(P)$ so $R(P)$is how much you are selling multiplied by the price of the good that is revenue, how much are you selling that is given by the demand because people will be demanding and that is why you will be selling. So, $R(P) = D(P)P$.

Now, we can use the product rule here. So, $R'(P) = D(P) + PD'(P)$ will be given by this so what is the thing that is going on in the background I have jumped one step. This is basically $D(P)$ multiplied by the derivative of P with respect to P. The derivative of P with respect to P is 1 so nothing is coming here plus P multiplied by the derivative of the demand function with respect to price which is $D^{'}(P)$, $PD^{'}(P)$ or we can take $D(P)$ common on the right hand side.

And what we should get is $R^{'}(P) = D(P)[1 + \frac{PD^{'}(P)}{D(P)}]$. Now this term is an interesting term. Let me write it separately It is $\frac{PD^{'}(P)}{D(P)}$. It has special significance, it measures the proportional change in demand with respect to proportional change in the price. It is called as the price elasticity of demand. So, what is this term $PD^{'}(P)$.

$D^{'}(P)$is the change in the quantity demanded with respect to instantaneous change in the price and whole thing is divided by the quantity demand which is $D(P)$. So, this is called the price elasticity of demand. Now there is something called the law of demand by dent of which the $D^{'}(P)$ is negative which means if price rises the demand actually falls so $D^{'}(P)$ is negative.

So, the above expression is generally negative. The above relation establishes that if revenue rises or falls with respect to change in the price it crucially depends on the values of price elasticity of demand. So, if I take the absolute value of this it is negative, but if I take the absolute value and if it is greater than 1, then this whole expression will be negative. So, that will mean that if price rises then the revenue will fall or if price falls then the revenue will rise.

So, whether the revenue rises or falls with respect to price it crucially depends on the expression this expression the price elasticity of demand.

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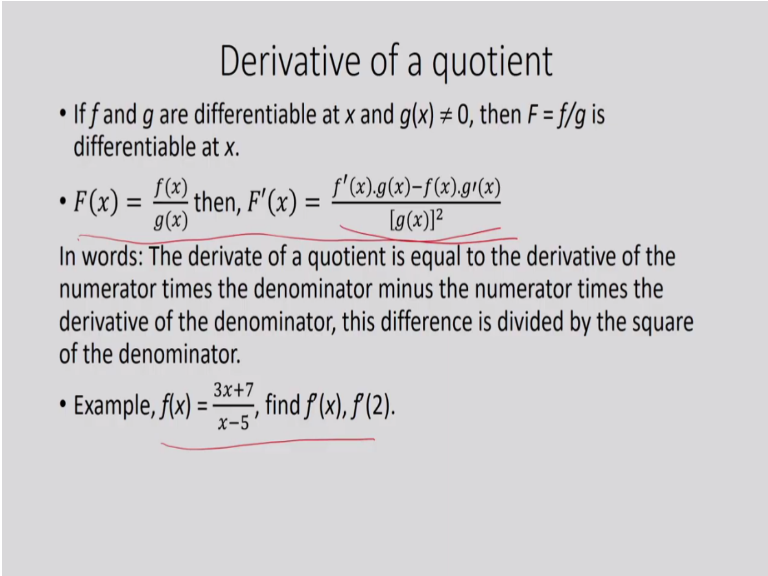

Now we come to another rule which is the derivative of a quotient. So, if f and g are differentiable at x and suppose $g(x) \neq 0$gx is not equal to 0, then capital $F = f/g$ is differentiable at x. Now, if it is differentiable, then what is the derivative? So, that is given in the next line so we have capital $F(x) = \frac{f(x)}{g(x)}$ and $F^{'}(x) = \frac{f^{'}(x).g(x) - f(x).g^{'}(x)}{[g(x)]^{2}}$.

In words the derivative of a quotient is equal to the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator. This difference is divided by the square of the denominator that is the explanation in terms of words. So, we take one example to actually grasp what is going on here we take an example where $f(x) = \frac{3x+7}{x-5}$.

