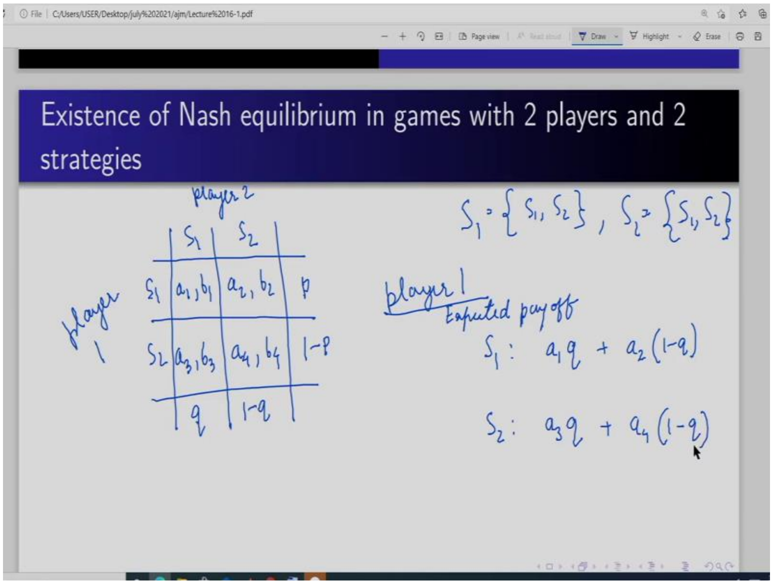
Introduction to Market Strategies
Professor Amarjyoti Mahanta
Department of Humanities and Social Sciences
Indian Institute of Technology, Guwahati
Module 5: Game Theory
Lecture 20
Existence of Nash Equilibrium in Games with 2 Players and 2 Strategies

(Refer Slide Time: 1:00)



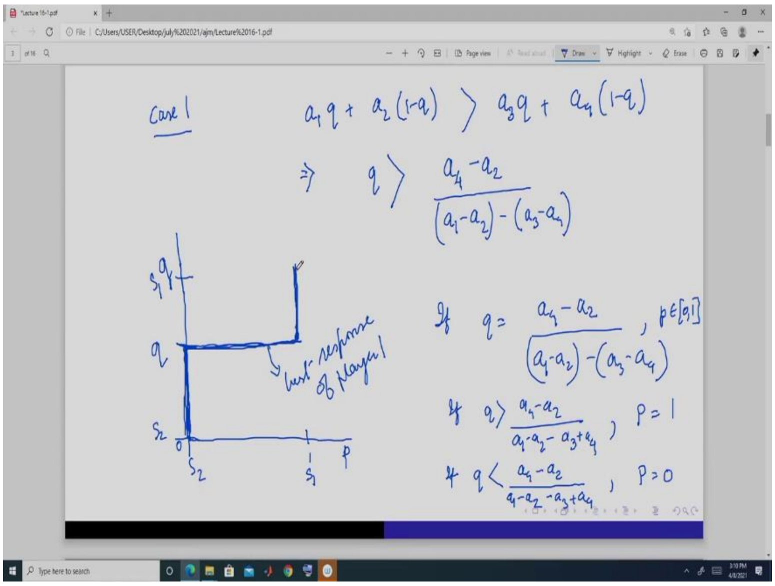
Hello, welcome to the course - Introduction to Market Structures. We were doing game theory so today we will show the existence of Nash Equilibrium in a game which has 2 players and 2 strategies and we will do it for a general case. So, the general game is this, okay, player 1 is the row player, player 2 is the column player.

The strategies are they have 2 strategies or 2 actions S1 S2, okay? So, you can take Si which is big A, it consists of $S_1 = \{S_1, S_2\}$ and S2 consists of 2 elements S1 and S2, i.e $\{S_1, S_2\}$, okay. And the payoffs are a for player 1, b for player 2, okay. So, this is the general game which has 2 players, player 1 and player 2. And each player has same strategies but they have 2 strategies S1 and S2 or 2 actions, okay.

We will show that in this kind of game there always exist a Nash Equilibrium, it can be a pure strategy Nash Equilibrium or it can be a mixed strategy Nash Equilibrium. So, here and the probability attached by each player is P to this strategy by player 1, 1 minus P to this strategy by player 1 to strategy 2, Q by player 2 on strategy 1 and 1 minus Q to strategy 2 by player 2, okay. So, I have specified the game, I have also specified the probabilities.

Now, let us look at the payoffs, okay expected payoffs. So, here let us look at the reaction function of player 1 first, okay. So, we will find all sorts all possible best response function or the reaction functions, okay. So, the expected payoff from S1 is a1 into q plus a2 into 1 minus q, i.e $S_1: a_1 q + a_2(1-q)$ and the expected payoff from S2 so this is for player 1, okay, player 1 is a3 q so this they get if player 2 attaches q probability to it and it gets a4 with 1 minus q4 probability, okay if it plays S2, i.e $S_2 = a_3 q + a\_4(1-q)$, okay.

(Refer Slide Time: 4:32)

