Introduction to Market Structures
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Module 05: Game Theory
Lecture 19
Mixed Strategy Nash Equilibrium

Hello. Welcome to my course Introduction to Market Structures. So, we were doing mixed strategy. And let us do one example.

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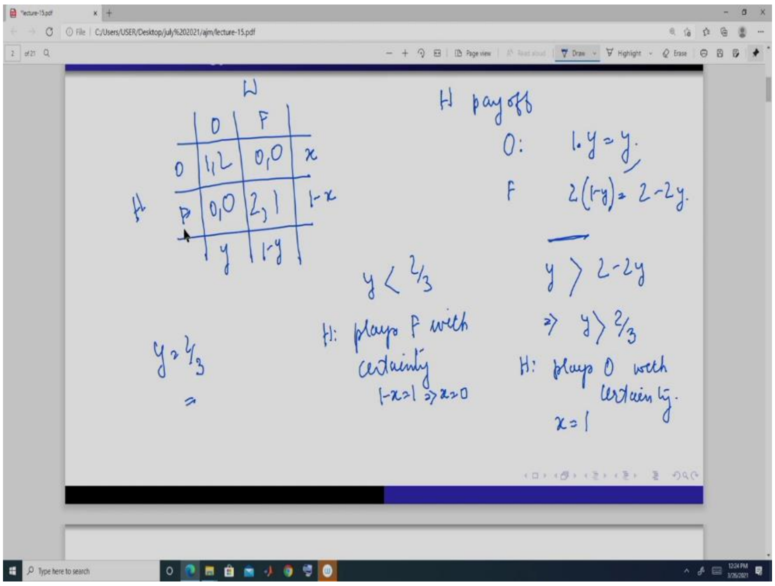

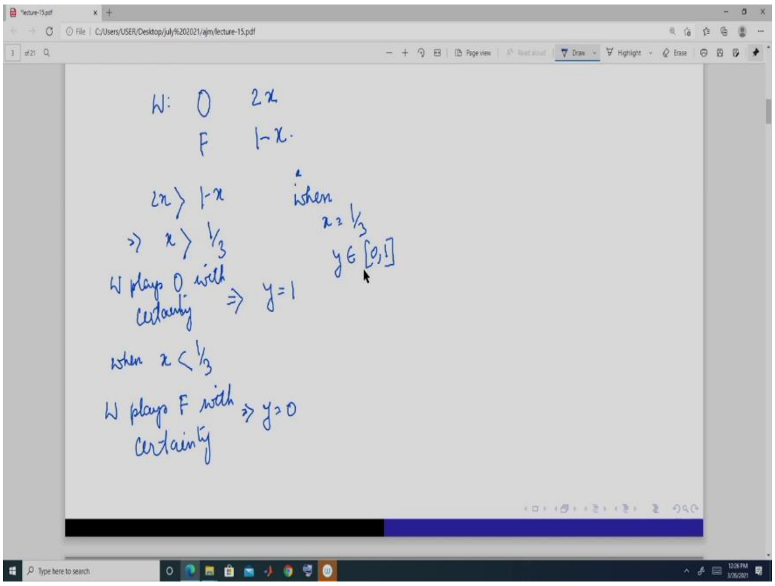

So, let us consider a game, and suppose this is a game of Battle of Sexes. We have done one version of it in the last class, right. And now we will try to find the mixed strategy in this case. So, these (1,2), (0,0), (0,0), (2,1) are the payoffs, right. Now, here suppose player 1 that is the husband attaches probability x to strategy O and attaches probability 1 minus x to the strategy F. And wife attaches y to the strategy O or action O, and 1 minus y.

Now, if you look at the, suppose husband's H payoff, okay so payoff from O is, if it plays O, if there is a chance that it will get 1 with probability y and 0 with probability 1 minus y. So, the A is payoff from playing O is 1 into y which is y. Payoff from playing F for player 1 is; it will get 0 with probability y and 2 with probability 1 minus y. So, 2, 1 minus y which is equal to, i.e $2(1–y)= 2–2y$, okay….

So, this is the payoff for the husband. And when husband is always going to play O? When y is greater than 2 minus y. And this implies, since this is greater than this, so then I am not going to attach any probability to O and F. I am always going to play O. So, this is when y is greater than 2 by 3.

