Advanced Linear Continuous Control Systems
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Lecture – 08
State Space Representation: Jordan Canonical Form

Today we start with Jordan Canonical Form.

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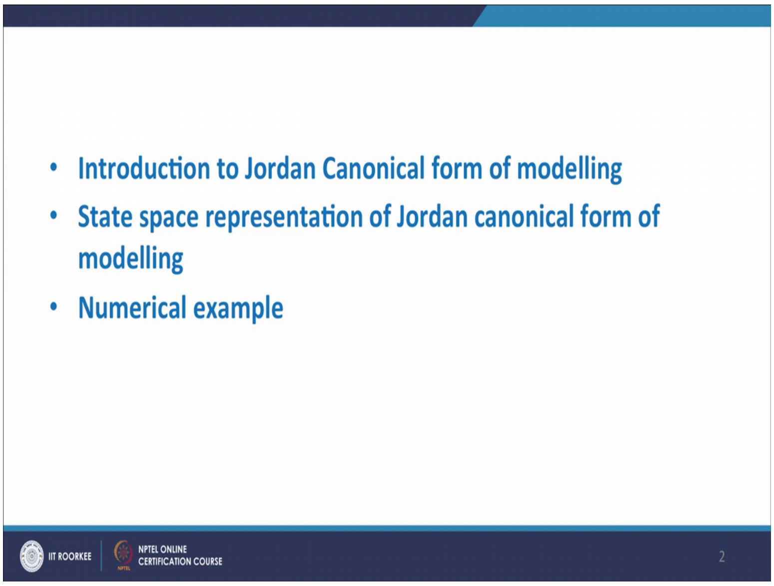

In this we will study, introduction to Jordan canonical form of modeling, state space representation of Jordan canonical form of modeling and numerical example.

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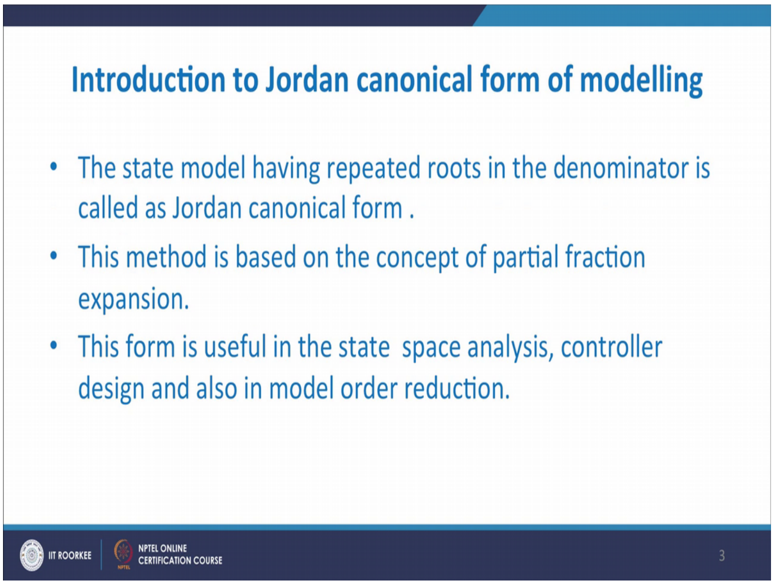

Now introduction to Jordan canonical form of modeling. The state model having the repeated roots in the denominator is called Jordan canonical form.

Last time we have studied diagonal canonical form. Diagonal canonical form means that Eigen values are distinct, they are not repeated. Here we are considering the case, where Eigan values are repeated. So, that type of model is called Jordan canonical form. This method is based on concept of partial fraction expansion. Similar to canonical form of modeling that is diagonal canonical form. This type of modeling is based on partial fraction expansion, and you know that why it is required because it is not possible to determine the inverse of such type of equations.

So, in order to determine inverse properly, we need to do partial fraction expansion. This form is useful in state space analysis, control design and also in model order reduction techniques. See here whenever we are learning any form it has some applications as I have told last time that canonical form that is, diagonal canonical form is useful in checking controllability observatory. So, this type of form is also useful in check in checking controllability, observatory and also it is used in model orderation technique.

So, last time I have told what is moderation? The model which is a higher order, which can be reduced to smaller size for that also this Jordan type of technique can be useful.