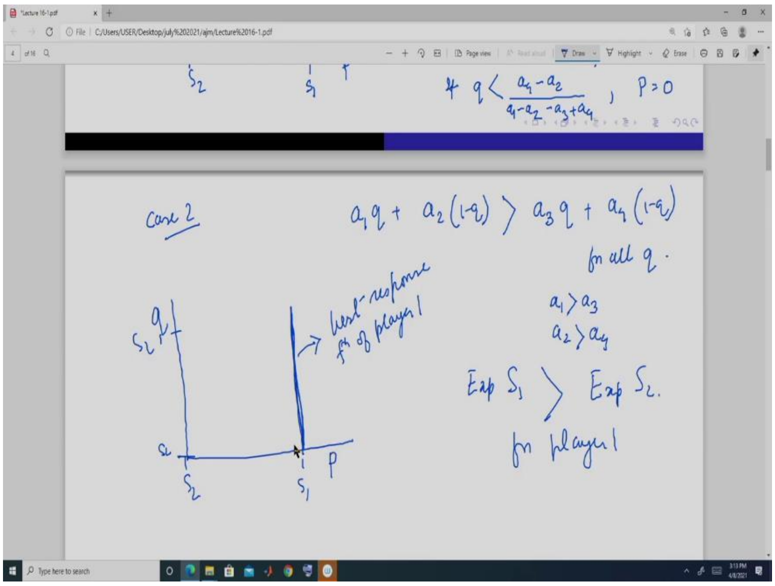


So, one possibility and let us that is case one is that, that a1 q plus a2 1 minus q is suppose greater than a3 q plus a4 1 minus q, i.e $a - 1\ q + a_2(1-q) > a_3 q + a_4(1-q)$. So, from here we get that $q > \frac{a_4-a_2}{(a_1-a_2)-(a_3-a_4)}$, okay, this. So, here we now plot the best response curve, so in this axis we plot the probability attached by player 1 and in this axis we plot the probabilities attached by player 2. If p takes value 0, that means if p takes value 0 then it is playing S2 for certainty and if p is equal to 1 then it playing S1 with certainty. So, this is suppose 1 so this is S1, this is 0, so this is S2. Similarly, for player 2 this is S1 because this value is 1 and this is S2, okay.

And suppose this is q and this q is this value, q is equal to a4 minus a2, a1 minus a2 minus a3 minus a4, okay this is q- $\frac{a_4-a_2}{(a_1-a_2)-(a_3-a_4)}$. So, if q is greater than this, then we have this situation, so this means that expected payoff from S1 is greater than expected payoff from S2 for player 1. So, if q is greater than this- q> $\frac{a_4-a_2}{(a_1-a_2)-(a_3-a_4)}$., then p is equal to 1. So, for here p is equal to 1 it means this, okay and if q is less than a4 minus a2 divided by a1 minus a2 minus a3 plus 4 this- $q < \frac{a_4-a_2}{a_1-a_2-a_3+a_4}$ then p is equal to 0, so it is here, right? And it is indifferent when q is this- $\frac{a_4-a_2}{(a_1-a_2)-(a_3-a_4)}$, so when q is….if q is this then p belongs to this range any value in this range so we get this as the…this is the best response of player 1 when we assume this that suppose this is greater than this one so that means whenever q is greater than this, that is this takes a positive value and this is lying between 0 and 1 then we get this, okay, this situation.

So, this is case one this is one possibility of best response function based on the payoffs. Suppose the payoffs are because this q is derived from the payoffs of player 1 and if q takes a value like this, suppose it takes a value like this if q is greater than this then p1 takes value here if q is less than this then p1 takes value 0 and p can takes any value lying between 0 and 1, if q is equal to this-$\frac{a_4-a_2}{(a_1-a_2)-(a_3-a_4)}$, if it is this, okay. So, this is case one.

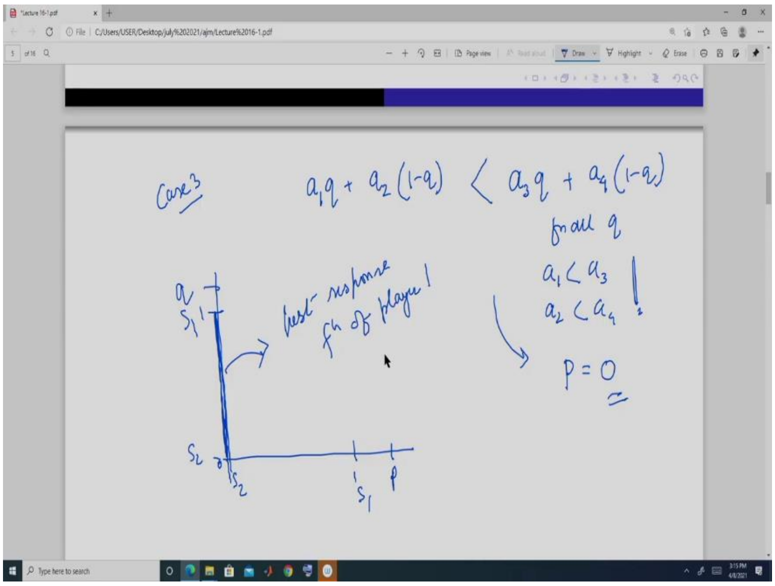
(Refer Slide Time: 9:32)



Now case 2, case 2 is, suppose, this- $a_1 q + a_2(1-q) > a_3 q + a_2(1-q)$ is true for all q, for all q this is true, whatever q takes value this is true this is possible when a is greater than a3 and a2 is greater than a4. So, a1 is greater than a3, a2 is greater than a4, right. So, we get this condition, so here again let us plot the best response function.

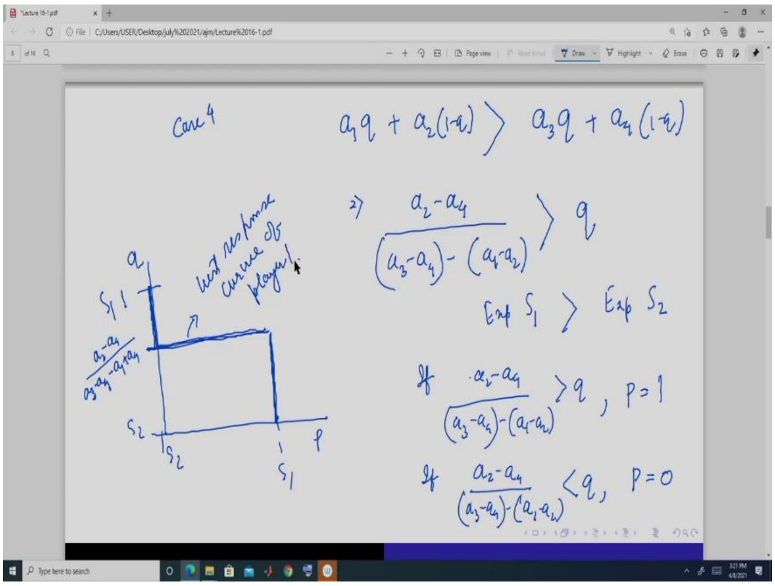
So, here whatever value q takes this is, so this is what expected payoff from S1, expected S1 is always greater than expected S2 for player 1. So, whatever q takes p is always going to be expected value of q and we know when p is 1 it is S1 when p is 0 it is S2. So, q any value it is this, so this is the best response function of player 1. Plug in any value of q you will get what is p, p is always 1 in this situation- $a_2 > a_4$. So, here you can say that S1 is a dominant strategy so S1 always dominates over S2, so that is why it always choose S1.