So, then in this case it will; H plays O with certainty, right? And if y is less than 2 by 3 that means F is, so this- $1–y=y$ is less than this-$2–2y$. So, then H plays F with certainty. What does this mean? This means that if O is played with certainty, that means x is equal to 1. And if F is played with certainty that means 1 minus x is equal to 1. So, this means x is equal to 0, right. And when it attaches some probability to it? It attaches some probability when y is equal to 2 by 3.

Whenever player wife plays O and F such that probability of playing O is 2 third and probability of playing F is one third then player 1 that is husband attaches some probability to O and F. What? We will come to it later, okay. But if y is greater than 2 by 3 then player husband is always going to play O. And if y is less than 2 by 3 then husband is always going to play F with full certainty. So, these are, okay.

Now, let us do it for wife. Wife if it plays O, if it plays O, it gets 2 with probability x and 0 with probability 1 minus x. So, payoff is 2x. And if it plays F it gets 0 with probability x and 1 with probability 1 minus x. So, it is 1 minus x. And when wife is always going to play A? When 2x is greater than this- $1–x$, so this means when x is greater than one third W plays O with certainty, okay. And when x is less than 1 by 3, W plays F with certainty. So, with certainty here it implies that y is equal to 1. Here it implies that y is equal to 0. And when it will going to attach some probability that y takes some value between 0 and 1? When and, when x is equal to one third y will take some value between 0 and 1.

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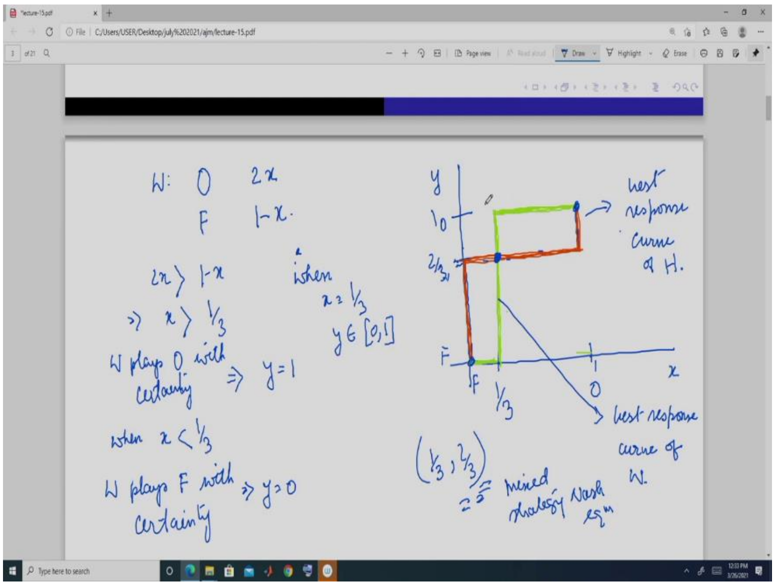

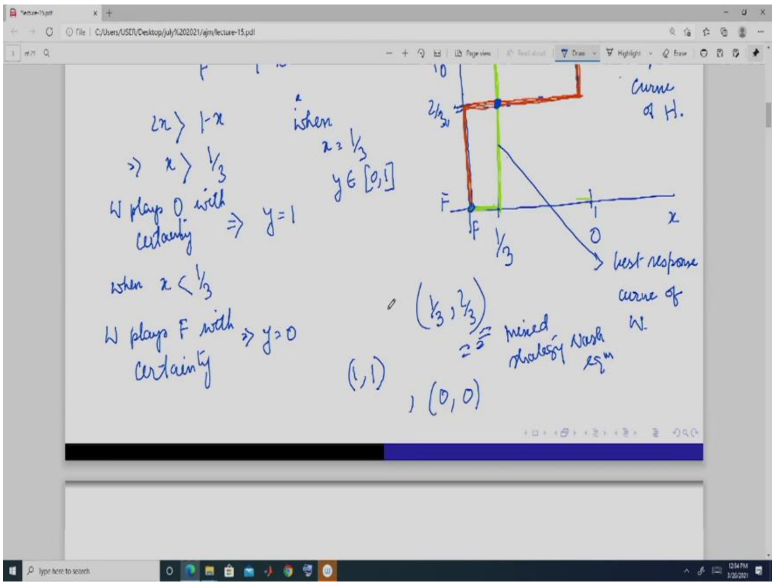

Now, we represent this whole information through a graph, okay. So, in this axis we plot x and in this axis we plot y. So, if x, so suppose this is 1, this is 1. If x takes value 1 it means what? That x is taking value 1. So, that means H is playing O with full certainty. So, this means it is playing O. So, this is O. And when x takes value 0 it means player H, husband is playing F with full certainty. So, this is F. This is 0.