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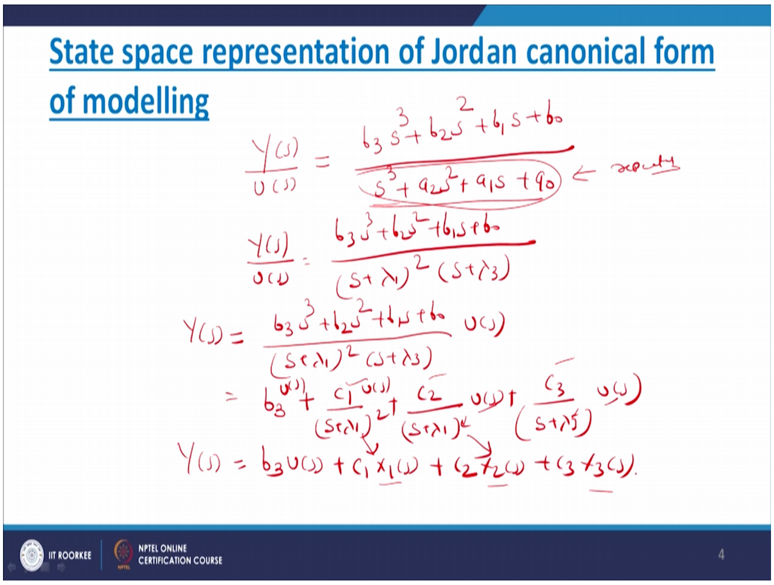

Now, we start with representing the any system in Jordan canonical, form for that purpose you write equation as Y of s divided by U of s that is equal to b 3 s cube plus b 2 s square plus b 1 s plus b naught divided by if you write as a s cube plus a 2 s square plus a 1 s plus a naught.

So, this is third order system, and you will find that here order of the numerator is also 3. So, this is a proper transfer function. And we assume that the roots of this denominator, are repeated; that means, the Eigen values are we can say roots of this denominator is repeated therefore, we write this equation as Y of s by U of s equal to b 3 s cube plus b 2 s square plus b 1 s plus b naught divided by s plus lambda 1 square plus s plus lambda 3. Here s plus lambda 1 square that is there are 2 roots or 2 roots that are repeated.

And now another root is s plus lambda 3. Now finally, write this equation as Y of s equal to b 3 s cube plus b 2 s square plus b 1 s plus b naught divide by s plus lambda 1 square into s plus lambda 3 into U of s. Now here this type of transfer function we cannot apply a inverse Laplace transforms. So, in order to apply inverse Laplace transform, this can this transfer function we have to write in this form as b 3 upon c 1 s plus lambda 1 square, b 3 into U of s here U of s then here c 2 s plus lambda 1 into U of s plus c 3 s plus lambda 3 into U of s.

So, Y s represented in terms of b 3 c 1 c 2 and c 3 c 1 c 2 and c 3 they are called residues. So now, what we do we can represent this transfer function Y of s that is equals to b 3 into U of s plus c 1 into X 1 of s plus c 2 into X 2 of s plus c 3 into X 3 of s. So, what we have done here? This U of s divided by s plus lambda 1 square we represent it as X 1 of s then this U of s s plus lambda 1 we represent it as X 2 of s and this U of s divided by s plus 3 we are represent by X 3 of s.

Now, here Y of s, but we have a state space model that is we require a state space model, that state space model is to be in a time domain; that means, for these equations we have to apply inverse Laplace transform. So, in order to get the model what we do we will consider each elements that is a X 1 s, X 1 of s X 2 of s and X 3 of s separately.

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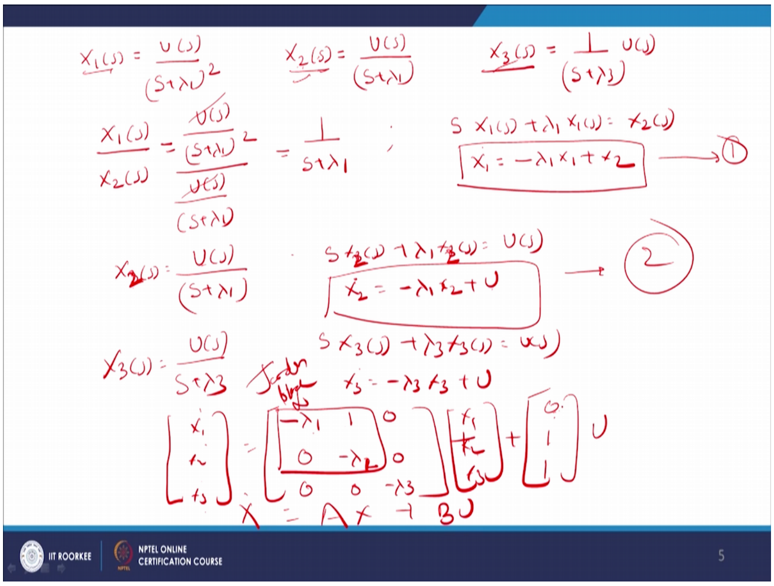

So now we write X 1 of s X 1 of s that is equals to U of s divide by s plus lambda 1 square. Now X 2 of s X 2 of s that is equal to U of s divided by s, plus lambda 1 and finally, X 3 of s equals to equal to 1 plus s plus lambda 3 into U of s.

So, we have written X 1 of s X 2 of s and X 3 of s now we want the state space model. Express this X 2 of s and X 3 of s we considered. If you consider, this X 2 of s and X 3 of s we will find that directly we can multiply it and we can get the model in time domain, but there is a problem with X 1 of s because s plus lambda 1 square is there.