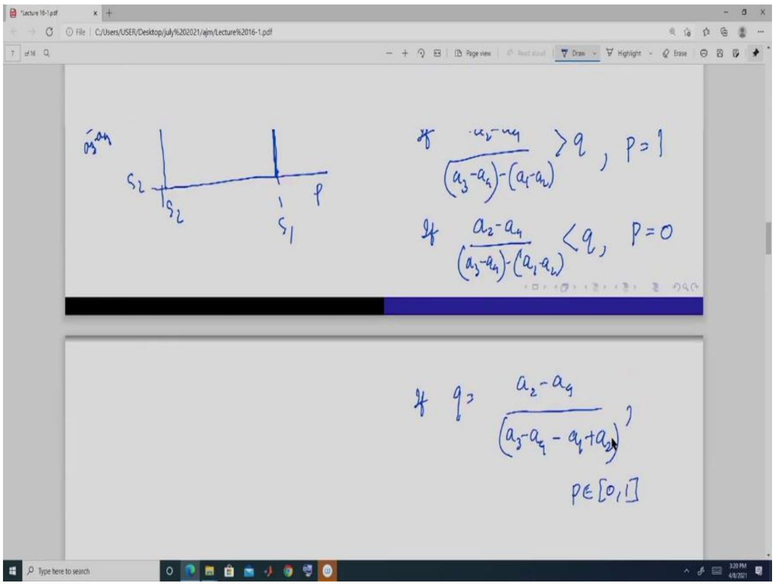
(Refer Slide Time: 11:59)



Next case, is this is always less than- $a_1q + a_2(1-q) < a_3q + a_4(1-q)$, okay, for all q, so this is possible when a1 is less than a3 and a2 is less than a4, so this is one situation where we will have such situation. So, in this case what is going to happen? Since the payoff from S2 is always greater than the payoff from S1, so player 1 is always going to choose S2, so here it is p, here it is q, this is p is equal to 1 and so this is S1, this is p is equal to 0, so this is S2, this is S2 for player 2, this is S1 for player 1 because this is 1, okay. Then whatever value q takes p is… so here p is always equal to 0. So, it is this… so this is the best response function of player 1, so this is the case.

(Refer Slide Time: 13:48)



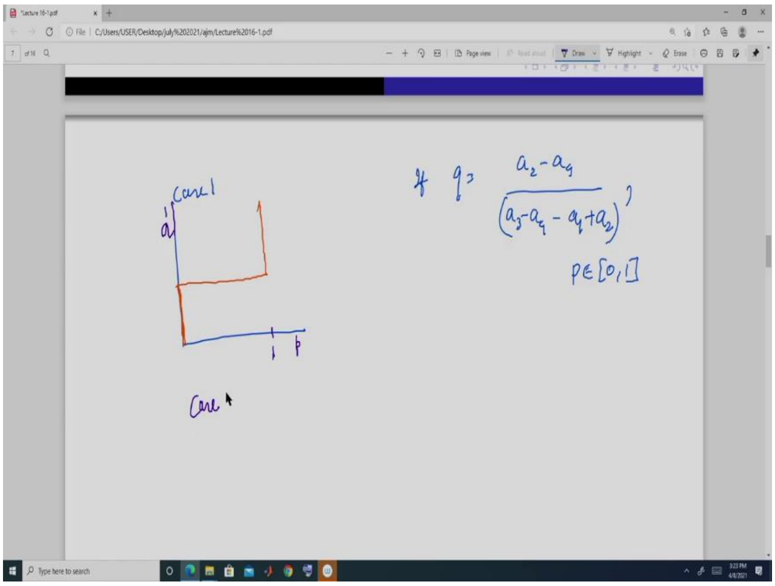


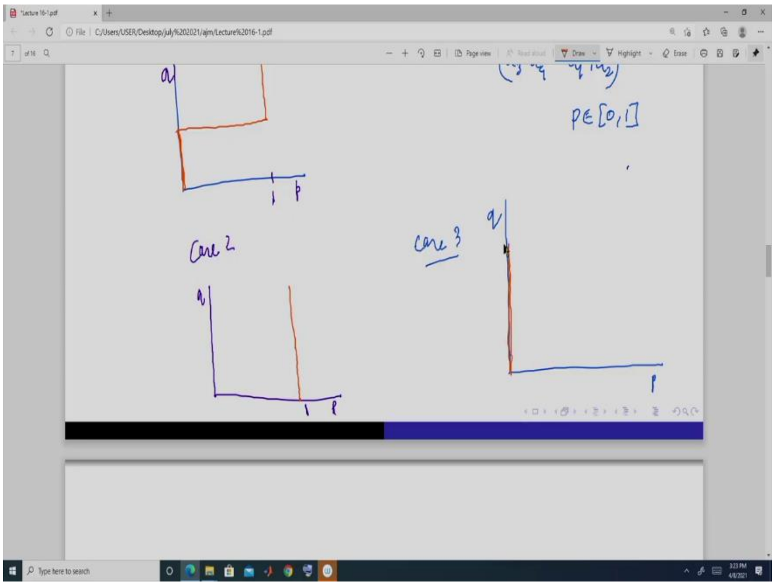
And now we do the last case, so again the expected payoff from S1 is this- $a_1q + a_2(1-q)$, expected payoff from S2 is this- $a_3q + a_4(1-q)$. So, if q is less than this number- $q < \frac{a_4-a_2}{(a_1-a_2)-(a_3-a_4)}$ and suppose this is a positive number lying between 0 and 0, so this number lies between 0 and 1, then if this is the case then if q is less than $\frac{a_4-a_2}{(a_1-a_2)-(a_3-a_4)}$ so we want the situation to be the opposite of… so if we have got this from here, we can write, suppose this is the case-$\frac{a_2-a_4}{(a_3-a_4)-(a_1-a_2)}$, okay, so if q is less than this number then the expected S1 payoff from S1 is greater than expected payoff from S2.