Similarly for player wife; when y takes value 0 it means that it is playing F with full certainty when F, y takes value 1 then it is playing O with full certainty. And suppose this point is two third and this point is one third, okay. From this argument if y is greater than two third, when y is lying here then x takes value 1 because it plays O with full certainty. And when y takes value less than two third it plays F with certainty. So, x takes value 0. So, it is this, this region. And when y takes value two third; it, x can take any value between 0 and 1 because it is indifferent between playing O and F, right. So, this is what we can call the, this is the best response curve of H given any y how it is going to attach probability to its actions or strategies. So, it is given by this red curve, this red line, okay this.

So, when y take value greater than two third x is 1. It is 1. When x takes, y takes value less than two third, x is 0. So, it is here, any point, y any point below two third, it is y, x is 0. And when y takes value two third x can take any value in this. So, it is this. Now, let us draw the, this best response curve of wife, okay. So, when x is less than one third, so when x is here, then when x is less than one third then this, when x is less then… wife plays F with certainty.

So, that means y is equal to 0. So, y is equal to 0 means here, whenever x is less than one third, x is less than one third it is going to play F. So, it is here, y taking the value 0, here. And when x is greater than one third so it is, x is here then y takes the value 1. So, it is here. So, it is this curve. And when x takes value one third y can take any value between 0 and 1. So, it is this.

So, this green line is the best response curve of player 2, of wife, okay. And here you will see that these two curves intersect at three points; at this point, at this point and at this point, okay. This point is (F, F); this point is (O, O) and this point is when x takes the value of one third, y take the value two third. So, this is the mixed strategy Nash equilibrium.

So, this, so that means when x take the value one third player 2 is indifferent between playing O and F. So, it is attaching some probability, okay. And when player 2 is playing, attaching probability two third then player 1 is going to attach some probability to F and O. And what is that probability? Out of all these probabilities it is going to attach only this, okay when player 2 is doing this.

Now, when player 1 y, sorry, when player 1 takes one third, player 2 is going to choose any of these, right? But if it chooses anything higher; then it is, player 1 is going to choose here. If it chooses anything lower player 2 is not going to attach any probability. It is going to play this. So, that is why player 2 when it is attaching any probability from this range it is going to attached two third, so this.

So, that is why this is the point of intersection of two best response curves and so this is the mixed strategy Nash equilibrium, okay. And these 2 are degenerated Nash equilibrium because here x is taking value 1 and y is taking value 1. So, if you represent it in terms of probabilities x taking value 1 y taking value 1; this is also a Nash equilibrium. And x taking 0 y taking 0 is also a Nash equilibrium because these two curves are intersects, okay. So, this is how we find the mixed strategy Nash equilibrium.

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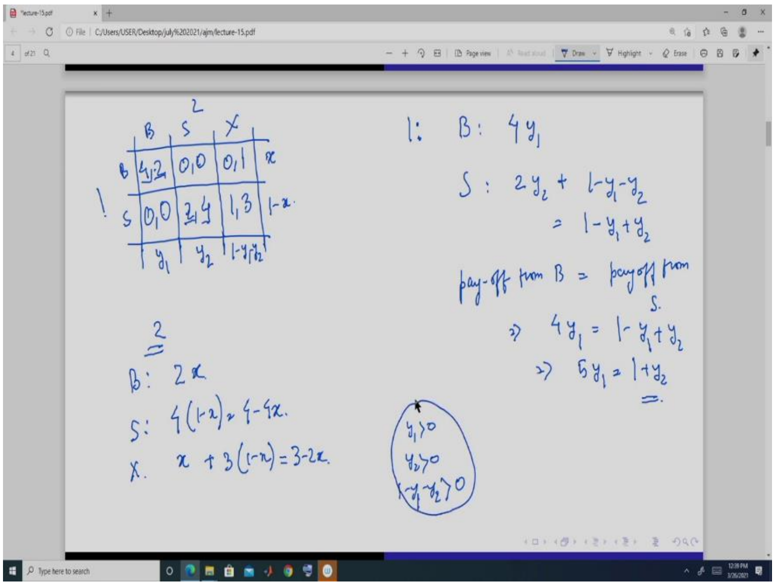