We have to find two things $f^{'}(x)$ *and* $f^{'}(2)$. So, if I can find out $f^{'}(x)$, then that is a function of x we have seen that the derivative itself is a function of x. So, to find $f^{'}(2)$ I just have replace x by 2. So, the first task will be to find $f^{'}(x)$.

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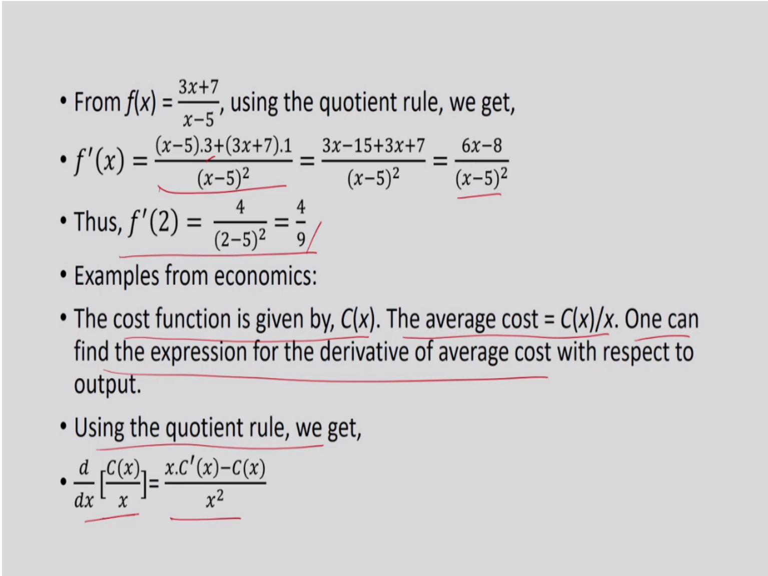

So, we use the quotient rule we get this. So, in the denominator you have the square of the denominator of the original function which is $(x - 5)^2$ and in the numerator you first have the denominator $(x - 5)$ multiplied by the derivative of the numerator which is 3 plus the numerator which is $(3x + 7)$multiplied by the derivative of the denominator which is equal to 1.

And then we simplify this it becomes $(3x - 15 + 3x + 7)$and that turns out to be $(6x - 8)$ and obviously this is going to be divided by $(x - 5)^2$. So, we have found out what is $f'(x)$, but there is another part which is we have to find out what is $f'(2)$ that is easily found out by replacing x by 2 so if we do that then on the numerator you have 12 minus 8 which is 4.

And divided by 2 minus 5 whole square which is 9. So, therefore it becomes 4 divided by 9. Again I talk about another example from economics. Now we know there is this thing called the cost function $C(x)$, x is the output, C is the cost. The average cost is given by $\frac{C(x)}{x}$. Basically we are dividing the cost function by the quantity produced that gives us the cost per unit of output and that is called the average cost.

Now one might be interested to find out the derivative of the average cost and here the independent variable is the output level. So, we have to take the derivative of the average cost with respect to output level. Now, we use the quotient rule. So, this is on the left hand side you have the $\frac{d}{dx}[\frac{C(x)}{x}]$ and if I use the quotient rule I get this expression $\frac{d}{dx}[\frac{C(x)}{x}] = \frac{x.C^{'}(x)-C(x)}{x^2}$. Now, let us simplify it further.

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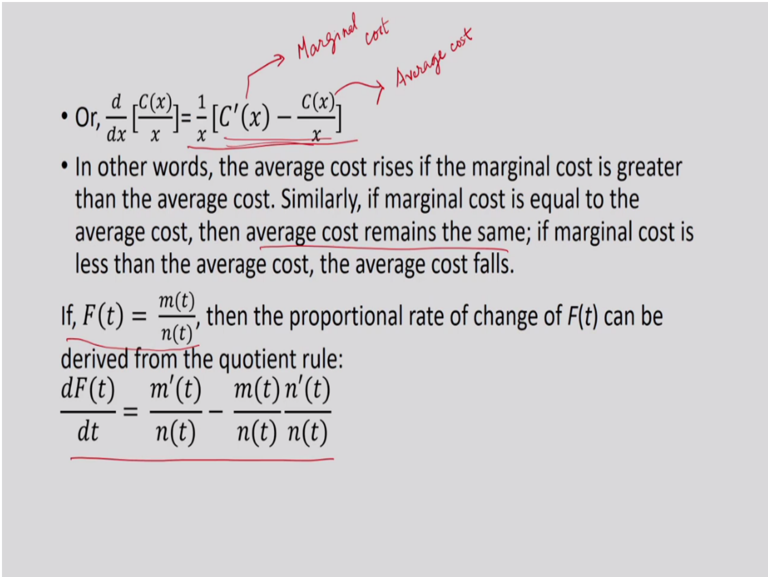