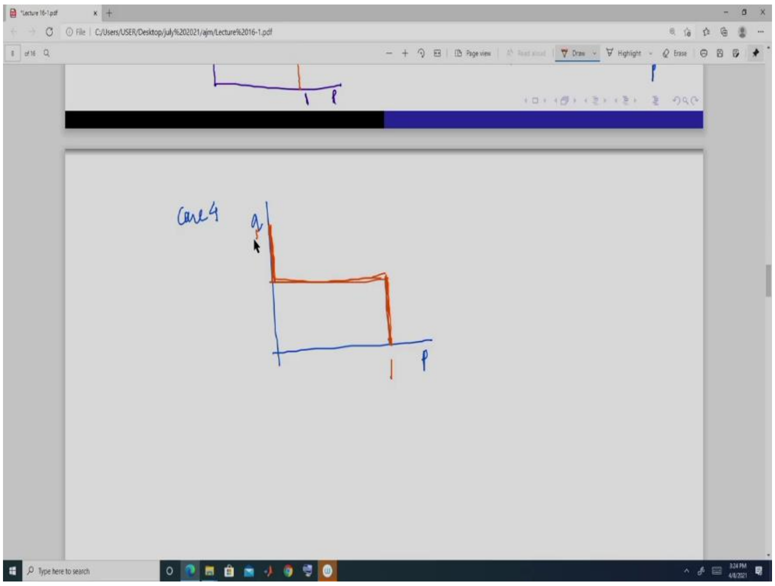
So, here if q takes a value like this- $\frac{a_2-a_4}{(a_3-a_4)-(a_1-a_2)}$, which is that q is less than this number then p is equal to 1, okay. And if this- $\frac{a_2-a_4}{(a_3-a_4)-(a_1-a_2)}$ is greater than q, then p takes a value equal to 0. So, here best response function, this is S1, this is S1, this is S2 so when q and this is suppose this value this is a2 minus a4 divided by a3 minus a4 minus a1 plus a4 this value so if q is less here then p is equal to 1. P is equal to 1 if q is less than here and q p is equal to 0 if q is greater than here and if q is exactly equal to this number that if q is equal to a2 minus a4, a3 minus a4 minus a1 plus a2, if this- $q = \frac{a_2-a_4}{a_3-a_4-a_1+a_2}$ then it is p, if this then p takes any value between 0 and 1. So, it is this, so this is the best response curve of player 1, okay.

So, this is not a function because this is a correspondence because for this point q taking a… this point we get a set and that set is this p= [0,1] so it is mapping from point to set. So, that is why it is not a function it is a correspondence, okay. But we do not need this all these things in detail but in case if you have any query regarding this, we get that it is a correspondence for each value of q we get this is the reaction of player 1. So, if q is less than this value, p is equal to 0, if q is greater than this value p is always equal to 0, if q is less than this value then p is always equal to one and if q is equal to this value, then p can take any value in this, so it is a set.

(Refer Slide Time: 19:41)