So, it is difficult to apply right now the inverse laplace for this equation, now how to tackle this? So, one possibility is that what we do, we can divide this X 1 of s and X 2 of s. So, if you divide this X 1 of s divided by X 2 of s, if we actually divide it what will happen? U of s divided by s plus lambda 1 square divided by U of s divided by s plus lambda 1. So, this U of s U of s will cancelled and one term of s plus lambda 1 will cancelled. And finally, we will get 1 upon s plus lambda 1.

Now, what we do we can do some cross multiplication? So, what we will get after doing cross multiplication we will get s X 1 of s plus lambda 1 into X 1 of s equal to X 2 of s. Now here we apply inverse laplace transform. So, X 1 of s that is X 1 dot equal to, now we will shift lambda 1 X 1 of s into right side, what we will get? We will get minus lambda 1 into X 1 plus X 2. So, this is a first state equation. This let us see equation number 1 X 1 dot equals to minus lambda 1 X 1 plus X 2.

Now, this X 2 of s X 3 of s is a simple. So, we can write this as X 3 of s equals to U of s s plus lambda 1. So, if you solve it, what we will get? We will get s into X 3 of s lambda 1 into X 3 of s into U of s. Now what we do? We can do it that is a X 3 dot sorry X 2 of s you can write X 2 of s U of s, s plus lambda 1 means we will get X 2 of s, here lambda 1 into X 2 of s equals to U of s. So, what we will get here X 2 dot equals to minus lambda 1 X 2 plus U.

So, what we have taken we have taken X 2 of s equal to U of s by s plus lambda 1. So, we cross multiplied it s X 2 of s plus lambda 1 into X 2 of s equals to U of s. So, inverse laplace we will get X 2 dot minus lambda 1 into X 2 plus U second equation we have got. So, we write like this is equation number 2 now about third is X 3. So, here X 3 of s equal to U of s divided by s plus lambda 3. So, how you write this equation again we cross multiply it s X 3 of s plus lambda 3 X 3 of s equal to U of s.

So, we got s X 3 of s lambda 3 X 3 of s plus U of s here also we apply inverse laplace transform. So, what we will get we will get X 3 dot equals to minus lambda 3 into X 3 plus U. Now what is the output equation? Now output equation is now after this we have to determine the state space model it can be written as X 1 dot X 2 dot X 3 dot that is equal to, what is X 1 dot? X 1 dot equals to minus lambda 1 X 1 plus X 2. So, we can write down as minus lambda 1 1 0.

Now, X 2 dot minus lambda 1 X 2 U. So, the terms of X 1 and X 3 are 0 in X 2 dot and similarly we find out the term of X 3 is 0 here that is why here we are written lambda 1 one, but in this case X 2 dot the term of X 1 and X 3 are 0. So, we write 0 minus lambda 2 0 and lastly X 3 dot equals to minus lambda 3 X 3 plus U. So, we can write down as 0

0 minus lambda 3 plus this is X X 1 X 2 X 3 plus s you see here for X 1 dot there is no terms of U.

So, we write 0, but as far as the X 2 dot is concerned, that is term of U that is U is not 0. So, we write one and similarly for X 3 dot we will get a term of. So, X dot equals to AX plus B U. So, if we represent at X dot equals to AX plus BU is this lambda 1 1 0 0 minus lambda 2 0 0 0 minus lambda 3 and whereas, B is 0 1 1. So, state equations we are determined.

Now, we have to determine the output equations. Now we have to determine the output equations. So, for a output equations, now output equation.

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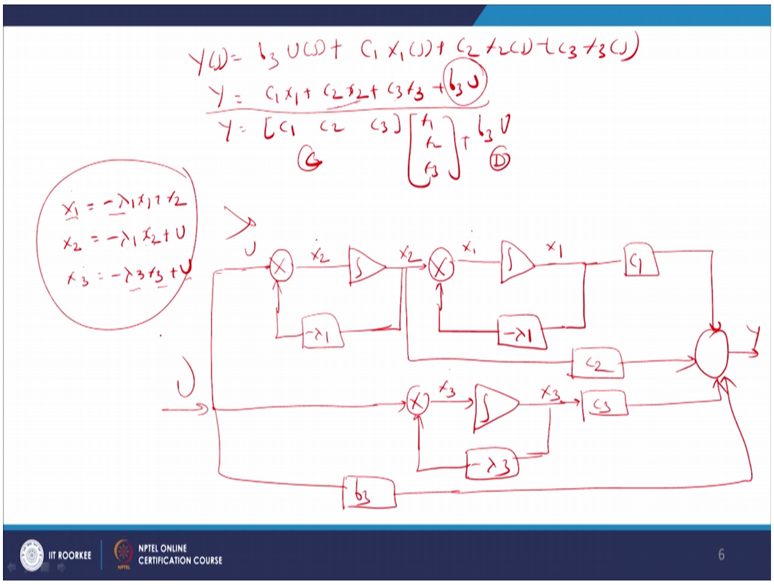

So, here output equation Y of s equals to write b 3 b 3 into U of s plus you just see here second is c 1 c 1 into X 1 of s, c 2 into X 2 of s, c 3 into X 3 of s. Now here we apply again a inverse laplace transform. So, we will get Y equals to c 1 into X 1 c 2 X 2 c 3 X 3 plus b 3 into U.

