Introduction to Market Structures
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Lecture No. 22
Dynamic Games, Backward Induction

Welcome to my course Introduction to Market Structures. So, we were doing game theory and we have completed the static game of complete information and now today we are going to do dynamic games of complete information.

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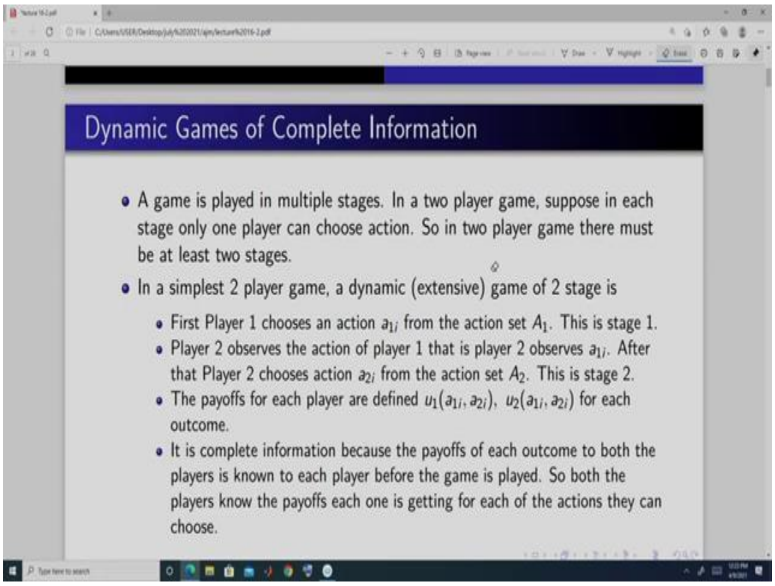

So, in a dynamic game actually game is played in multiple stages. So, in a static game, game is played only once and it is played simultaneously. Here the players may play the game sequentially one after another.

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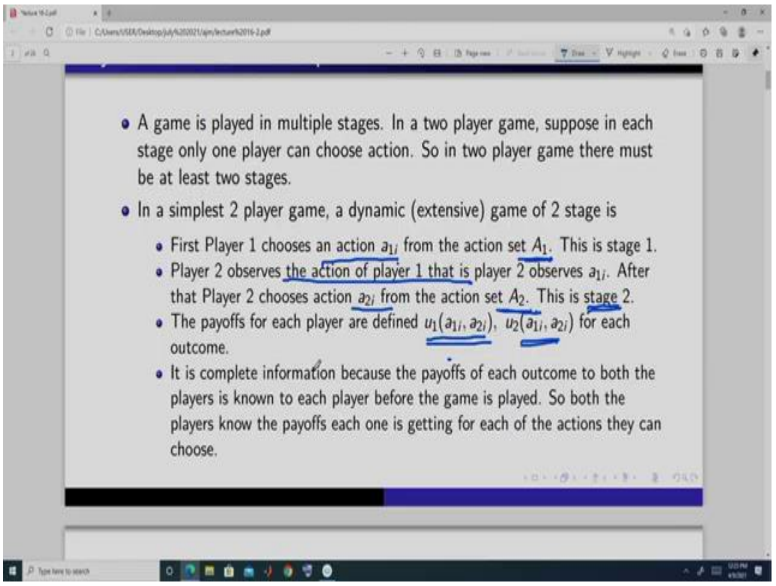

So, in a simplest possible game, if we have two player, so it can be something like this. So, Player 1 first chooses an action from, chooses an action ai from a set of action that is **A**i. So, that is stage 1. And Player 2 observes the action chosen by Player 1 and then Player 2 chooses an action a2i from its set of action that is capital A2. So, this is simplest, you can say, two stage game, where this is, in second stage Player 2 makes the decision.

And we define the utility function or a payoff function for this for each player and for each outcome, okay. And now here what I have said, Player 1 moves first, Player 2 moves second and Player 2 while making decision knows what is the action chosen by Player 1. So, this is a complete information and perfect information game. Why complete information, because the payoff of each player for each outcome is known by both the players.

So, Player 1 knows all the possible outcomes of Player 2 and its own outcome payoffs and also Player 2 knows what are the possible payoffs in each of the outcomes in, of Player 2, Player 1 and also of itself, okay. So, they know what each other's payoff. So, that is why it is a complete information and perfect because when Player 2 is choosing it knows the history, so it knows what is the action of Player 1, okay because it is moving second. So, because of that it is a perfect information.

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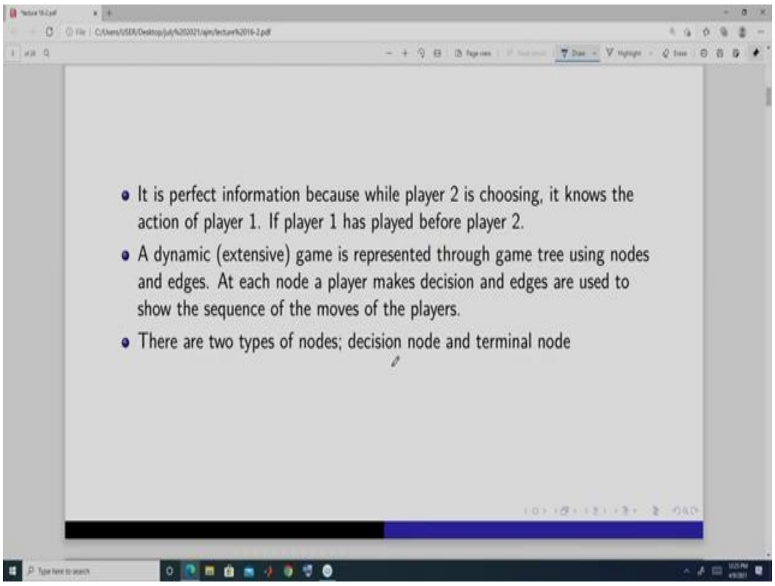

So, again dynamic games are generally represented through a game tree and in the game tree we denote it through edges and nodes. Nodes are where each player is making some decision or taking, choosing their action and edges are to denote the movement of the game, okay.

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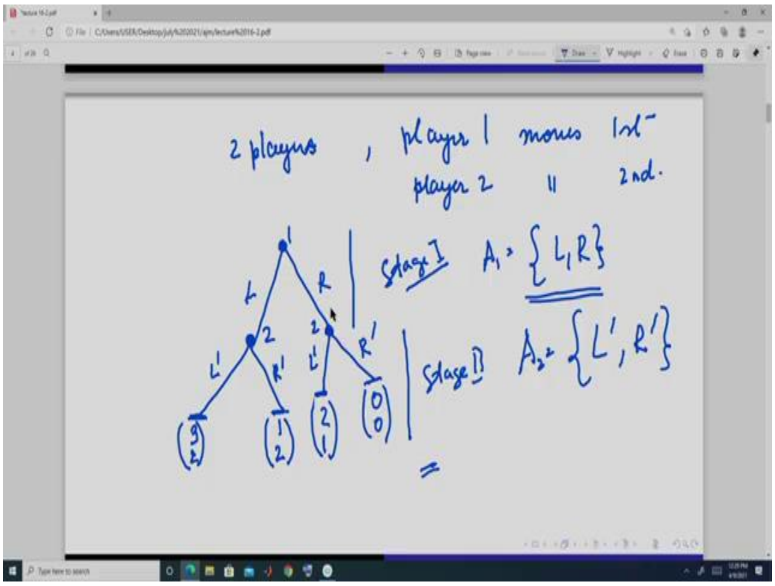

So, you will see one example here. So, this is suppose there are two players, okay. And player 1 moves first, player 2 moves second. So, we represent the game in this way player 1. Suppose the action set of player 1 is L and R, okay, so this is you can say a two player game, where player 1, action set of player 1 is this L and R. So, player 1 moves first, so it is this node. It is the starting point of the game and so this is the stage 1 you can say and this nodes from here is stage 2.