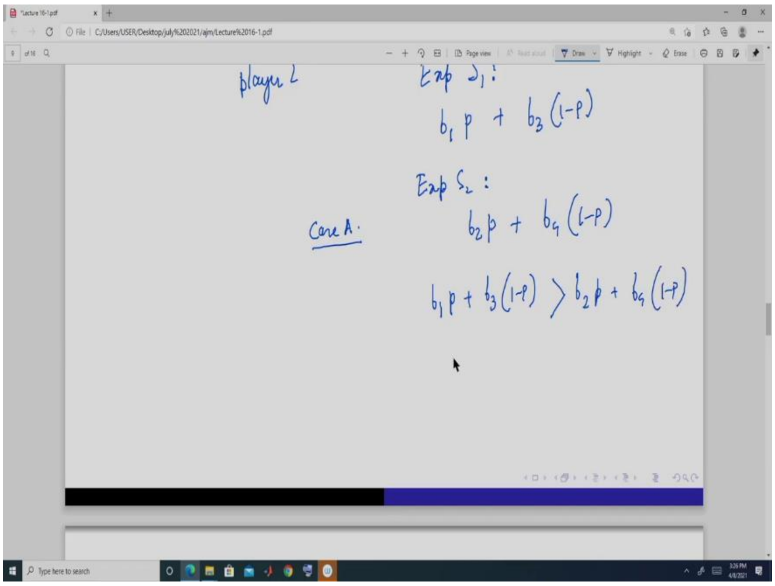


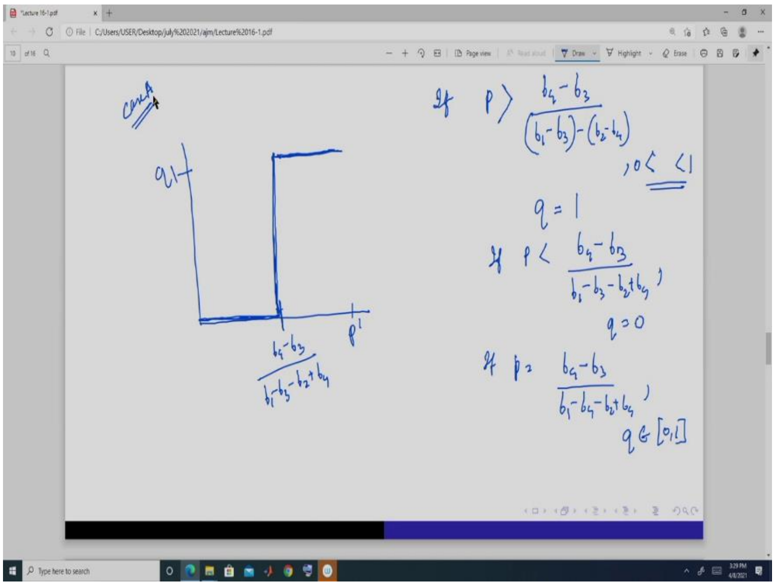




So, we get 4 possible cases what are these 4 possible cases? Case 1, we have got this ok let me draw it by using a different colour. So, this is the, right? case 2, so this is p and this is q, p is 1, q is 1 here so we have got this. Case 2, plug any value of q p is always 1, okay. Case 3, p and q… so any value of q, p is always equal to 0, this is the case. And case 4 we get this, is the best response, okay. So, these are the all-possible situations depending on the different payoffs, okay.

(Refer Slide Time: 22:12)



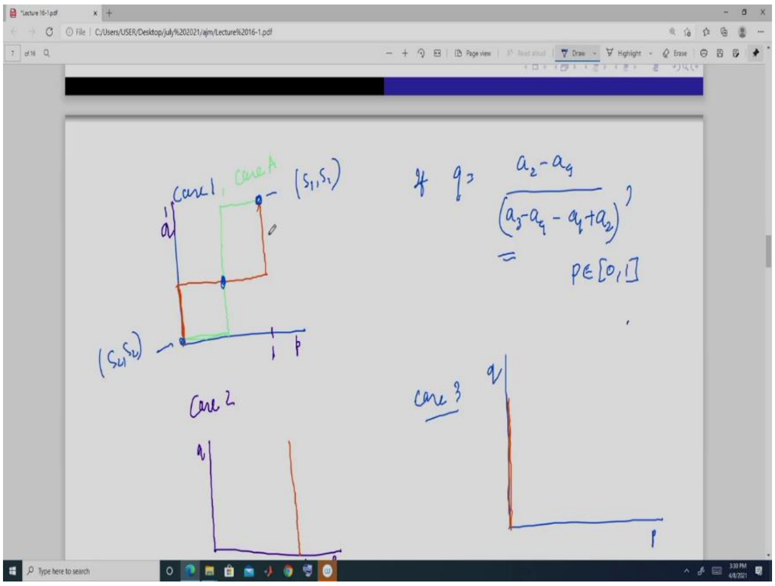


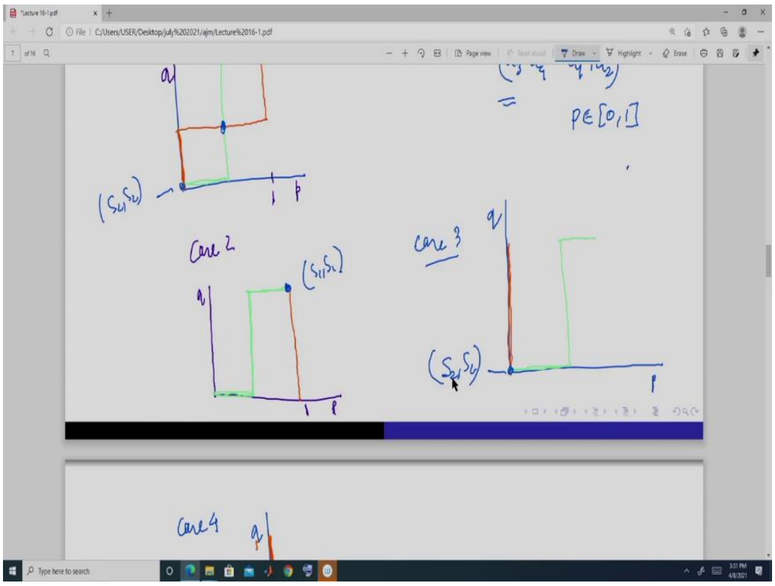
Now we look at the case of player 2, so if you look at this game and the payoffs of player 2 the expected payoff, so now we are looking at player 2, so expected from S1 is b, 1, p plus b3, 1 minus p, i.e Exp $S_1 : b_1 p + b_3(1-p)$ and expected payoff from S2 is b2 p plus b4 1 minus p, i.e Exp S_2: $b_2p + b_4(1-p)$ and we may have a situation where b1 p plus is so this is suppose case A- $b_1p + b_3(1-p) > b_2p + b_4(1-p)$, okay, this and we have got,

We have got- $p > \frac{b_4-b_3}{(b_1-b_3)-(b_2-b_4)}$, so here in this situation if p is greater than this value and suppose this lies between 0 and 1. So, the payoffs b1, b2, b3, b4 are such that this lies between, this is greater than 0 and this is less than 1, if it is like this. If this value lies in this range- $0 < \frac{b_4-b_3}{(b_1-b_3)-(b_2-b_4)} < 1$, then q is always equal to 1, right? And if p is less than this- $p < \frac{b_4-b_3}{b_1-b_3-b_2+b_4}$ then q is equal to 0 and if p is equal to this- $\frac{b_4-b_3}{b_1-b_3-b_2+b_4}$ then q lies between 0 and 1.

