Advanced Linear Continuous Control Systems
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Lecture – 35
Pole Placement by State Feedback (Part-III)

Now, we start with the Pole Placement by State Feedback - Part III.

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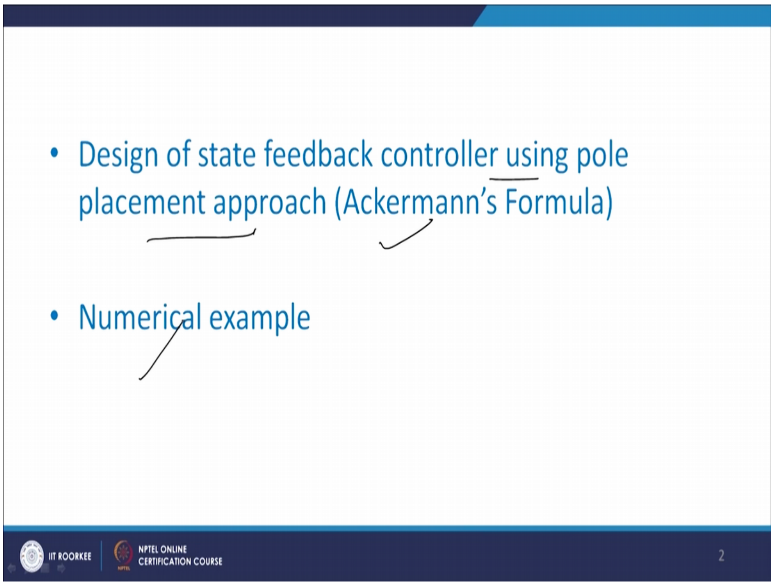

In this we will study design of state feedback controller using pole placement approach that is Ackermann's formula and whatever technique we will study that is based on Ackermann's formula that base that, that means, we will solve a numerical example based on this Ackermann's formula.

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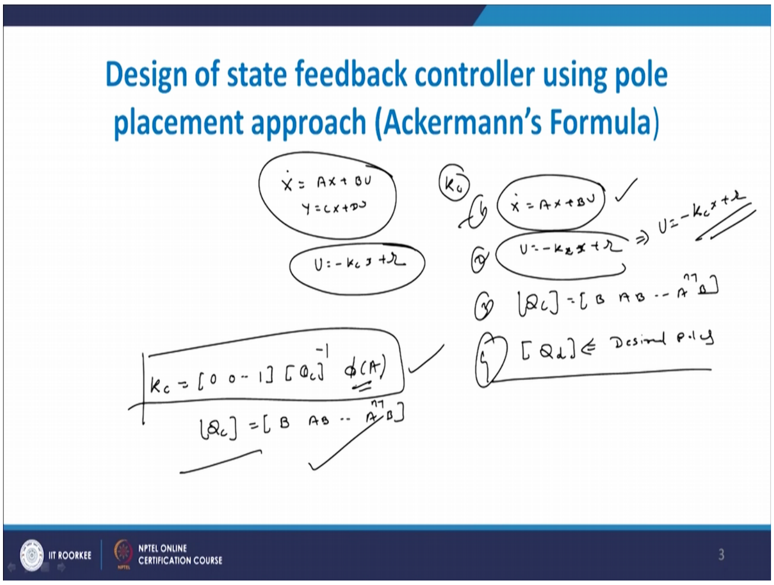

Now, design of state feedback controller using pole placement approach that is Ackermann's formula.

As our system is a state space system. So, that state space system is X dot equal to AX plus BU and here output is Y equal to C X plus D; C X plus D. this is your plant. And for this plant we have to design a controller that is; we are designed controller that is k c and previously we have seen that your control law U equal to minus k c x plus r this is our control law.

And here, all this. methodology designing of the controller is basically concerned with this equation only: X dot equals to AX plus B only. And law is concerned with this same U equal to minus k X plus k c x plus r this is your control law. So, these are relationship between these. And these and in that there is a 1 part coming a controllability issues Q C; controllability B AB A raise to n minus 1 B.