And here in stage 1 player can choose any action from this set, either it can choose L or it can choose R. And in stage 2 player 2 knows either it can be in this node or it can be in this node. If L is chosen, it is in this node, if R is chosen it is in, player 2 is in this node. And here, player 2's set of action is less L dash and R dash. It is same for, in each situation or in contingency, okay and these are the payoffs. So, this tree is, the first element is the payoff of player 1 and the second element is the playoff of player 2. So, like this. So, this is a terminal node.

In the terminal node, we specify the payoffs. So, these are the end of this game. And these nodes are the decision nodes and these are the edges. And based on these edges we specify the movement or how the sequence in which the game is played like first player 1 will choose and player 2 is going to observe that and then player 2 is going to choose, okay. So, this is an example of two stage game which is played between two players and it is a perfect information. Why it is perfect, because it is, player 2 knows what is, in which node it is, when it is choosing and making a decision. Here again player 2 knows in which node it is lying when it is making a decision, okay.

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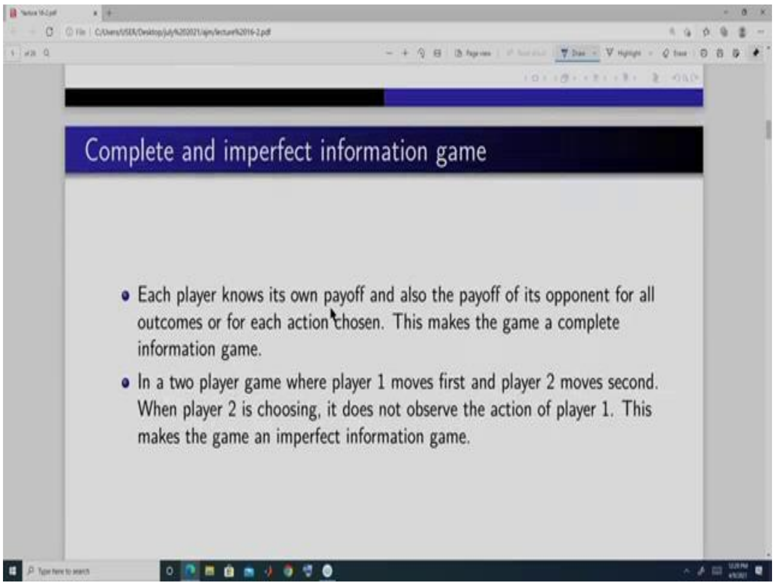

Next there can be another type of game and that is complete information and imperfect information, okay. So, it is a complete information. Why, because player 1 knows the payoffs of each outcome of itself and also the payoff of player 2. Similarly, player 2 knows what its payoff for each outcome and also the payoffs of player 1 for each outcome, okay. So, that is why it is a complete information.

But it may be imperfect, because while player 2, suppose the game is played between player 1 and player 2 and it is a two stage game, where player 1 moves first and player 2 moves second, but while player 2 is moving or choosing its action, then player 2 does not observe the action chosen by player 1.

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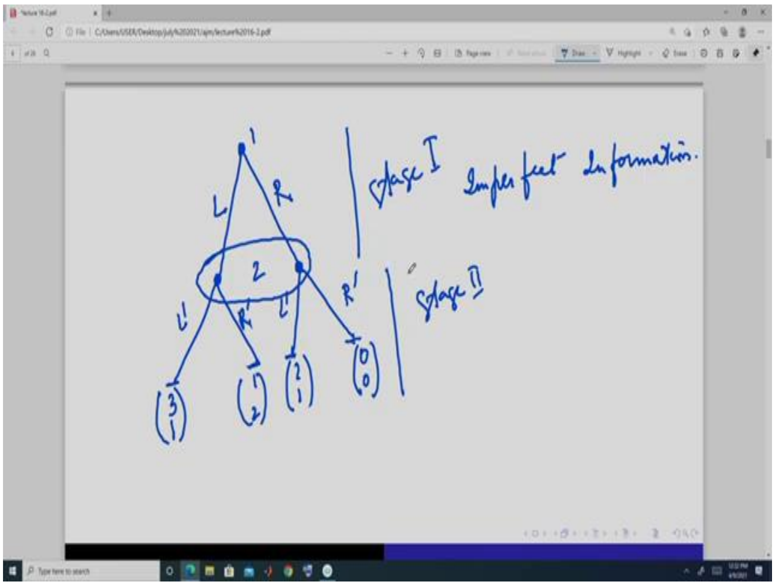

So, if we take the similar game as we have discussed, so it will look something like this. This is player 1, this is player 2, L, R, L dash, R dash, okay. So, here player 1 has chosen and player 2 is supposed to choose here and then we represent it through a, like this and we write here. So, player 2 knows that player 1 can choose either L or it knows or it can choose R. It knows all these outcomes, all these payoffs. But he does not know what is the exact choice of player 1 either L or R. So, player 2 knows that it can be either in this node or it can be in this node, okay.

So, in this sense it is imperfect information, because it does not know in which node it is lying, okay or what is the action chosen by player 1 or by the previous player. So, here also it is, this is stage 1 and this is stage 2, okay. And these are the decision nodes and this is the terminal nodes. The game ends in this way, okay. So, this is the way to represent games in a complete information dynamic games that is it can be of perfect information and it can be of imperfect information, okay. So, first we will discuss perfect information and then later on we will do imperfect information, okay.

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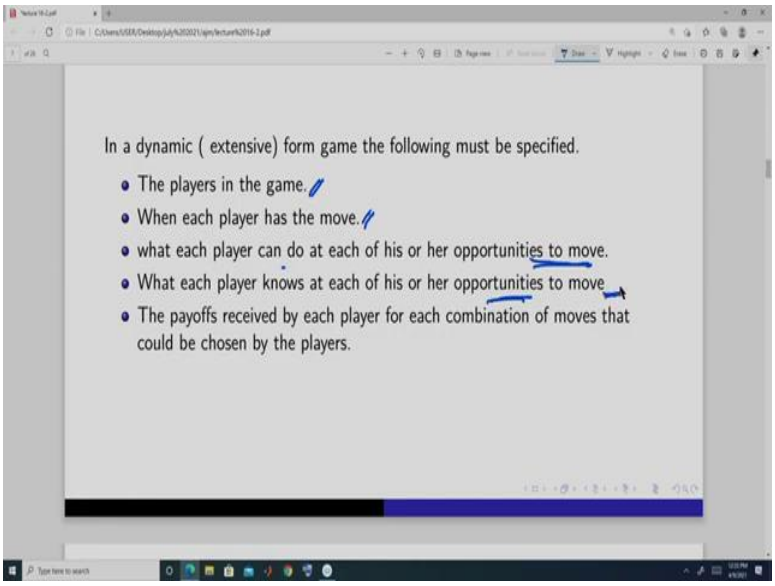

So, from here what do we get that in, while specifying a dynamic game or an extensive game, we should first specify the players, we should specify when each player has the move that is in this game we know that or in this game we know that player 1 moves first and player 2 moves second, similarly, here also we know player 1 moves first and then player 2 moves. And then we have to specify the action set so each player can do at, what each player can do at each of his or her opportunities to move when it is making a decision what it can choose from the set of actions, okay that has to be specified. And also we have to specify what each player knows at each of his or her opportunities.