So, here in case 1, if we take plot p here, q here, 1 so p, suppose this is the p this is b4 minus b3, b1 minus b3 minus b2 plus b4, okay, this value is this. Then if p is greater than this, then q takes value 1, q is 1, okay. If p is less than this q is 0, it is this, okay. So, this is one possibility of which is case A.

(Refer Slide Time: 26:00)

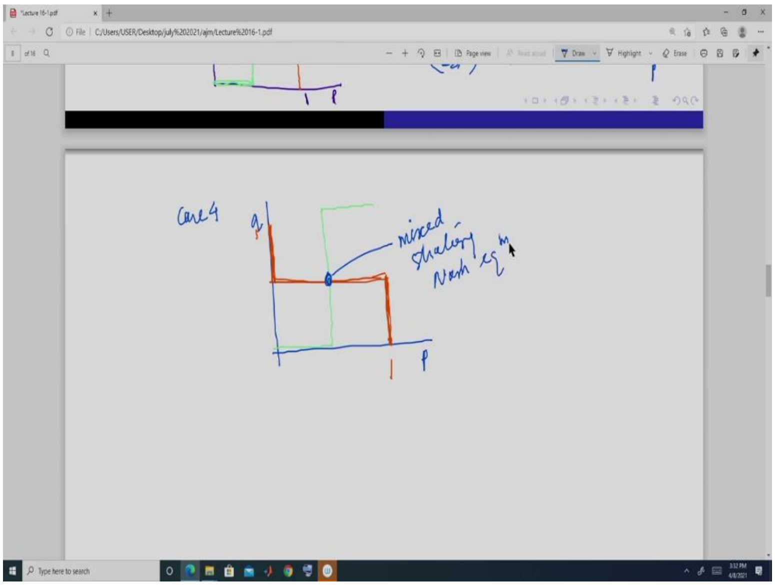




Now here just look at this so we have this case and we can have any one of these 4 cases of player, here player 1. So, now just in this graph just look at this here so if case 1 and so we can have a combination like case 1 and case A, so in that case this so we have 3 Nash Equilibrium, this is 1, where it is (S1 S1), this is another which is (S2 S2), and this is the mixed strategy where q takes this value- $\frac{a_2-a_4}{a_3-a_4-a_1+a_2}$ and p takes this value-$\frac{b_4-b_3}{b_1-b_3-b_2+b_4}$, okay. So, this is the situation so we have 3 Nash Equilibrium in this case.

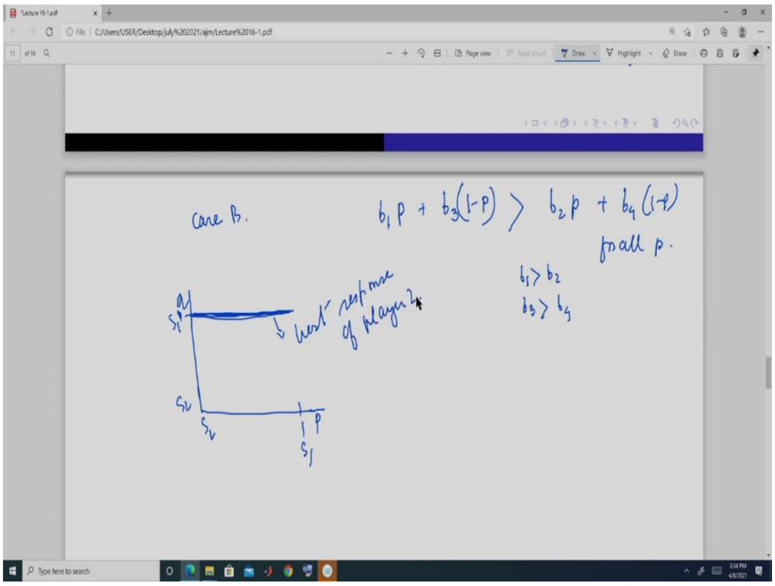
And suppose instead of case 1, we have suppose case 2, then this is the reaction function of player, best response function of player 2, so we have only 1 Nash Equilibrium in this situation and that is this and this is S1, and this one S1, okay. In this situation, if suppose we have case 3 of player 1 and case A of player 2, so this is the point of intersection of best response function, so we have 1 Nash Equilibrium it is (S2, S2).

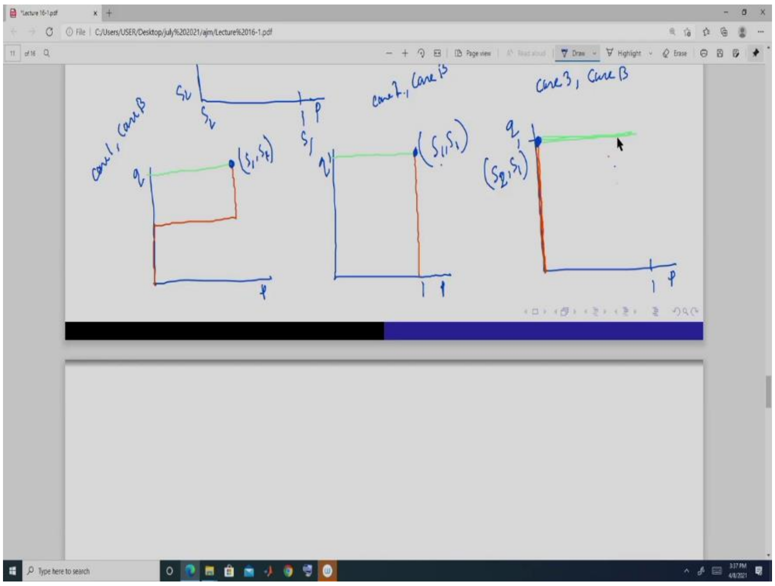
(Refer Slide Time: 28:08)



In this situation, we have this we have only 1 Nash Equilibrium in this situation and it is the mixed strategy, q so this is the mix strategy Nash Equilibrium, where p, this p takes this value sorry p takes this value and q takes this value, okay. So, what we have got if we if the payoffs are such that best response of player 2 is case A and take any case of player 1 either case 1 or case 2 or case 3 or case 4, we have at least 1 Nash Equilibrium, okay. So, this is one possible outcome.