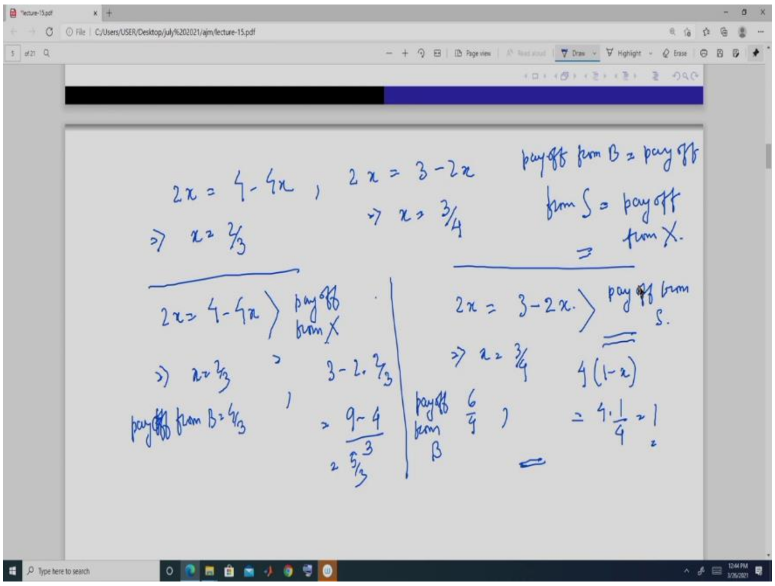

And we have shown in one case in the last class where we have, again where we do not have any pure strategy but we have a mixed strategy Nash equilibrium, okay. Now, let us do another example. So, there are 2 player. This is player 1. It is column player. And player 2 is the, sorry this is the row player and this is the column player, okay. So, this is the game. if you look at this game then this B, if player 1 plays B then the best response for player 2 is to choose B because 2 is greater than 0 and 1. And if it chooses B then it is this. So, this is one Nash equilibrium. If player 2 chooses S player 1 is going to choose S because 2 is greater than 0. And if it chooses S then it is going to choose S. So, this (3,4) is one Nash equilibrium.

If player 1 chooses this we know, so this is not a Nash equilibrium. But player 2 choose x then it is going to choose this S. And when it chooses S, player 2 is going to choose… So, there are only two pure strategy Nash equilibrium. And here we are going to check whether there is any mixed strategy Nash equilibrium or not, okay. So, this, suppose player 1 attaches probability x to this action or strategy B, and to action strategy S it at attaches 1 minus x. And player 2 attaches suppose y1 to B, y2 to S and 1 minus y1 minus y2 to x.

So, now let us write the payoff for player 1. Player 1's payoff, if it plays B it is going to be 4, it is simply going to be 4 y1, because rest are (0, 0). If it plays S then it is going to be 2 y2 plus this, i.e $S = 2y_2 + 1 - y_1 - y_2$. So, this is equal to- $1 - y_1 + y_2$, okay. And from here we know, when know when player 1 is going to attach some probability; x taking some positive value and 1 minus x also taking some positive value when payoff from B is equal to payoff from S.

Otherwise if suppose payoff from B is greater than payoff from S then it is always going to choose B. If payoff from S is greater than B then it is always going to… So, we will attach some probability only in this sequence. So, this means that 4 y1 is equal to 1 minus, i.e- $4y_1 = 1 - y_1 + y_2$ So, this means 5 y1 is equal to 1 plus y2, okay. So, remember this- $5y_1 = 1 + y_2$

Now, let us find the payoff of player 2, player 2. If player 2 plays B its payoff is 2 with probability x so it is 2x. And it is 0 with 1 minus x so it is 2x. If it plays S its payoff is 0 into x and 4 into 1 minus x, i.e $S = 4(1 - x)$. So, this is 4 minus 4x, okay. If it plays X, capital X then it is 1 with x; plus 3, 1 minus x. So, it is 3 minus 2x, i.e $X = x + 3(1 - x) = 3 - 2x$, okay.