And 1 important thing which will come that is a desired pole; in this we will come a desired pole that is say Q D which is the desired poles. That means the designer of controller in a state space it depends upon the state space model X dot equals to AX plus BU, second it depends upon the control law that is U equals to minus k c x this is k that is control law is U equal to minus k c x plus r this is a control law. Then it depends upon the controllability matrix B AB A raise to n minus 1 B and Q D is the desired poles. That means now, whatever the formula which will get it depends on these 4 condition: 1 2 3 and 4. So, we will find at base on these conditions. the Ackermann's has proposed a formula for the designing of the controller. So, controller design using the Ackermann formula gives as 0 0 1.

Now, here this is Q C which is controllability matrix inverse into phi of A this Q of C is nothing but B AB A raised to n minus 1 B through controllability inverse into phi of A. So that is very important formula, we can directly get the controller there is no need for any iterative approach; but now how to get these particular formula that means, particular this formula; how to get this formula. Now, we start deriving the Ackermann formula and the technique which was used here to get this is a Cayley-Hamilton theorem.

As we have seen that earlier in a getting the state transition matrix that is Cayley-Hamilton theorem says that every square matrix is satisfied own characteristic equations. So, this phi of A is nothing but the older. characteristic equation which is in terms of eigenvalues, or in terms of lambda, in terms of characteristic coefficient that has been repaired by phi of A, so that is basically a Cayley-Hamilton theorem. That is, Ackermann formula is nothing but the a. use of or base basic of this is Cayley-Hamilton theorem.

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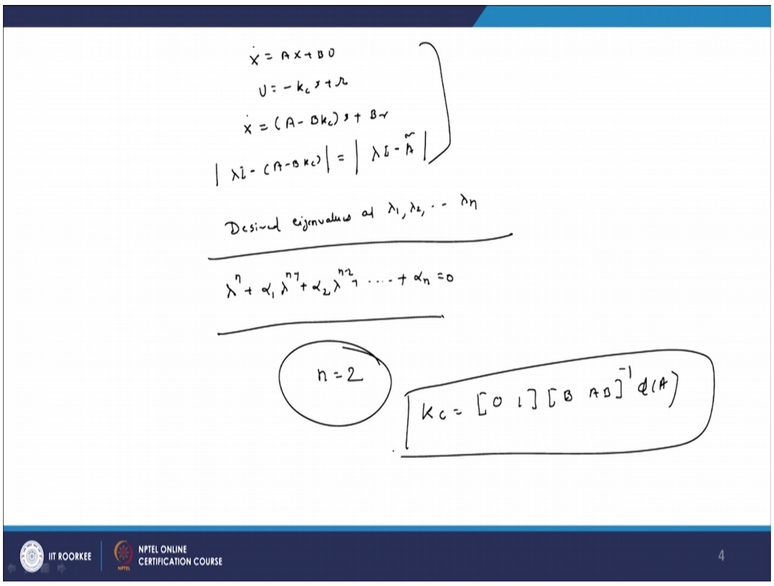

Now, we start deriving the results. So, so again we start writing down. original plant X dot equal to AX plus BU. U equal to minus k c x plus r, and here your X dot A minus B k c into X plus B into r. Now, the characteristic equation in terms of lambda, lambda i minus A minus B into k c. Now here this lambda i, this A minus B k c we can write down as A bar.

Now, we have desired. eigenvalues or the poles which has to place. So, that desired eigenvalues or poles we can write down as: desired eigenvalues at lambda 1 lambda 2 up to say lambda n. So, here this part which we have got based on the control law, so these control law is working like this and we have desired poles: so desired poles are lambda 1 lambda 2 lambda n. So, based on the desired poles we can also write down the characteristic equations. So, based on the desired eigenvalues the characteristic equation we can write down as: lambda raise to n. alpha 1 lambda raise to n minus 1. Alpha 2 lambda raise to n minus 2 plus alpha n equal to 0 this we have written, we have got.