So, here this is required, because it can be of this nature, because here player 2 does not know whether it is in this node or in this node because player 2 has not observed the action taken by player 1. But in this game player 2 has observed the action of player 1 in stage 1. So, player 2 knows whether it is in this node or it is in this node while making a decision in stage 2. So, we have to specify that this thing and we call this as the information set of each player. We will specify the, specifically define the information set later on, okay. And we have to also specify the payoffs. If we do not specify the payoffs, then we will not be able to play the game, okay.

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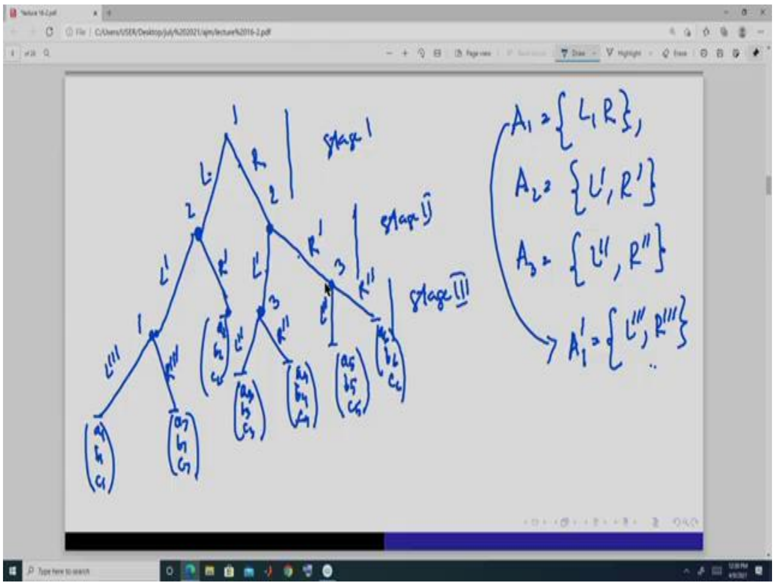

So, it is something like this. So, we have to specify that, for example, if we take another game like this player 1, player 2, player 2, player 1's action set is L, R. So, game ends here. Suppose this is a game between three players, player 1, player 2, player 3. So, the sequence is, this is stage 1, this is stage 2 and this is stage 3. So, stage 3 may be played or may not be played, okay depending on what is the action chosen by player 1, okay. So, this is stage 1, this is stage 2, and this is stage 3, okay.

So, we have specified the players player 1, 2, 3. We have specified their action set. So, player 1's action set is L, R, i.e $A_1 = \{L_1, R_1\}$ player 2's action set is L dash, R dash, i.e$A_2 = \{L', R'\}$ again player 3's action set is L double dash, R double dash, i.e $A_2 = \{L'', R''\}$in this game. And now we specify and again we have specified the moves, who moves first, who moves second, who moves third. We have also specified who knows what, like when player 2 is choosing, making a decision here it knows the action of player 1.

When player 2 is making a decision here, it knows the action of player 1. Again, player 3 knows the action of player 1 and also knows the action of player 2. Again, player 3 knows the action of player 2 and also player 1, so all this, and we specify the payoffs here suppose (a1, b1, c1), (a2, b2, c2), and this is (a3, b3, c3) and this is (a4, b4, c4) and this is suppose (a5, b5, c5) and this is (a6, b6, c6), okay, so this.

And here interestingly we make, can make this game further complicated. So, we can do like this. Suppose here we again, so if player 1 chooses L dash, then player 1 can again move and suppose it is and the game ends here. So, it is (a1, b1, c1) and this is (a7, b7, c7), okay. So, then the action set of player 1 is a1 and we have another action set and that is A dash that is L. This action set $A_1' = \{L''', R'''\}$ is here in stage 3. So, in stage 3 we may have either this game being played or this game or this game, okay. So, we can have different ways in which we can represent a game depending on what is our objective, okay.

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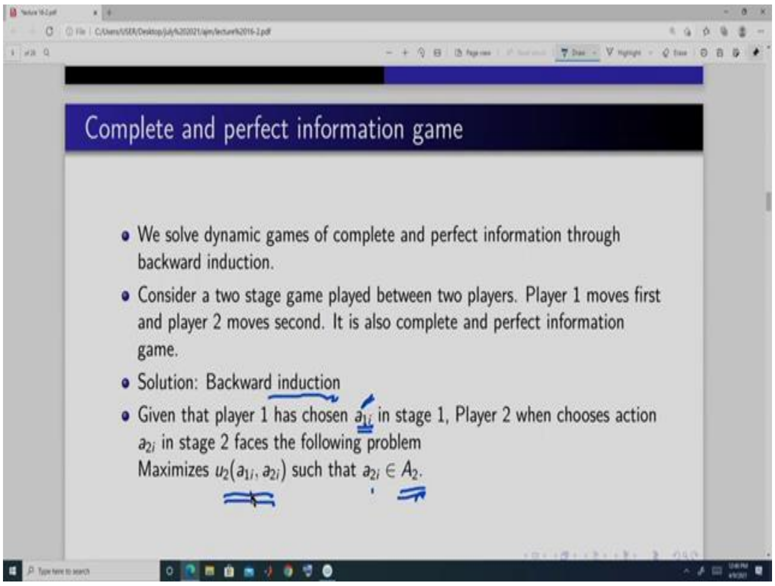

Now, the question is how to solve this problem? What is the solution concept? So, we use something called a backward induction method, okay. What do we do in a backward induction? So, let us take the simplest case that is we have two players, player 1 and player 2. Player 1 moves first and player 2 moves second. While player 2 moving or choosing, it knows the action of player 1. So, this is a simple game like this. Player 1 has moved first, player 2 has moved second in stage 2. It knows the action chosen by player 1, right? So, it knows that it is either in this node or it is in this node, okay.

So, here what player 1 will do? So, player 2 first, suppose player 1 has chosen an action from its set that is a1 then, in stage 2, so when we say we are using backward induction, then it means that we are moving from backwards. So, we will first go to the last stage that is in this case stage 2. So, we know suppose player 1 has chosen an action this in stage 1, then player 2 is going to do what, it is going to maximize its payoff function such that by choosing an action which belongs to his set of action, okay. So, given a action of player 1 that is a1i in stage 2 player 2 is going to maximize this, okay with, such that this action belongs to this set of action.