So, we have got this as a output equation c 1 X 1 c 2 X 2 c 3 X 3 b 3 U, and we have seen that here we got X 1 dot X 2 dot X 3 dot lambda 1 here is actually we write as lambda 1 of lambda 2 lastly we have taken a it is actually lambda 1. So, lambda 1 0 0 0 lambda 3 X 1 X 2 X 3 0 0 1 U, because here this particular block this call as Jordan block and here in this block this lambda 1 are repeated and again we find that in this particular case you have 0 and 1 is there.

But if you see the actual canonical form actual diagonal canonical form here we will get 0 here we will get 1, but in Jordan that 0 and 1 has been interchanged, you can easily observed here. Now here Y of s equals to b 3 of s c 1 X 1 of s c 2 X 2 of s and c 3 X 3 of s and now why we are written like this, Y equals to c 1 c 2 c 3 X 1 X 2 X 3 plus b 3 into U now this is an output model this is C matrix and this is an a D matrix C matrix and D matrix and Y here d has come D has come; because if you see your original transfer function than a proper transfer function and last and in many of our previous classes, we are already seen that whenever there is a proper transfer function that this is there exists a term of b.

So, in this way we have determine the model a state space model in Jordan canonical path. Now after this model has been determined our main purpose is to verifies performance. So, this performance can be verified through any software we can use a mat lab also, but before that we have to develop the model in stimuli. So, we have directly now a block diagram of a Jordan canonical form. Now how to develop the model last time you have seen the, the order of the system or number of states which are present equals to number of indicators.

So, we will find out the order is 3 therefore, how many integrators is required we require 3 integrators. So, first of all what we do first of all write the equations and then we try to formulate the model. So, what is the equation we have derived it? We writing down the equations lambda 1 equals to X 1 dot equals to lambda 1 X 1 plus X 2 X 2 dot that is equals to minus lambda 1 X 2 plus U and X 3 dot equal to minus lambda 3 X 3 plus U see the equations.

X 1 dot lambda 1 X 1 plus 2 X 2 dot lambda 1 X 2 plus U X 3 dot equals to minus lambda 3 X 3 plus U. Now we have to develop the a model. So, first of all you use one integrator. So, one integrator for these equations, here I am using an integrator like this. Now here this is X 1 dot, and now here is X 1 and now you and now this X 1 dot equals to minus lambda 1 X 1 that is we will have to show here minus lambda 1 and now here we have to show X 1.

So, X 1 dot equal to minus lambda 1 X 1 plus X 2. So, I am showing X 2 here now about the second equation second equation is X 2 dot equals to minus lambda 1 X 2 plus U. So, here again we have to use one integrator. So, I am using one integrator here. And here I am showing because of the integrator will we will show X 2 dot and now here we using again a summation block. So, here X 2 dot equal to minus lambda 1 X 2 plus U.

Now, about the third X 3 dot equals to minus lambda 3 X 3 plus U; that means, here also we have to use a another integrator. So, here I am showing integrator like this it is another integrator summation block. So, here this is X 3 here X 3 dot and here we are showing minus lambda 3. So, you we have shown lambda 1 minus lambda 1 minus lambda 1 then another is minus lambda 3 and here X 3 dot equals to equal to minus lambda 3 X 3 plus U.

Now, here U this is also U this is also U. So, it is common so we can combine it. So, all the states we have shown see all these states; that means, this one these equation we are represented here, but complete state the model involve state equation as well as the output equations. So, here also we have to show output equation. So, what is the output equation Y equals to c 1 X 1 c 2 X 2 c 3 X 3 b 3 into U. So, therefore, now here now here we will show output Y.

So, what the which signal is coming here that is c 1, c 1 into X 1. Second is c 2 into X 2, so c 2 into X 2. So, let us see this is X 2. So, here we take c 2 and we will connect here. Now about c 3 c 3 into X 3. So, we will connect c 3 here and this is output. So, this part we have completed Y equals to c 1 equal to X 1 c 2 X 2 c 3 X 3.

But now here is another term b 3 U now we have to use connect a b 3 U. So, U is there. So, what we do simply we will connect the block b 3 into U. So, this is U. So, we will find that we have determined the state space model in Jordan form when Jordan form is when there are repeated Eigen values.