Now, we have to develop a formula, but when we go for n-th order you will find that to determine inverse and determine the characteristic equations it may be difficult particularly for a valuation point of view. So, normally when we do a derivations or to get a generalized results for in it is sometimes what we are doing, we are going for some lower order; once we get a result for low order then we can prove that since that the result similar result we can obtain for the n-th order system. So, here for simplicity what we are doing, we will design the controller for n equals to 2 only. That is, the controller k c which we should get 0 1 B AB into phi of A. This is to be determined.

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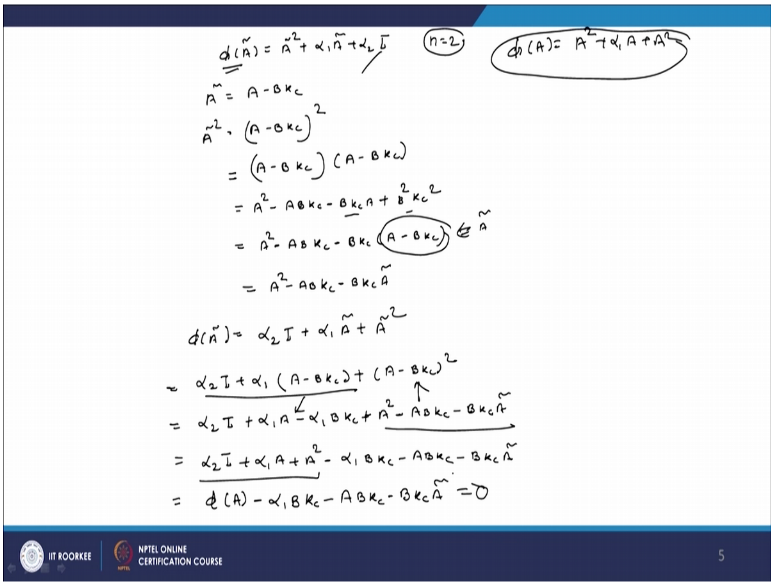

Now, according to Cayley-Hamilton theorem what we can write down, this phi of A equal to A bar square plus alpha 1 into A bar plus alpha 2 into I. Here we will find that,. this is your the characteristic equations. lambda n and now here our original plant is A but because of the control law these A has been changes to A bar. So, A bar is plays an important role and therefore based on the desired eigenvalues we can write down as phi of A A bar square plus alpha 1 A bar alpha 2 into I; this is basically for when n equals to 2.

Now, we will find that this A bar. equal to A minus B into k c. Now this A bar, you see this A bar is nothing but A minus B into k c. But in the equation we required A bar square; so what we can write down this A bar square. equal to A minus B into k c whole square. So, now, there is need to solve this equations. So, here we can write down this as: A minus B into k c. multiplied by A minus B into k c. Now, if you multiplied A with respect to A B k B k C, so you will get A square minus AB into k c. minus B k c into A plus B square into k c square.

Now, here after solving this will get A square. minus AB into k c minus here B k c B k c is common. So, we can write down as B into k c let A minus B into k c. And we know that this A minus B k c is equivalent to A bar. So, we can write down this equation as: A square minus AB into k c minus B into k c into A bar.

So, we have got A bar square. Now, here this phi of A actually it basically is phi a bar. Now, we have got these equations. Now, what will do? We replace this value and see what type of results we can achieve. So, here phi A bar that is equal to alpha 2 into I plus alpha 1 into A bar. plus A bar square. So, A bar we know A minus B k c that is our now, the desired. Desired means that we are getting after the controller it come into existence.

So, now this is the equations we are written according to the Cayley-Hamilton theorem. So, alpha 1 I alpha 1 A bar A bar square, now you replace the (Refer Time: 11:33) A bar and A bar square in this equations. So, we can write down this as alpha 2 into I plus alpha 1 what is A bar, A bar is A minus B k c. B k c k c is a controller. plus A minus B k c whole square. Now, we solve this equations we will get alpha 2 into I plus alpha 1 into A minus alpha 1 B into k c. plus. Now, this is a A minus B k c square. So, we can write down as A square minus this is A bar A minus. square that is A bar AB into k c minus B into k c into A bar.