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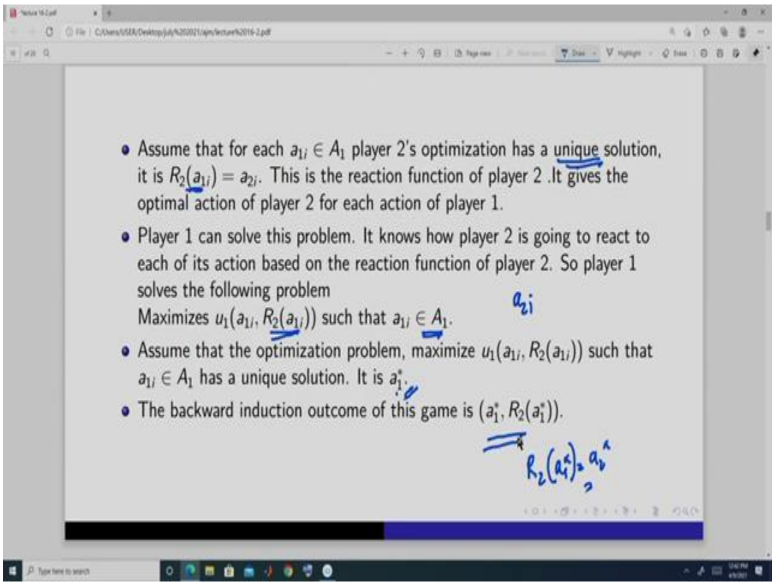

So, then it means what, then we will have a solution to this optimal problem and we assume that there is a unique solution, okay. So, then what do we get, we get a function like this- $R_2(a_{1i}) = a_{2i}$ and we call this a reaction function. So, we plug in the value of a1i that is a choice of action or the action of player 1 and then we get the optimal choice of player 2. So, this is a reaction function of player 2. So, it is a function of the action of player 1, okay.

Now, since it is a complete information game, so player 1 will know this reaction function. So, player 1 knows that if I choose a1i then what is going to be the decision of or what is going to be the choice of the player 2 in stage 2 based on this reaction function. So, in stage 1, what player 1 will do, will plug in this- $R_2(a_{1i})$ in case of a2i, okay. And then it will choose a1i such that it maximizes its payoff function and also it should belong to the set of actions, okay. And we assume that it has a unique solution, otherwise, it is a problem, and then we get a like this- a_1^*, okay an optimal solution.

So, the backward induction outcome is that a1 is always, a is always going to choose a1 star, and player 2 is going to choose a2 star which is always equal to a1 star like this. So, this is the backward induction outcome, okay.

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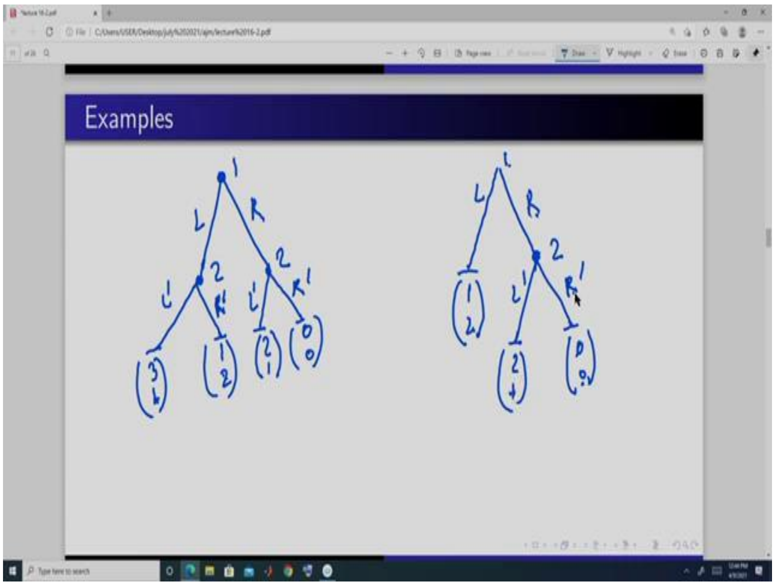

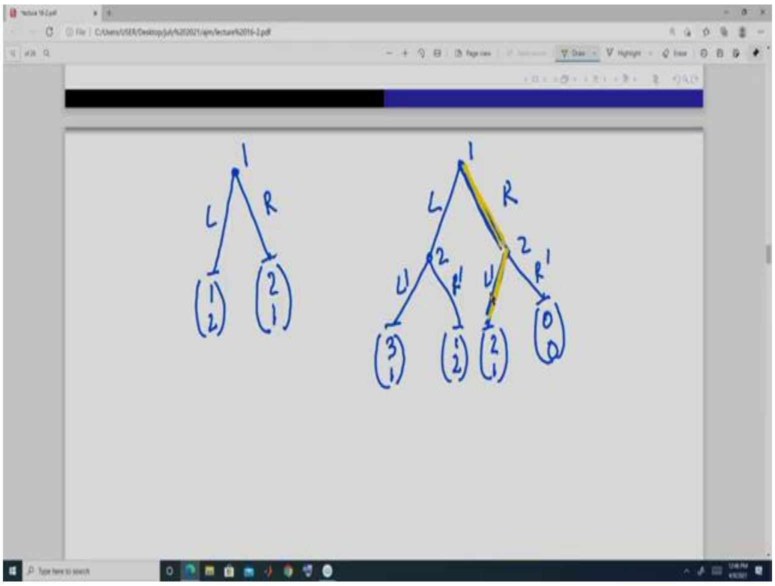

So, now, let us solve some examples. So, we have already specified one type of game. Let us take that game. Player 1 moves first, L, R, player 2 moves second L dash, R dash, okay. So, this is the game. So, this is the, in stage 1 player 1 makes a decision and in stage 2 after observing the action of player 1 player 2 makes a decision, okay. So, let us start with the second stage here or here. So, in second stage we can have decision can be taken in this node or decision can be taken in this node, okay. So, suppose decision is taken in this. So, player 2 knows that player 1 has chosen L. So, it is in this node.

So, here player 2 is always going to choose R dash, because it is compared to 2 and 1. So, R dash is this. So, what we do in this portion we represent that the optimal choice of player 2 given that player 1 has chosen L is to choose R dash, so the outcome is this. So, we delete this portion and we write it is going to be this, okay. So, it is this. Now, suppose player 1 has chosen R, then player 2 is here. If it is here, it will compare between 1 and 0. If it plays L dash it gets 1, if it plays R dash it gets 0. So, player 2 is going to choose L dash.

So, so player 1 while it is choosing it knows how player 2 is going to behave. If it chooses L player 2 is going to choose R. So, this is the outcome. If it chooses R player 2 is here, player 2 is going to choose L dash. So, it is outcome is this. So, player 1 will choose based on this. So, it will choose R, this. So, backward induction outcome in this game is actually, okay so backward induction outcome is this and this. This path is the backward induction outcome, okay.