So, if I simplify it a little bit it becomes $\frac{d}{dx}[\frac{C(x)}{x}] = \frac{1}{x}[C^{'}(x) - \frac{C(x)}{x}]$. In other words the average cost rises if the marginal cost is greater than the average cost. Why am I saying this, because this first expression if you remember this is called the marginal cost. We have talked about this before and this itself is average cost. So, $\frac{d}{dx}[\frac{C(x)}{x}] = \frac{1}{x}$ multiplied by marginal cost minus average cost.

So, if it is the case that marginal cost is greater than the average cost, then the right hand side is positive. And if the right hand side is positive, then the left hand side is positive which means that as output level rises, average cost is rising. Similarly, if marginal cost is equal to the average cost, then on the right hand side you have 0 and if it is 0, then the $\frac{d}{dx}$ of average cost is equal to 0 that means the average cost does not change, it remains the same.

And finally, if the marginal cost is less than the average cost, then from the right hand side you are getting a negative expression, marginal cost is less than the average cost, in that case the left hand side is also negative that means the average cost is declining as the output level is rising. We will see that these things are quite important if one talks about what is called the theory of the firm.

Now we can also talk about proportional change. Here, suppose you have a function $F(t) = \frac{m(t)}{n(t)}$. So, both m and n are functions of t. We use the quotient rule here and we will get this expression that $\frac{dF(t)}{dt} = \frac{m'(t)}{n(t)} - \frac{m(t)n'(t)}{n(t)\,n(t)}$. I have jumped some steps here, but this is what you are going to get by using the quotient rule.

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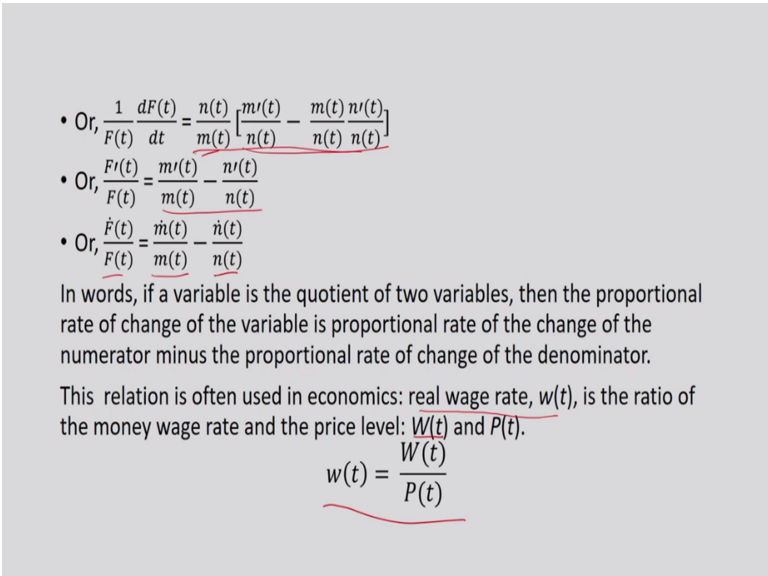

Now from the above expression, then I divide both sides by $F(t)$ and I will get this expression $\frac{1}{F(t)}\frac{dF(t)}{dt} = \frac{n(t)}{m(t)}[\frac{m'(t)}{n(t)} - \frac{m(t)n'(t)}{n(t)\,n(t)}]$ because $F(t) = \frac{m(t)}{n(t)}$ and then I multiply this term with the terms in the third brackets and I will get this expression $\frac{F'(t)}{F(t)} = \frac{m'(t)}{m(t)} - \frac{n'(t)}{n(t)}$ and which means that $\frac{F'(t)}{F(t)} = \frac{m'(t)}{m(t)} - \frac{n'(t)}{n(t)}$.