Now, first here we have to identify. Suppose player 2 attaches some probability to all these three strategies or all the three actions. When, when it will attach? Some positive, so some number which is y1 is positive, y2 is positive and y minus, this- $1 - y_1 - y_2$ is also positive, fine, when the payoffs are same in all these three strategies. Because if suppose a situation is that if payoff from B is greater than these two then player 2 is going to play B with full certainty. That means y1 is taking value 1 and the rest (0, 0).

So, we have to now check for this situation- $y_1 > 0, y_2 > 0, 1 - y_1 - y_2 > 0$. First we will check this. So, it is 2x must be equal to 4 minus 4x, i.e $2x = 4 - 4x$. So, then this means x is equal 2 by 3, i.e $x = \frac{2}{3}$. Again we need x is equal to 3 minus 2x. So, this means x is equal to 3 by 4, i.e $2x = 3 - 2x => x = \frac{3}{4}$. So, this is not equal. So, that means we cannot have a situation where payoff from B is equal to payoff from… payoff from S is equal to payoff from X. So, this payoff from all the three actions or strategies cannot be equal. This is for player 2. This is clear from this thing.

But now suppose we may have a situation that this is- $2x = 4 - 4x$, we attach some probability to these two. And this is greater than payoff from X, i.e$2x = 4 - 4x > \text{pay} - \text{off from X}$. So, this means that x is equal to 2 by 3. If x is equal to two third then payoff from from A, from B is equal to 4 by 3. And payoff from X is 3 minus 2, i.e $3 - 2.\frac{2}{3} = \frac{9-4}{3} = \frac{5}{3}$ It is 5 by 3. So, again this is not possible. When payoff from B and payoff from S is equal but it is greater than payoff from X it is not possible. So, we ruled out this also, okay. Next, look for a situation where payoff from B is equal to payoff from X and this is greater than payoff from S, right. So, if we plug in here this to the a so it will be what? It will be 6 by 4, payoff from B. But what is going to be the payoff from S? So, it is 4(1-x), this, so it is $4.\frac{1}{4} = 1$

So, this is possible because 6 by 4 is greater than 1. So, this is one possibility. So, that is why player 2 attaches some probability to B and some probability to X, okay. And when it attaches some probability to B and X? When X player 1 plays B with this, with probability 3 by 4; and S with probability 1 by 4, okay.

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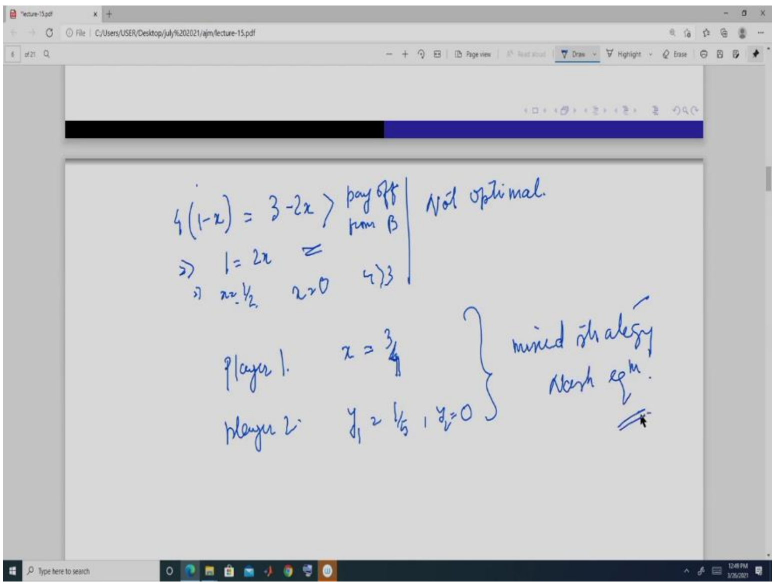