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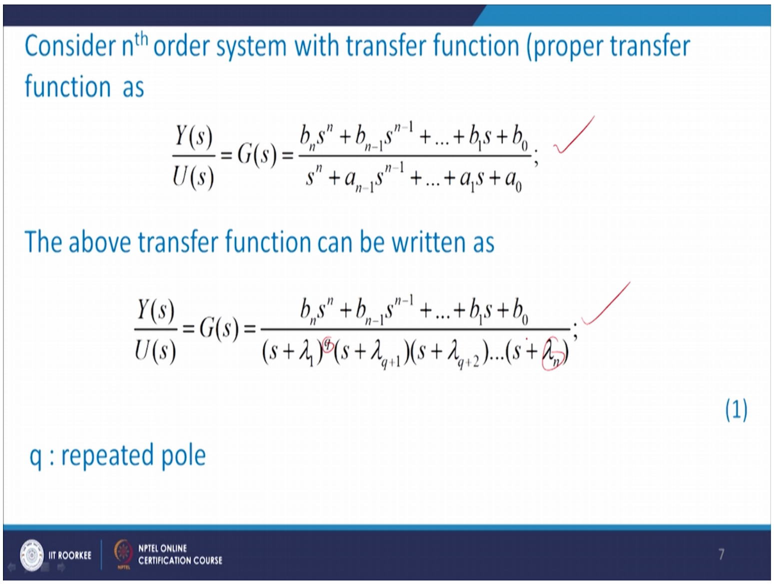

So, whatever we have seen. So, I am again I will again explain in a systematic manner. So, for that purpose what we I have done here I have taken a same transfer function Y s by us equals to bn sn b of n s minus 1 b 1 s plus b naught s plus lambda 1 q s plus lambda q plus 1 s lambda q plus 2 s lambda n.

That is this is the generalized form, generalized form example we have taken the third order only and the here I have shown q q are repeated pose it can be 3 4 5 and again, but finally, it comes lambda n.

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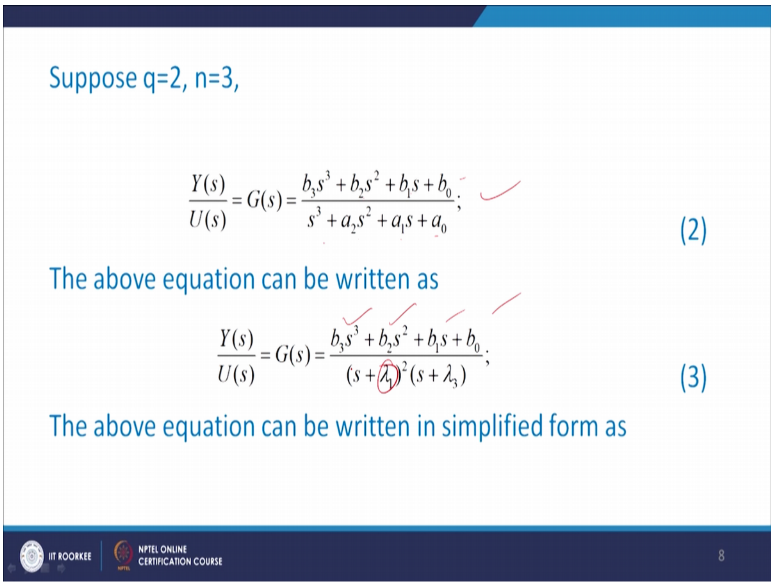

Same thing suppose q equals to 2 n equals to 3. So, here b 3 s cube b 2 s square b 1 s b naught s cube a 2 s square a 1 s a naught the transformation view are described earlier same thing is here.

Now, above equation can be written as b 3 s cube b 2 s square b 1 s b 0 and now here lambda 1 is the repeated Eigen value.

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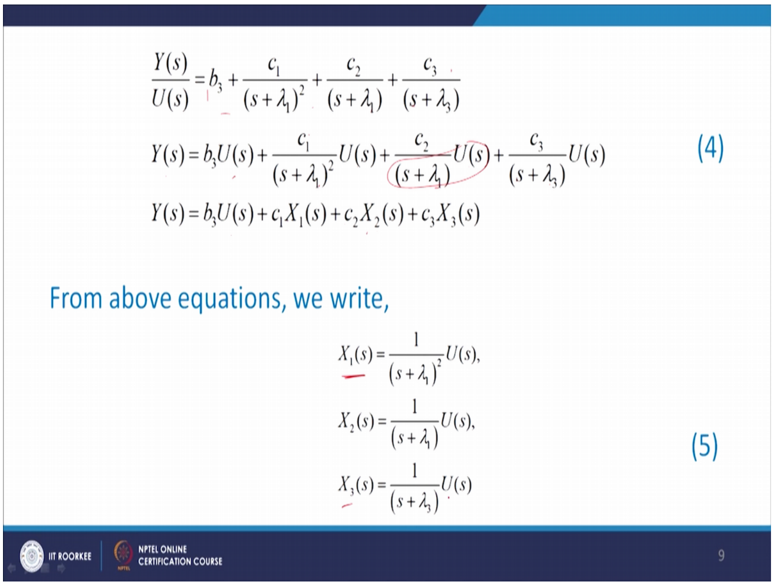

Now, above equation can be written in simplified form as b 3 c 1 s lambda 1 square c 2 s lambda 1 c 3 s lambda 3. Now we have solved this equation Y of s equals to b 3 into U of s, c 1 s plus lambda 1 square into U of s, c 2 s plus lambda 1 into U of s, c 3 s plus lambda 3 into U of s and finally, Y of s equals to b 3 U of s and this U of s s plus lambda 1 square you have taken as a X 1 of s.