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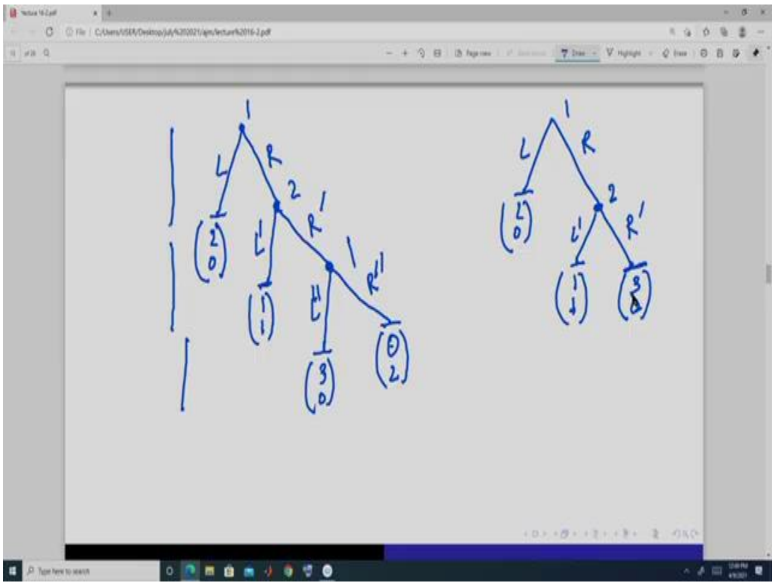

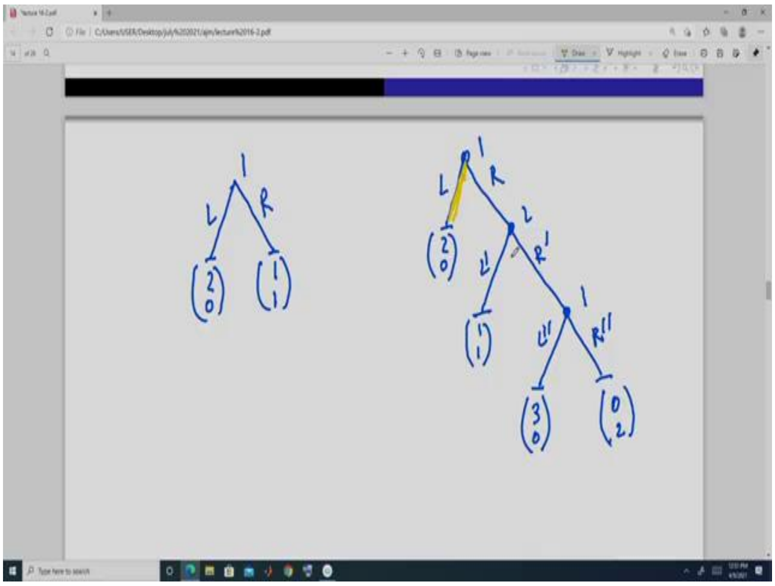

Now, let us do another example. So, suppose the game is played between two player, player 1 moves first and its action is L and R. Suppose, so this is a game played between two players player 1 and player 2. Player 1 moves first. It is, this is stage 1, this is stage 2 and this is stage 3, okay. So, if player 1 chooses L, then the game ends. If player 1 chooses R then player 2 can move. It can choose either L dash or R dash.

If player two chooses L dash, then again the game ends. If he chooses R dash, then again player 1 get a chance to move and it can choose from L dash, L double dash and R double dash, okay and these are the payoffs- (3,0), (0,2), okay. So, the first element is the payoff of player 1 and the second element is the payoff of player 2.

So, let us, since we are using now backward induction, let us start from this stage, so that means we have to start from this-L'', R'', okay. Here, so when player 1 is here suppose, it is making a decision from here, it is always going to choose this (3,0), because 3 is greater than 0. So, here we can write this. So, here it is going to get, player 2 is going to get, because player 1 is going to choose L double dash, so it is going.

Now, here player 2 is making a decision. Player 2, it will compare, if it chooses L dash, it is going to get 1. If he chooses R dash, it is going to choose, going to get 0. So, here player 2 will choose L dash, because 1 is greater than 0. So, it is, this becomes this- L= (2,0), R= (1,1). So, player 1 is always going to choose L, this rather than this, right? because 2 is greater than 1.

So in this game, the backward induction outcome is that the game is only played by player 1 and that is also only once. So, these edges represents its action, okay and movement who moves what, okay. So, the backward induction outcome here it is this. So, player 1 is always going to choose R and the game ends. Instead the game could have been played for three stages, but actually if we use backward induction as a solution concept, the game is played only once and that is in only one, first stage, okay. So, this is one example.

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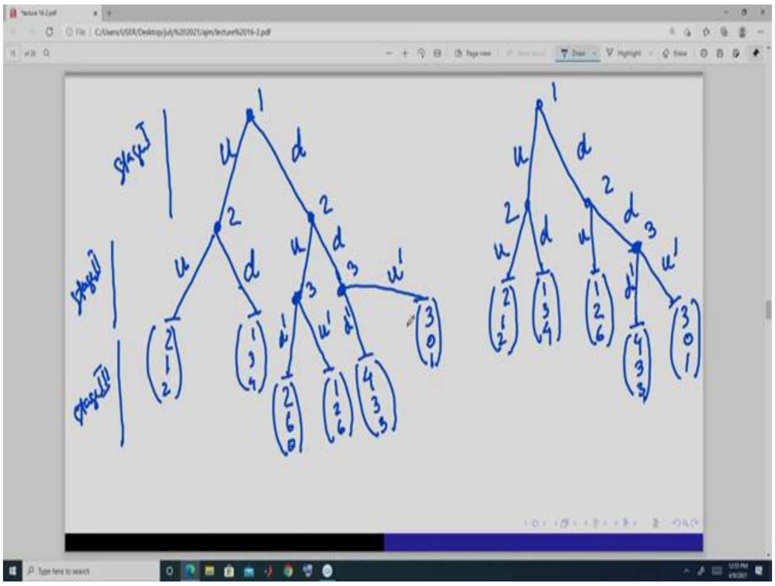