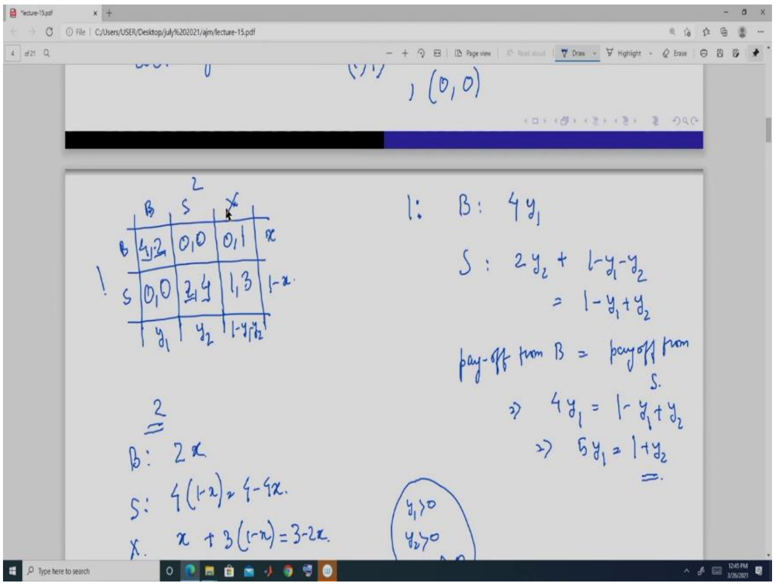

Now, let us look at another possibility that is when payoff from S is equal to payoff from X and it is greater than the payoff from B, i.e $4(1-x) = 3 - 2x >$ pay off from B. So, this is equal to $1 = 2x \Rightarrow x = \frac{1}{2}$when x takes value half. But here now remember this. Look at the game. If player 2 plays S and Y with some probability and plays B with 0 probabilities right? then player 1 knows that if it attaches some probability to B it will get 0; and it will get positive always when it attaches some probability to S. So, it will always play S. So, if it always play S in this situation then the best response for player 2 is to play 4. So, that is why this situation is not going to happen. So, this is not optimal, okay.

This is not optimal because if player 2 attaches some probability to S and X then player 1 is always going to play S. So, x is always going to be 0. The moment x is always going to be 0, here you plug in x is equal to 0, so 4 is greater than 3. So, that is why player 2 is always going to play S and not X and not going to attach any probability to X.

So, from these what do we get? So, we get that only this- $2x = 3 - 2x >$ payoff from S, is one possible outcome, okay. So, so if this is one possible outcome that is when player 1 attaches probability 3 by 4 to this strategy and 1 by 3 to this strategy then player 2 plays this with some probability and this with some probability and this with 0 probability.

So, here plug-in y is equal to 0. So, it means that x y1 is equal to 1 by 5, i.e $y_1 = \frac{1}{5}$. So, it means that player, so what is going to be our mixed strategy? Mixed strategy for player 1, that is x is going to be 3 by 4 and for player 2 y1 is going to be 1 by 5, y2 is going to be 0, okay. So, this is the optimal outcome, because if player 1attaches probability 3 by 4 to this and 1 by 3 to this then player 2 is indifferent in playing B and X. And the payoff it gets from playing B with some Probability and X some probability is greater than S. So, it is going to attach 0 probability to y is equal to, y. So, y2, 0 probability to y, 0 value to y is equal to 2, sorry y2.

So, since payoff from S is less than payoff from B and X so it is attaching 0 probability here. So, y2 is equal to 0. And y1 and this is going to take some value. What is that some value? Now, for player 1 to be indifferent so that it is going to attach some probability to B and S x y1 should take this when y2 is taking value 0. So, so that is why this- Player1: $x = \frac{3}{4}$, player 2: $y_1 = \frac{1}{5}, y_2 = 0$ is a mixed strategy Nash equilibrium, okay.