C 2 is here U of s s plus lambda of one we have taken as X 2 of s, this c 3 U of s divided by s plus lambda 3 we have taken as X 3 of s. Now again from above equation we write this X 1 of s same thing 1 upon s plus lambda 1 square into U of s X 2 of s equals to 1 upon s plus lambda 1 into U of s X 3 equals to 1 upon s lambda 3 into U of s and now for this equation.

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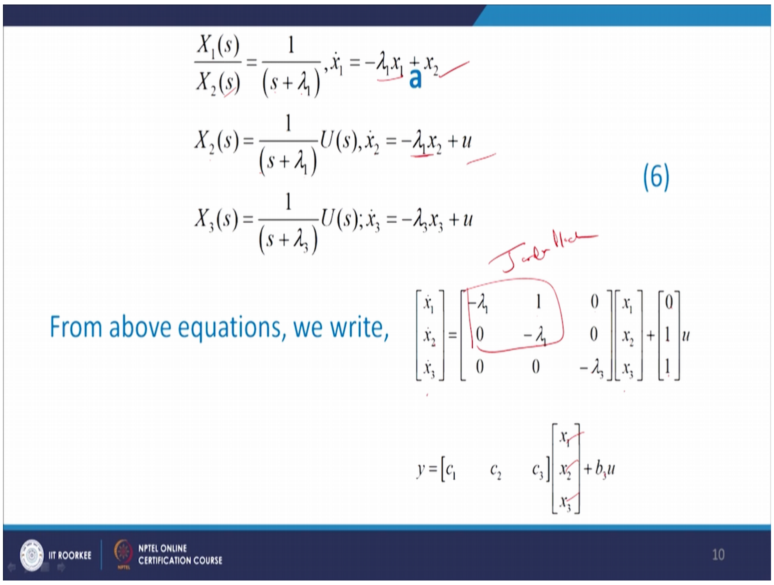

We apply an inverse laplace transform and finally, we have got result X 1 by X X 1 of s by X 2 of s equal to 1 upon s plus lambda 1. So, X 1 dot lambda X 1 plus X 2 X 2 of s equals to 1 upon s plus lambda 1 U into U of s you write as X 2 dot minus lambda 1 X 2 plus U, and third X 3 of s equal to 1 upon s plus lambda 3 into U of s. So, we are written as X 3 dot equal to minus lambda 3 X 3 plus U and finally, this has been written in the state space model as X 1 dot X 2 dot X 3 dot equal to minus lambda 1, 1 0 X 2 dot 0 minus lambda 1 0 X 3 dot equal to 0 0 minus lambda 3 X 1 X 2 X 3 and you see here a 0 1 1 you know a discussion I have told that that the 0 has been appeared here in a diagonal canonical form.

But in case of Jordan canonical form this instead of 1, here we have comes instead of 0 here comes one and here instead of 1, 0 has come and this particular block is called as Jordan block. And then finally, output Y equal to c 1 c 2 c 3 X 1 X 2 X 3 plus b 3 of U.

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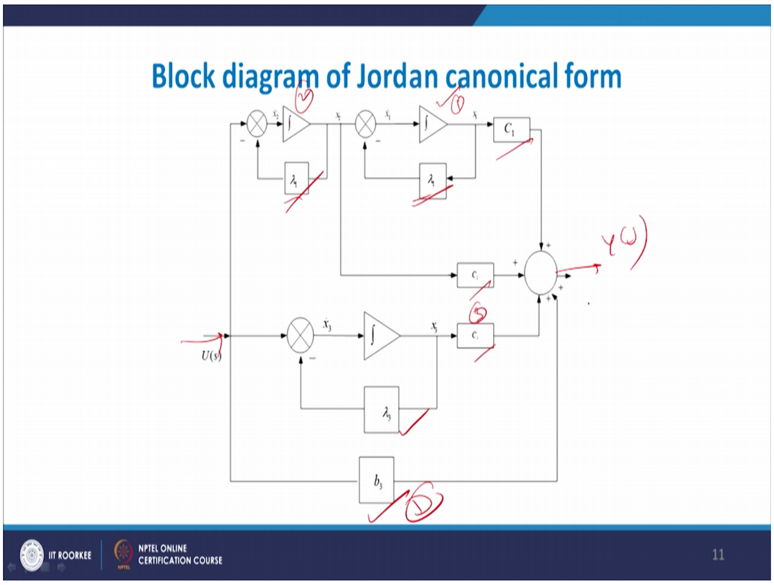

So, now we see the block diagram of Jordan canonical form. So, 3 integrators integrator 1, integrator 2, integrator 3 then here lambda 1 lambda 1s here minus then here is lambda 3 3 Eigen values this is lambda 1 lambda 2 are repeated Eigen values.