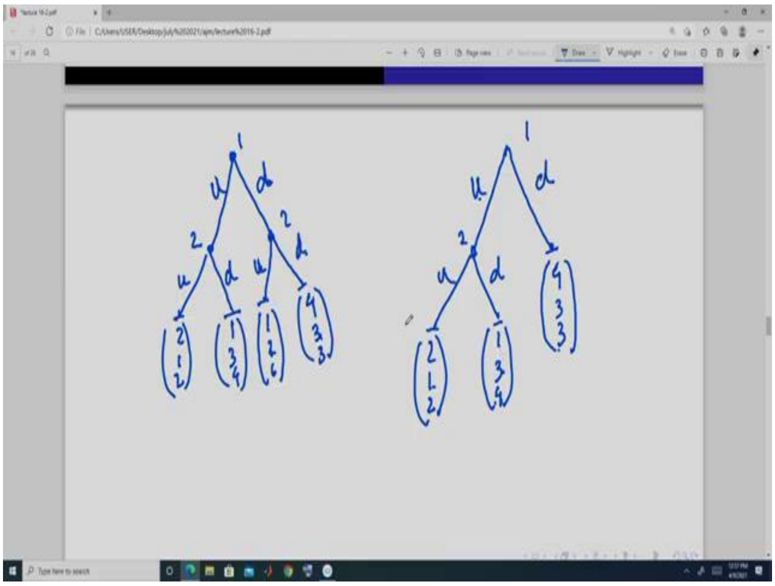

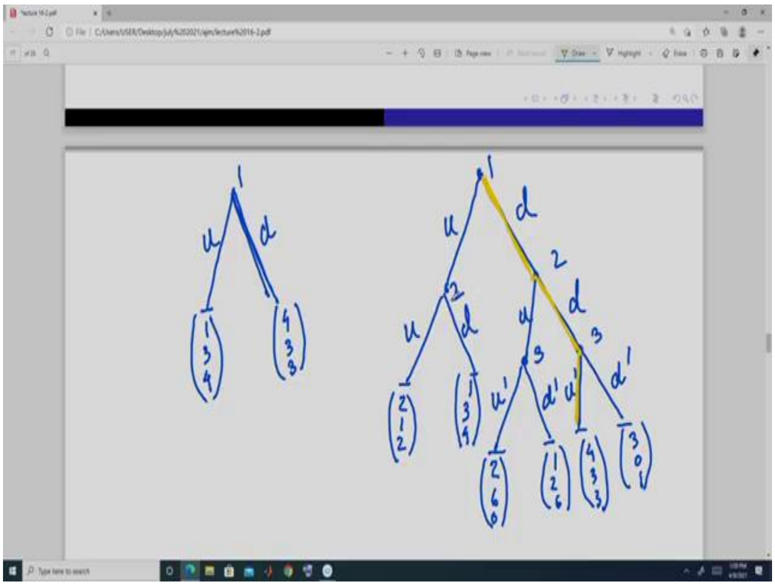

Now, let us do another example and this is a slightly bigger game. Let me use blue color, okay. So, this is the game. And this is a three stage game way. So, in the first stage, player 1 makes a decision, so this is stage 1. In stage 2, player 2 makes a decision either in this node or in this node. So, this is stage 2. So, the game may end at stage 2 if player 1 chooses u and player 2 is choosing either u or d. But here we can have another stage and that is from here for this. In stage 3 player 2 chooses or moves if player 1 chooses d and so then they will be in this path and player 2 will get the chance to make the decision. So, then we will have third stage, okay.

So, since we are using backward induction, let us start from the third stage. So, suppose here, suppose player 1 has chosen d and player 2 has chosen u and so it is here. So, player 3, so first element is the payoff of player 1, second element is the payoff of player 2 and third element is the payoff of player 3, so 0 to 6. So, player 3 is always going to choose u dash, because 6 is greater than 0. So, this game becomes. So, here if player 1 chooses d and player 2 chooses u, player 3 is always going to chose u dash, because 6 is greater than 0. So, here it is (1, 2, 6), okay.

Now, we have this portion. Here, player 3 if it chooses u dash it gets 1, if it chooses d dash it gets 3. So, player 1 is going to choose d dash, okay. So, the game is now like this. We have only two stages now. We have played the third stage. So, in stage 2, if player 2 is, if player 1 has moved, chosen d if it is here, it knows if it plays u, it is going to get 2, if it plays d it is going to get 3. So, player 2 is going to choose d. Here it is going to choose d. So, it is this. And here if player 1 chooses u, player 2 is going to choose d, because 3 is greater than 1, it is this and player 2 is going to choose d here, because 4 is greater than 1.

So, the backward induction outcome of this game is. So, the backward induction outcome here it is like this. Player 1 chooses d, player 2 chooses d and player 3 chooses u dash, okay. So, this path soon by these yellow colored edges gives me the backward induction outcome in this game, okay. So, these are some of the examples of, to solve dynamic game with complete information and also perfect information using backward induction, okay.

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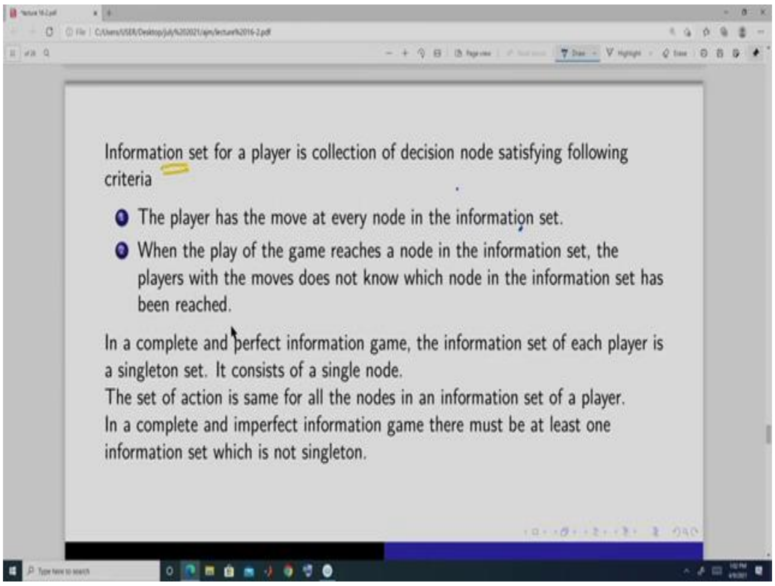

Now, we will specify the complete information game of perfect and imperfect information in a more precise way by using a concept called information set. And what is an information set? Information set is actually a collection of nodes for each player, okay. So, each player will have some information node. So, for each, so you can understand information set as that for each stage we will have one information set and for one player, okay depending on the player who is moving or making decision in that stage. So, it is a collection of nodes.

So, it gives the player has the move at every node in the information. So, since it is a collection of nodes, so in each node which belongs to an information set a specific player has to move or has to make a decision in each of these nodes. And if there are many nodes in a information set, the player, so information set is specific to a player. So, the player will never know in which node it is lying or because player 1 will not observe the history if there are multiple nodes in an information set, all the previous histories.

So, when the play of the game reaches a node in the information set, so you have an information set and the game while playing it has reached a node then the players with the moves does not know which node in the information set has been reached, okay. So, these definitions have been taken from Gibbons, A Primer in Game Theory, you can follow from there also, okay.

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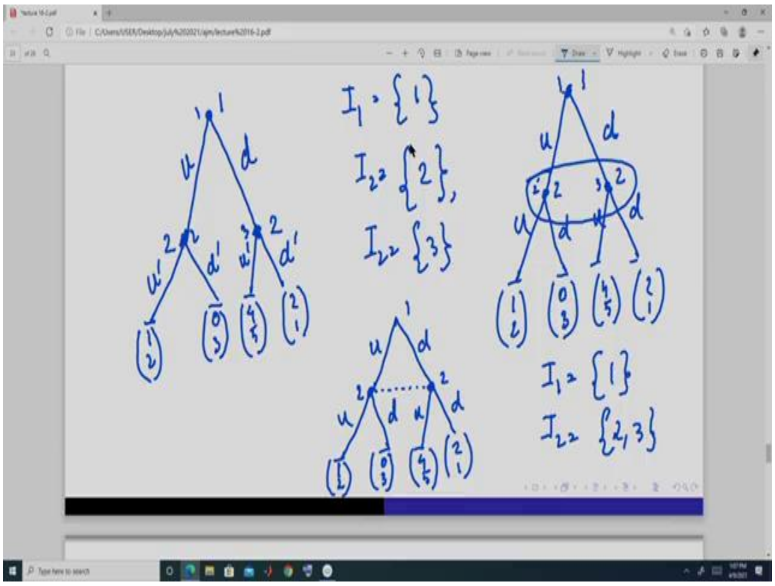

So, if we take this game, suppose, okay and suppose the payoffs are like this-(1,2), (0,3), (4,5), (2,1) okay. So, this is the two stage game and it is a game between two players, player 1 and player 2. Player 1 moves first, player 2 moves second and player 2 observe this. So, what we can say that the information set of player 1 is only this node. So, here we have to number the nods also. Suppose nodes are numbered in again in a 1, 2, 3, this way, okay. So, player 2 it is only node 1 information set. Information set of player 2 it has 2 information set, this and this.