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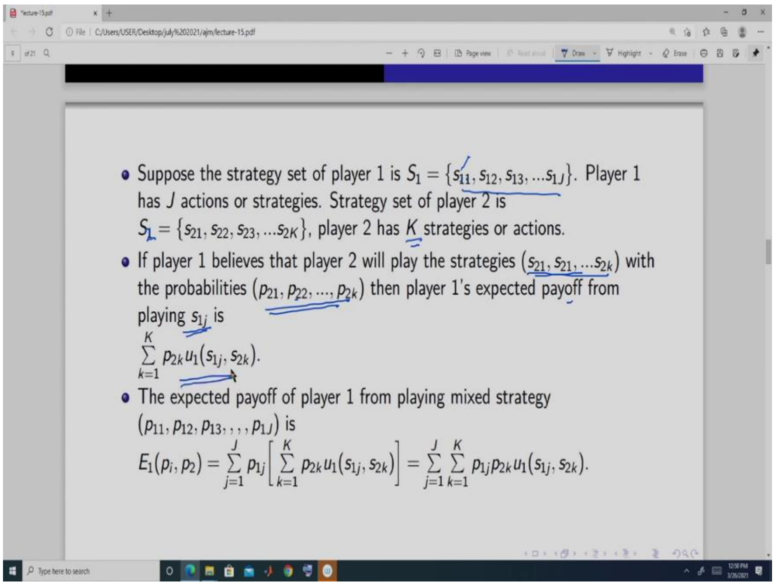

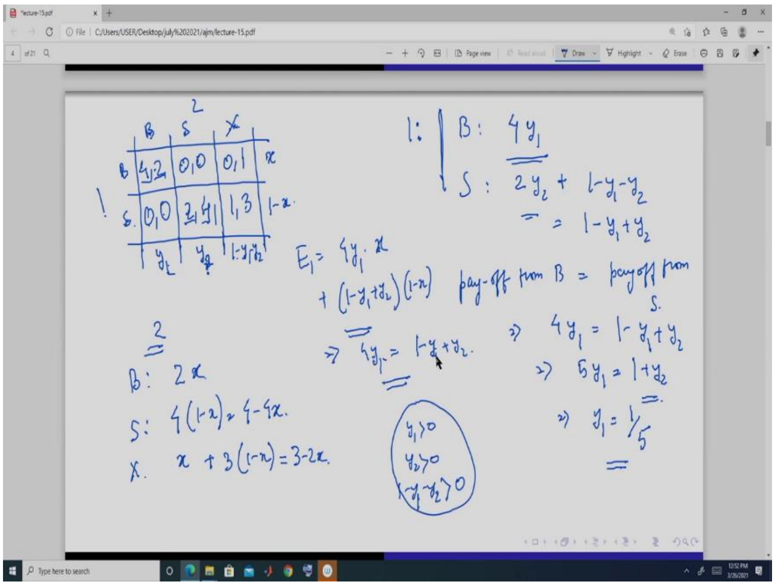

Now, more formally what do we define? That suppose there are 2 player and the strategy set of player 1 is this- $S_1 = \{s_{11,}s_{12} \dots . s_{1j}$. So, player 1 has J strategies from 1 to J. So, all this small, so this is for, signifies player 1 and this is the strategy 1 or action 1. So, there are J actions, capital J. And for player 2 this should be S2$= s_{21}, s_{22} \dots \dots s_{2K}$. First subscript is player and the second is the strategy.

So, s21, s22 s23, s2k. So, these are the strategies of player 2. And it has k strategies, okay. And player 1 believes that player 2 will play strategy this- $\{s_{21}, s_{22} \dots \dots s_{2K}\}$with probabilities this- $\{p_{21,}p_{22} \dots . p_{2k}\}$, okay. So, it is attaching some probability. So, this can be any number between 0 and 1, okay each of these. Then player 1's expected payoff from playing any strategy j from this set is this- $\Sigma_{k=1}^{k} = \mathrm{p_{2k}u_1(s_{1j}, s_{2k})}$. So, this is actually you can say here; when I say that player 1 is playing B, player 1 is playing B, so it is summing over all the possible outcomes from this action that is B with probability y1, S with probability y2, X with probability 1 minus x1 minus x2. So, it is this- $B = 4y_1$. And when it plays S it is 0 with probability y1, 2 with probability y2, 1 with probability 1 minus x1, y1 minus y2. So, it is this. So, this is what this definition, this says.