This c 1 c 2 c 3 the elements of the output matrix, then here b 3 which concerned with the coupling matrix D input and here is a output Y of s.

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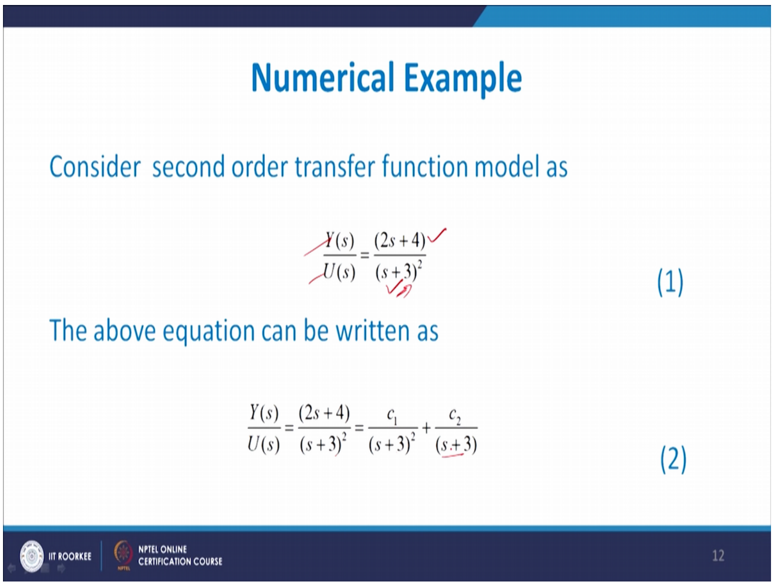

So, all theory we have seen now we solve an example this simple example. So, here I have taken a Y s by us equal to 2 s plus 4 divided by s plus 3 whole square. So, how to solve this problem, as I have told that, that this has repeated pole X plus 3 whole square.

So, here we have to use the concept of the partial fraction expansion. So, what we are done 2 s plus 4 s plus 3 square equal to c 1 in divided by s plus 3 square plus c 2 s plus c.

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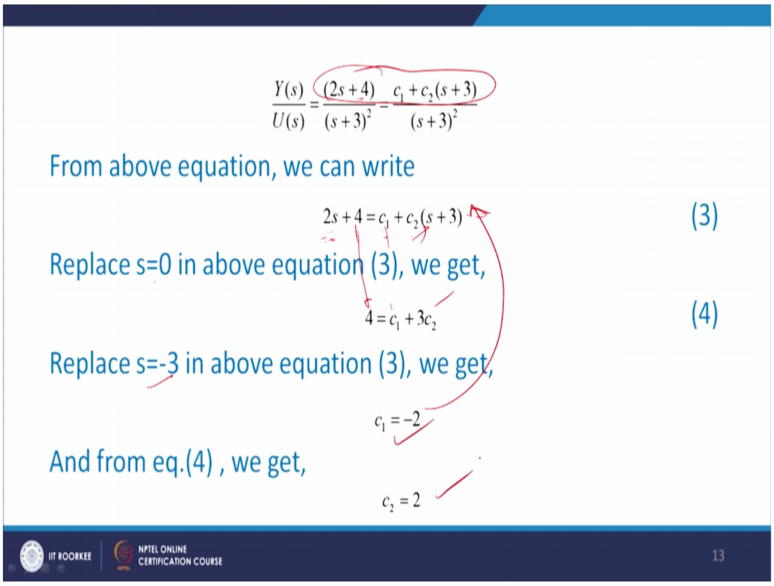

Now, we have to solve it. So, Y s by us equal to 2 s plus 4 s plus 3 whole square equal to c 1 plus c 2 s plus 3 into s plus c square. Now what we do ? This equation this part we will compare. So, 2 s plus 4 equal to c 1 plus c 2 s plus 3.

So now, we have to solve it. So, what we will do we replace s equals to 0 in this equation. Then we replace the s equals 0 in this equation what will happen it becomes 4 c 1 and 0 means 3 into c 2. So, we will get equation 4 equal to c 1 plus 3 c 2. Next now what we do we replace s equal to minus 3 in this equation. So, when we replace the s equals to minus 3 equation this will become 0. So, what will get these 3 minus 6 plus 4 finally, we will get c 1 equal to minus 2.

And now what happened you have got c 1 equals to minus 2 we also required the value of c 2. So, what will do we replace c 1 equals to minus 2 in this equation and we will get c 2 equal to 2.