(Refer Slide Time: 29:31)

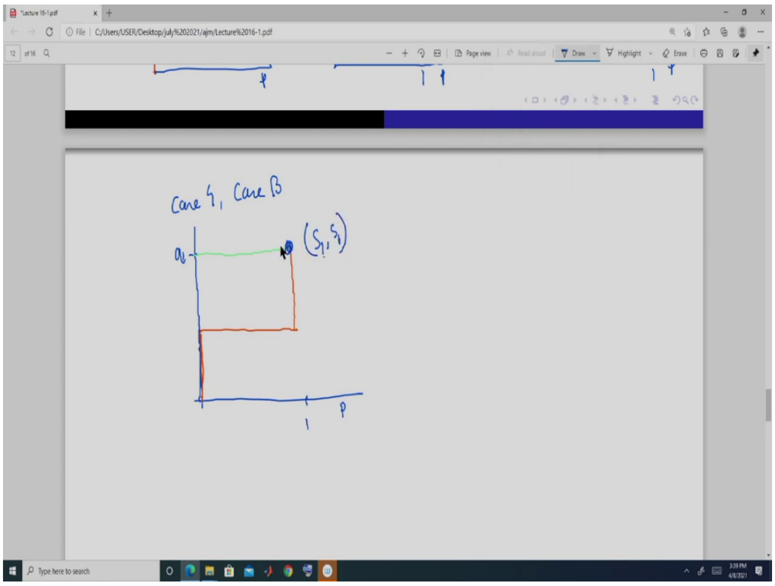




Now, let us look at case B of player 2. Case B is, suppose this is always, i.e $b_1 p + b_3(1-p) > b_2 p + b_4(1-p)$ for all p, so then it means one possible way is b1 is greater than b2 and b3 is greater than b4. So, it always chooses S1. So, in this whatever be the value of p player 2 is always going to choose S1 in this case. So, here it is, this is suppose 1, is S1, S2, this is S2, so whatever be the value of p player 2 chooses S1. So, this is the best response this is the player 2, best response curve or you can say best response function, here it is a function, right?

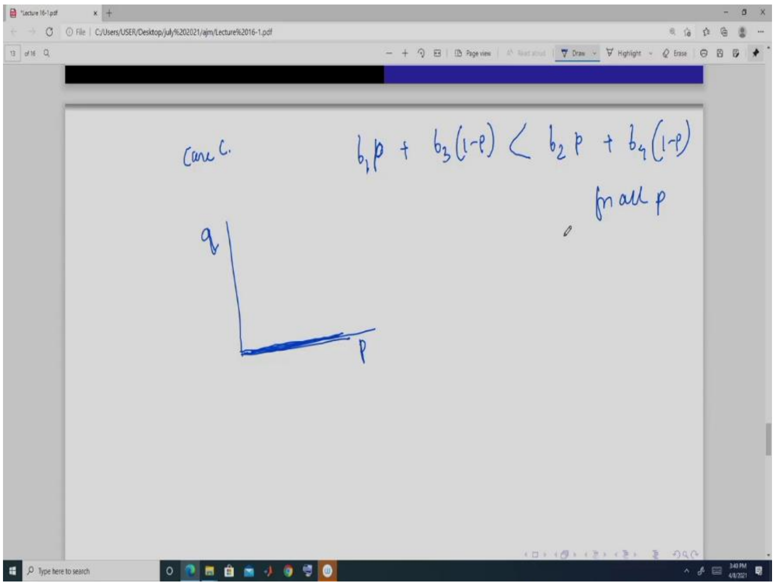
So, now if you look at all the possible, so this is case B, okay. So, suppose this is case 1 and case B, okay, so then this is the reaction function of player 1 and the reaction function of player 2 is this. So, this is the Nash Equilibrium and Nash Equilibrium is (S1, S1), okay in this case. So, this is for case 1 and case B. Now, we have may have case 1, case 2 and case B. Case 2 is this, this right? and case B is this, so the Nash Equilibrium is this and it is (S1, S1) okay. Next is suppose case 3 and case B, so here this is the reaction function of player or the best response of player 1 and the best response of player 2 is this case B. So, the Nash Equilibrium is this point and this is (S2, S1). S2 of player 1 and S1 of player 2, okay.

(Refer Slide Time: 34:12)



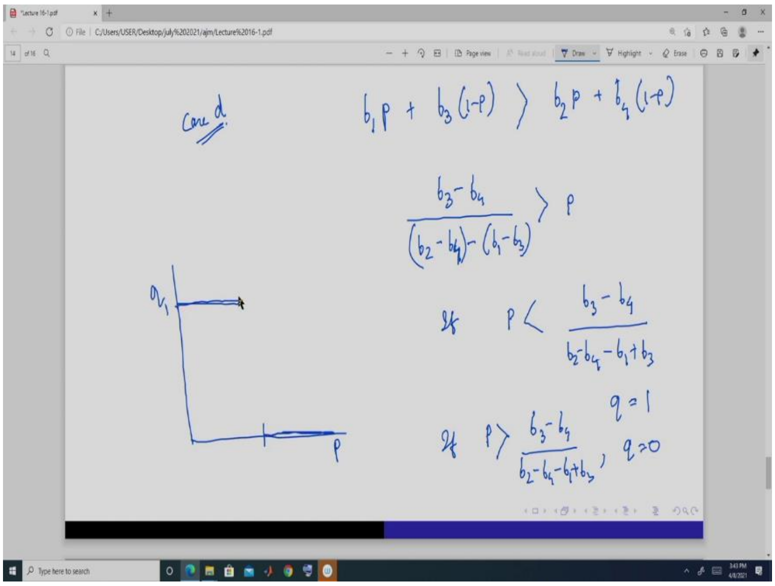
Now, let us look at this case that is case 4 and case B. So, in this situation we have best response of player is this and best response player 2 is, sorry, okay and best response of player 2 is this. So, the Nash Equilibrium here is again this, so this is S1 of player 1 and S1 of player 2, this. So, here we have shown that in this case also we always have a Nash Equilibrium.