So, why player 2 has 2 information set, because see player 2 knows that it is in this node. Again, here player 2 knows that it is in this node. But if we place all these two nodes, because player 2 has two nodes in which it is, it can make decision in a same information set, then it means that player 2 will not know whether it is in this node as the game is being played, whether it is, it has reached this node or it has reached is node, okay. But in this game player 2 knows that it is in this, whether it is in this node or in this node. So, that is why we have to separate this and these are singleton sets.

So, if all the information sets are singleton, then it is a game of perfect information. And if it is not singleton, then it is a game of imperfect information. So, it is like this suppose, same game, but it is like this. So, here player 1's information set is only this node that is node 1, this node is node 2 and this node is node 3, okay. So, but here player 2, player 1 can choose u or it can choose d. If it has chosen u, player 2 is in this node. If it has chosen d it is in this node. The game has played will move in this path, again game move in this path.

But player 2 does not observe the action of player 1, but it knows player 1 can choose either d or it can choose u. It can be either in this node or in this node, but it is not sure whether it is in this node or it is in this node, so that is why we mark a like this, okay. We mark it in this form.

Another way to mark it is in this form is to make a dotted line like this. Connect these nodes by a dotted line. Then it means that player 2 does not know whether it is in this node or it in this node. That means player 2 has not observed the action of player 1.

So, here, information set of player 1 is node 1, this- $I_1 = \{1\}$ because player 1 moves first, okay. Then player 2 information set is this node 2- $I_2 = \{2,3\}$. So, there are two nodes. It does not know whether it is in, whether, while making a decision, whether it is in this node or in this node. But here player 2 knows that whether it is in this node or it is in this node. So, that is why we have to make two separate information sets for player 2. But here we do not need. So, then in this way we can define the complete information, imperfect information game and information, perfect information game.

How, if all the information set of each player is singleton then it is perfect information. If there exist at least one information set which is not singleton then of any player, then it is an imperfect information game, okay.

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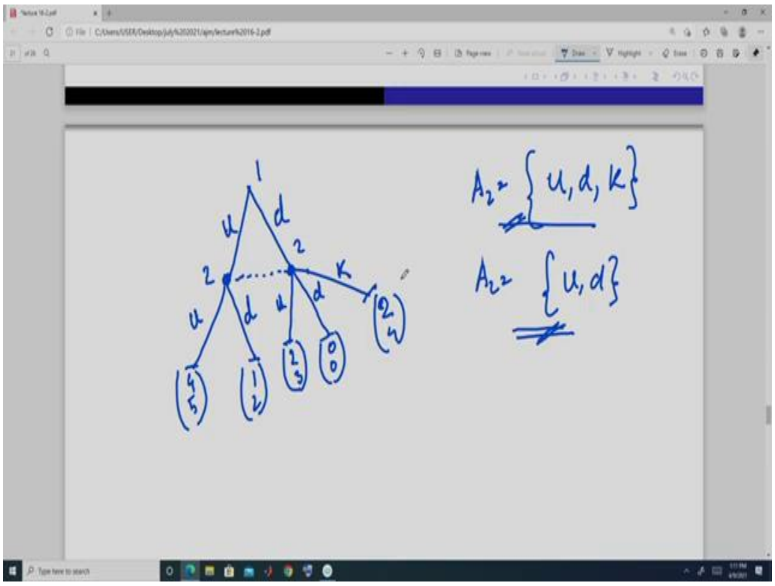

And further we require one more thing. Suppose if there are two nodes in one information set and in each node the number of actions are different then also player will be able to distinguish. So, the action set should be same for each nodes in a information set, okay. So, it is something like this. And suppose, so this is an imperfect information game, because player 2 does not know whether it is in this set, node or this node.

But suppose player 1 knows, player 1 plays this u, then ideally since it is connected by the dotted line these two nodes, so player 2 should not be able to distinguish between this node and this node. But here if it is here, then the action set is this- u= (4,5), d= (1,2). So, that is player 2 will know that, okay. But if it is here, the action set is u, d, k. So, it has two different action sets, right? So, if player 1 plays this d, then the action set that the player 1 has to choose from is d, k u, d, k, i.e. $A_2 = \{u, d, k\}$. And here this action set is only u and d. So, it does not matter whether player 2 observes the action of player 1 or not.

If it knows that whether it has to choose from this-$A_2 = \{u, d, k\}$. or it has to choose from this- $A_2 = \{u, d\}$, then because it will come to know, right? moment it is here, its action set is this- $A_2 = \{u, d, k\}$ or you can say action space is this. If it is here, its action space this- $A_2 = \{u, d\}$. So, it can distinguish between these two. So, here, even if player 2 has not observed the action of player 1, it can make this distinction and it will, from here it will know whether it is in this node or it is in this node.

So, that is why if we are in a information set, there are multiple nodes or nodes more than one, then the action space or the set of actions available should be same. So, here we should remove this k, okay. So, the action space in each node should be same if these nodes belongs to the same information set, okay.

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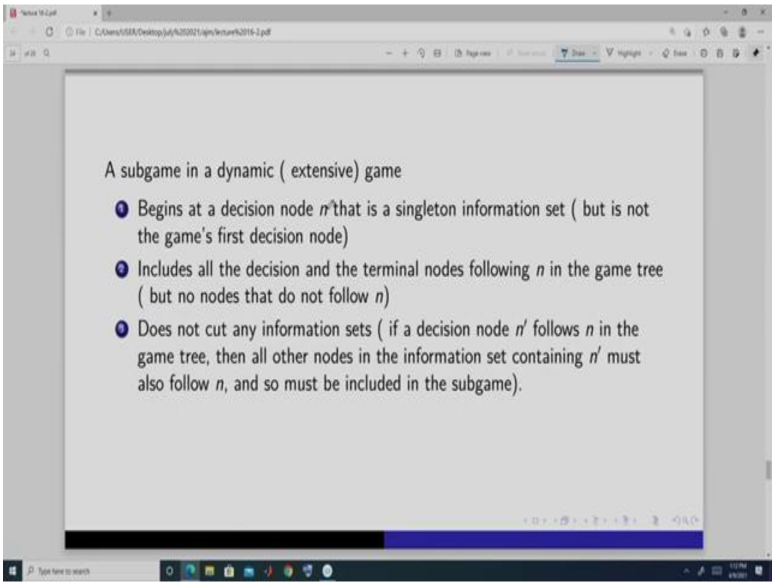