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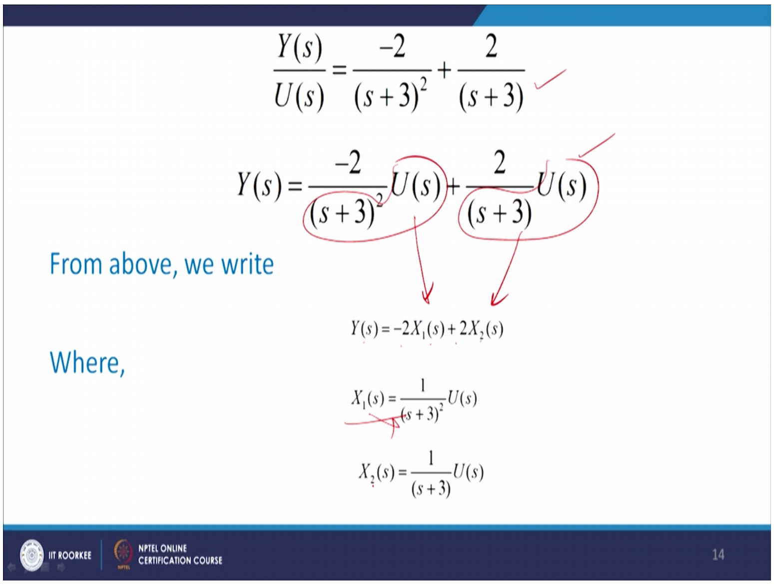

Then Y s by us equal to minus 2 divided by s plus 3 square plus 2 divided by s plus 3. Now you solve it Y of s is equals to 2 s plus 3 whole square U of s plus 2 s plus 3 into U of s, what we are done? Now what we have to do U of s s plus 3 square you right, as X 1 of s and then U of s X plus 3 we have to write as X 2 of s.

Therefore what we will get Y of s equal to minus 2 X 1 of s plus 2 X 2 of s and this Y of X 1 of s equals to 1 by s plus 3 whole square into U of s X 2 of s equal to 1 upon s plus 3 U of s they will find that again this equation difficult to solve by inverse laplace transform. So, what we have doing here we are dividing X 1 s and X 2 s.

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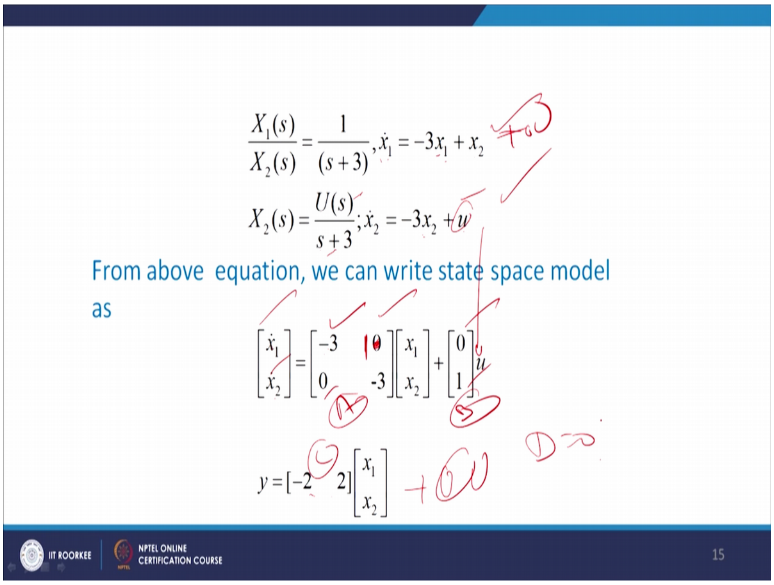

So, when we divide X 1 s by X 2 s what will get? 1 upon s plus 3, now you cross multiply it. So, after a cross multiplication what we will get we will get X 1 dot equal to minus 3 X 1 plus X 2. And X 2 dot X 2 of s equal to U of s plus s plus 3 again after a cross multiplication, what we will get? X 2 dot minus 3 X 2 plus U.

So, 2 state equation you have got. So, X 1 dot X 2 dot then A matrix minus 3 0 0 minus 3 and here 0 1 and Y equals to minus 2 2. This is the C matrix this is a matrix and here is the B matrix here, there is no D matrix because this transfer function is strictly proper transfer function ok. Now we see X 1 of s divided by X 2 of s equal to 1 upon s plus 3 X 1 dot minus 3 X 1 plus X 2 X 2 dot X 2 of s equal to U of s s plus 3 that equals to X 2 dot equal to minus 3 X 2 plus U.

Now we have to replace the equation. So, we will write as X 1 dot equal to minus 3 X 1 plus X 2. So, your minus 3 and here instead of 0 we will get 1. So, X 1 dot equals to minus 3 X 1 plus X 2 and 0 U here out 0 U and here X 2 of s equal to U of s by s s plus 3 we will write as a X 2 dot equal to minus 3 X 2 plus U. So, we write as X 2 dot equal to 0 0 into X 1 minus 3 X 2 whereas, term of U has come. And finally, Y equals to minus 2 2 X 1 X 2 that is 2 X 1 plus 2 X 2 and here 0 into U D is 0. Now these are the references.

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Thank you.