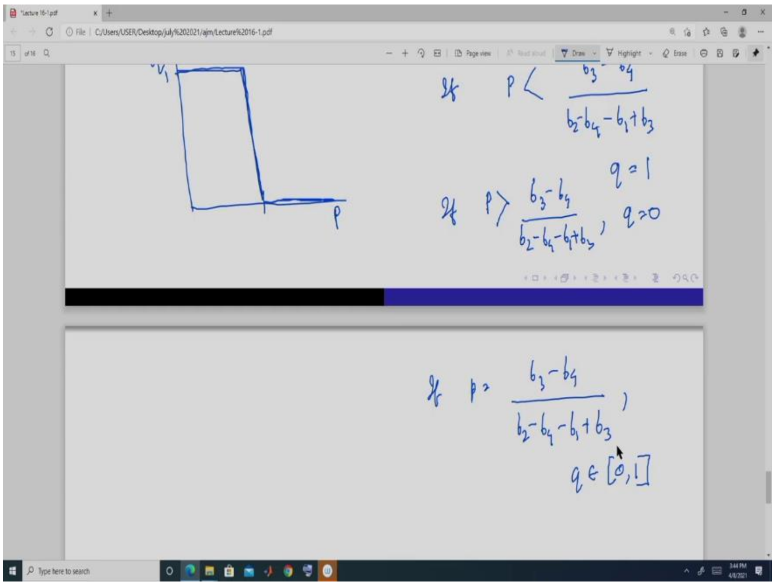
(Refer Slide Time: 35:33)



So, we can go on doing this so we will get 2 more cases and this are case C where we have b1 p plus b3 1 minus this is always less than b2 p plus b4 1 minus p for all p, i.e $b_1p + b_3(1-p) < b_2p + b_4(1-p)$, okay. So, in this case the whatever be the value of p q is always this, this is the reaction. Now, here you compare this with all the force case of player 1 and you will see that there exists at least one intersection point. So, we always have at least one Nash Equilibrium in this case.

(Refer Slide Time: 36:28)





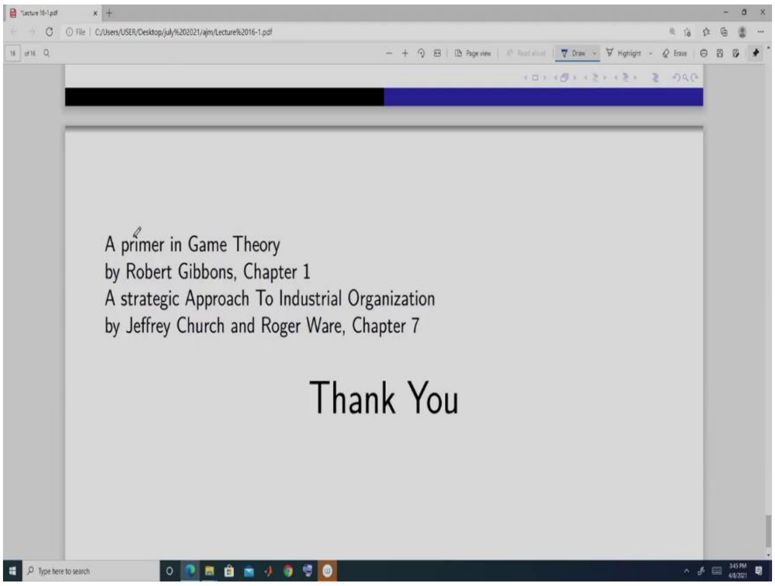
Again, the case D is suppose this- $b_1p + b_3(1-p) > b_2p + b_4(1-p)$, okay, so this is, so if p is less than this number, i.e $p < \frac{b_3-b_4}{(b_2-b_4)-(b_1-b_3)}$ then q is equal to 0 and if p is greater than this number, was this the case A that we have done, this is what we have done here is this, if p is like this if p is greater than this number-$\frac{b_4-b_3}{b_1-b_3-b_2+b_4}$, p is greater than this number then we have this situation.

So, here if p is less than this number then we have this situation, so here it is q is not equal to 0 but q is equal to, q is equal to 1 and in this situation if q is this, then q is equal to 0. So, here if we plot p and if we plot q, if p is less than, then q is 1 and if p is greater than then q is 0, this- $p > \frac{b_3-b_4}{(b_2-b_4)-(b_1-b_3)}, q = 0$. And if p is equal to this number-$\frac{b_3-b_4}{(b_2-b_4)-(b_1-b_3)}$, then q can take value, then q takes any value between 0 and 1, so this is the case 4, case D of player 2.

So, now here again in this case we can look at all the 4 possibilities of payoffs of player 1 and then we will see at least in one point the reaction functions are going to intersect. If we take this case and we take all the 4 possible cases of player 2, player 1 then again, we will see that at least in 1 point they are going to intersect the best response functions are going to intersect. So, this means that there always exist a Nash Equilibrium when we have 2 player and 2 strategy game.

So, actually Nash prove this for n player, any arbitrary number of player and any arbitrary number of strategies and he has proved it using both Brouwer's Fixed Point Theorem and Kakutani Fixed Point Theorem, okay. So, we do not require those things so it is beyond the scope of this course, so we will skip this portion.

(Refer Slide Time: 40:47)



So, this is the end of Static Game Theory and you can read it from chapter 1 of, A Primer in Game Theory by Gibbons or you can read it from chapter 7 of a Strategic Approach to Industrial Organization by Jeffrey Church and Roger Ware. Thank you!