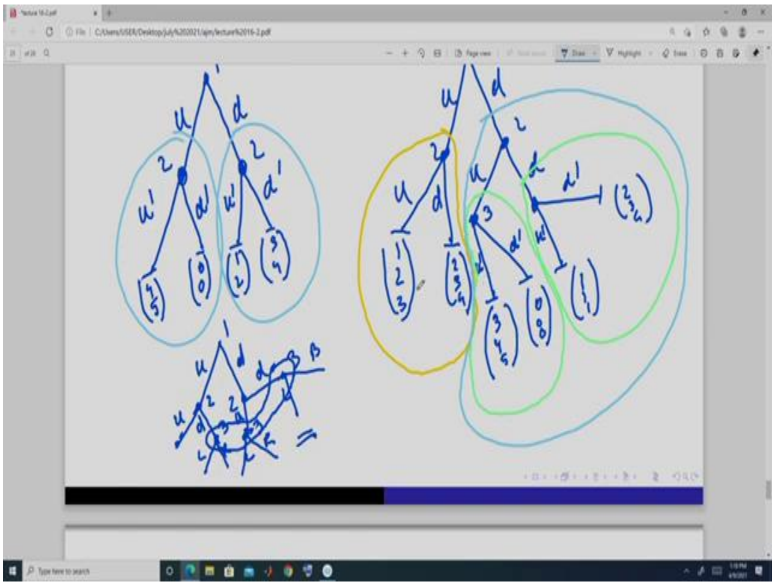

So, in this game, in this complete or dynamic games, we defined it through a game tree and we have multiple stages. Now, here we bring in another concept that is sub-game. And sub-game here, if we take this game, okay sub-game here it is defined in this way that the sub-game should begin at a decision node, node numbered n, any number that is a singleton information set, okay. So, a sub-game should start from a game here, either it can start from here or it can start from here. But it should not be the topmost that is the first node, okay. So, it should not be the first decision node, okay.

And it should include all the decision node following from it. So, if we have a game like this, suppose it ends here, suppose this is a game, okay. So, it is a three stage game. So, here, a game, this is a sub-game, because it starts from a decision node which not the first one and again this is a sub-game, again this is a sub-game and this is also a sub-game, because from here all these nodes which follows it belongs to this game only. So, it will have this. This is one sub-game- u= (1,2,3), d=(2,3,4), this is another sub-game- $u' = (3,4,5), \mathrm{d}' = (0,0,0)$, this is another sub-game-$u' = (1,1,1), \mathrm{d}' = (2,3,4)$, and we will have another sub-game like this- $u' = (3,4,5), \mathrm{d}' = (0,0,0), u' = (1,1,1), \mathrm{d}' = (2,3,4)$, okay.

So, this game has 1, 2, 3, 4, sub-game. This has only two sub-game. This is one and this is one, okay. So, sub-games are that it includes all decisions and terminal, decision node and terminal node following n in the game tree. So, if you start from any node then all the nodes that follows it, it should belong to that game, okay and it should not cut any information set. So, this means that if the game is suppose of this nature, now suppose player 3 also moves here, okay player 2 always moves u and d, player 1 moves at 3, L, R, L, R, L, R, so the game is something like this, okay.

So, here player 1 moves, either it can choose u or d. Then player 2 moves, it can choose u, d. If it is here, player 2 if it chooses d then player 3 can make a choice. Again, player 2 here it can choose either u and d and then player 3 can choose from here. So, in this game what you can see the sub-games are only one sub-game that is, actually it has no sub-game, because if you start from here what happens, this and this. But this game ends here. But if you start from there, it belongs to this information set. And this information, here we do not have any information set.

But this, if we start the game from here, then this node also belongs to this information set, but this node is not in this path of the game played. This is, this node is from this path, right? So, that is why this is, there is no sub-game in this game, okay because if you start from this, then we move here or we move here. But here this node it belongs to this information set and these nodes are not in the path that follows from here.

But if you look at this, if the game starts here, then all the paths, you can go to all the nodes that follows u. From here it, but from here you cannot go to this which are in the information set, this information set. So, that is why here is no sub-game in this game, okay. So, this is way we define the sub-game.

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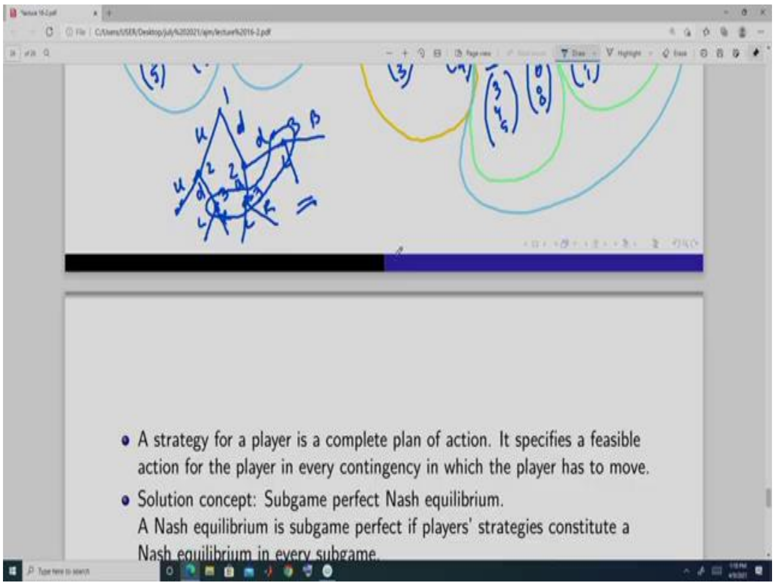

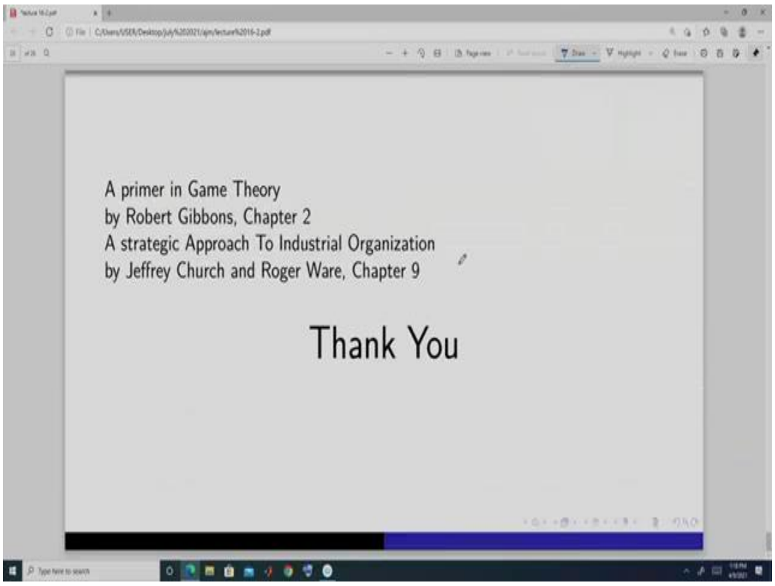

And then we will use a concept that is sub-game perfect Nash equilibrium and that we will do in the next class, okay. So, thank you. And for this you can read chapter 2 or chapter 9 from this book, okay.