Or in other words since $F'(t) = \dot{F}(t)$ so $\frac{\dot{F}(t)}{F(t)} = \frac{\dot{m}(t)}{m(t)} - \frac{\dot{n}(t)}{n(t)}$. So, if I express this in words if a variable is the quotient of two variables then the proportional rate of change of the variable is the proportional rate of change of the numerator minus the proportional rate of change of the denominator.

This is the proportional rate of change of the numerator and this is the proportional rate of change of the denominator. You take the difference of these two you get the proportional rate of change of the ratio. This relation is again quite important in economic analysis. So, for example there is something called the real wage rate so $w(t)$. It is the ratio of the money wage rate and the price level.

So, suppose W is the money wage rate and P is the price level then the real wage rate is defined by this so $w(t) = \frac{W(t)}{P(t)}$ So, the total amount of money people are getting as wages divided by the price level that is called the real wage rate. Now, we can use the rule here that we have just got.

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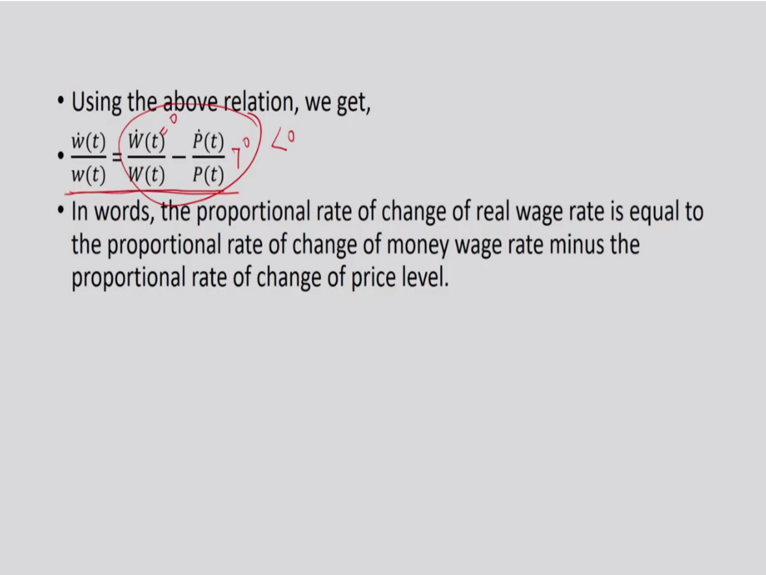

Then we should get the following expression. In words what we are getting is that the proportional rate of change of the real wage rate is equal to the proportional rate of change of money wage rate minus the proportional rate of change of price level and this is not very difficult to understand if you think about it intuitively the real wage rate of the worker suppose what does it mean? It means the purchasing power.

We are dividing the amount of money that they are getting by the price level. Now that real wage rate can rise because of two reasons. One is that the money that they are getting that is rising or it is possible that price is declining. So, in both cases their real wage rate will rise. So, that is why you have the proportional rate of change of the real wage rate is equal to the proportional rate of change of the money wage rate minus the proportional rate of change of the price level.

So, it also says that suppose the actual money that the people get is constant that is not changing, but the prices are rising in that case on the right hand side this will be equal to 0 where this is positive. So, the entire term becomes negative which means that in that case the real wage rate of the workers will be declining and this is quite intuitive. We shall start with this topic in the next lecture partial differentiation.

So, we have today covered the topic of differentiation proper, we have not talked about partial differentiation and we have talked about different rules of differentiation, the summation of two functions if we take the derivative of that, then what happens. If we take the product of two functions and take the derivative if we take the quotient of two functions and take the derivative, then what are the expression that one is likely to get.

We would also talked about power rules, we talked about the derivative of a constant term etcetera, etcetera and let us call it a day. So, I shall see you tomorrow with another topic which is partial differentiation. Thank you.