Now, the expected payoff of player 1 from playing mixed strategy, this. Now, player 1 is playing the mixed strategy, okay. So, it is attaching this. So, this, if it attaches this probability- $\{p_{11}, p_{12}, \ldots . p_{1j}\}$ to this-$E_1(p_i, p_2) = \Sigma_{j=1}^{j} p_{1j}[\Sigma_{k=1}^{k} p_{2k} u_1(s_{1j}, s_{2k})$, it is going to be this- $\Sigma_{j=1}^{j} \Sigma_{k=1}^{k} p_{1j} p_{2k} u_1(s_{1j}, s_{2k})$. So, where, in this situation, so the expected payoff of 1 is 4 y1 into x plus 1 minus y1 plus y2. So, this is the expected payoff- $E_1 = 4y_1.x + (1 - y_1 + y_2)(1 - x)$. And player 1 is going to choose the expected payoff in such a way that it should be maximum with respect to x.

So, it will choose such that… so if you simply maximize this with respect to x you will get that 4y should always be equal to this- $4y_1 = 1 - y_1 + y_2$. So, it will attach probability x in such a way that these two are equal, okay. Or you can say it is only going to attach some probability when they are equal, okay. So, the expected payoff of player 1 when it is playing this mixed strategy is this-$E_1(p_i, p_2) = \Sigma_{j=1}^{j} p_{1j}[\Sigma_{k=1}^{k} p_{2k} u_1(s_{1j}, s_{2k})$. And when we… because it has J strategies so we have J probabilities. These are numbers lying between 0 and 1, and we get this, and we have this as the expected value which is this- $E_1(p_i, p_2)$.

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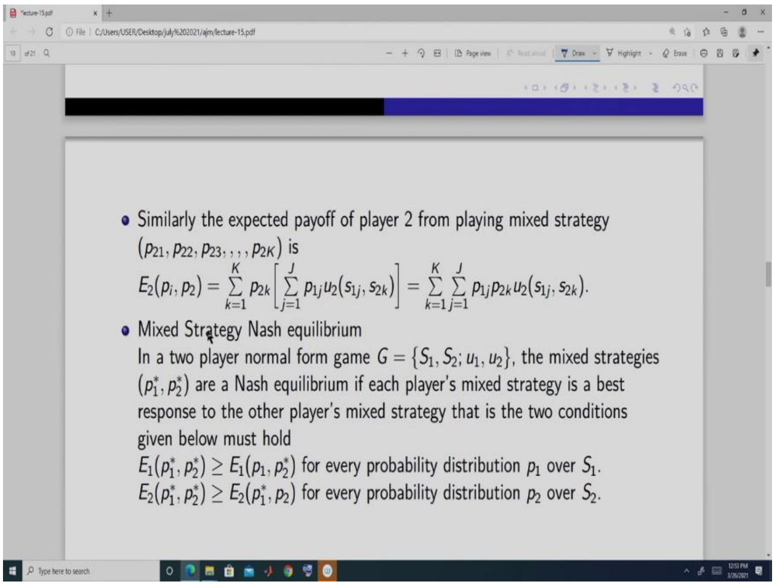

Similarly for player 2 we can write the expected, now it will be sum over first, the player 1 when player 1 is attaching some probability, and then it will be sum over its own probabilities it is this-$E_2(p_i, p_2) = \Sigma_{k=1}^{k} p_{2k}[\Sigma_{j=1}^{j} p_{2j} u_2(s_{1j}, s_{2k}) = \Sigma_{j=1}^{j} \Sigma_{k=1}^{k} p_{1j} p_{2k} u_1(s_{1j}, s_{2k})$, okay. And how do we define the mixed strategy? We define the mixed strategy that the mixed, in the 2 player game this, where S1 as we have defined above, S2 as we have define above, and these are the payoff functions; the mixed strategy, this p1 some probability attached to each strategies, p2 some probability attached by player 2 to each strategies are a Nash equilibrium if each player's mixed strategy, if each players, so like player 1 the p1 is the best response to the other players mixed strategy to p2 star. That is this, expected value of this should be greater than any other probabilities that we are assigning here- $E_1(p_1^*, p_2^*) \geq E_1(p_1, p_2^*)$. And this should also be greater- $E_2(p_1^*, p_2^*) \geq E_1(p_1^*, p_2)$..

So, together, these two defines the mixed strategy. And one way to find out the mixed strategy from these two conditions you will see that whenever the players are indifferent between their actions or strategies then only it is going to attach some probabilities. And so that gives me the probability of the other player. And for the player 1 it will attach probability in such a way so that player 2 is indifferent among its strategies. So, that gives me the mixed strategy Nash equilibrium. Thank you very much.