Here A minus B k c. square that is nothing but A bar square, so A bar square we have already determined. So, simply without solving here we are replace the value of A bar square as A square AB into k c minus B k c into A bar. Just see here, this part. This part is nothing but this 1. And this portion is nothing but this one. Now, again we simplify these equations so we will get: alpha 2 into I plus alpha 1 into A. Now, what we are doing? That terms concern with the A we have taken a one side particularly we have taken alpha 2 I alpha 1 A plus A square. Now, here alpha 1 B into k c. minus AB into k c minus B k c into A bar.

As we have got phi A bar as A square plus alpha 1 A plus alpha 2 A, but if you write the equation as phi of A is nothing but A square plus alpha 1 A plus A square. So, this phi of A this particular equation it is of the original system A, because original system also satisfy the Cayley-Hamilton theorem. Because, here we have seen there Cayley-Hamilton theorem condition we have written fine find A bar. So, A bar means we have got this portion and when we take talking about the A, A is the original system the Cayley-Hamilton means theorem says phi of A equal to A square plus alpha 1 A plus A square. So, we will find that.

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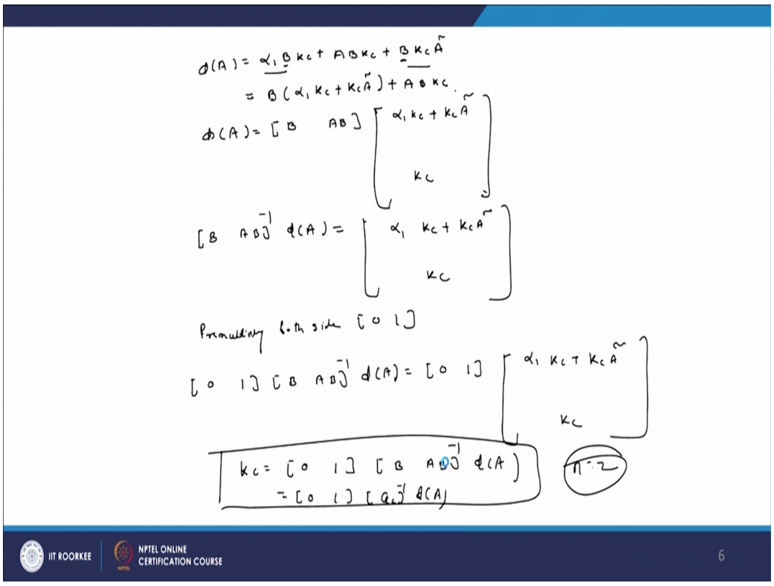

This particular equation we write down as phi of A minus alpha 1 B. into k c minus AB into k c minus B k c A bar and this is. 0. Now we will see, phi of A that is equal to alpha 1 B into k c plus AB into k c plus B into k c into A bar. So, what actually we have done?

This phi of A bar that equal to 0. So, we can write down this equation as: phi minus this equal to 0. So. finally, we have written the condition in terms of phi of A equal to alpha 1 B into k c plus AB into k c plus B k c into A bar.

Now here, what we can do? We can taken out this B common here from this part, this and this. If you can take B common, so we will get alpha 1 into k c plus k c into A bar plus AB into k c. And therefore, this phi of A. you can write down as B. AB as alpha 1 into k c plus k c A bar and here as k c. So, we written equation in terms of A phi of A equal to B AB alpha 1 k c k C A bar into k C. Now, what will do? We pre multiply both these equation by 0 1, and prior to this 1 we write the equation as B. AB inverse. phi of A equal to alpha 1 k c plus k c A bar into k c.

Now, here we pre multiply; pre multiply the both side by say 0 1 matrix. So, we can write down as say 0 1 B AB inverse say phi of A to say 0 1, and here we will get we can write down as: alpha 1 this k c plus k c into A bar it has equal to k c. Now, after solving it what will get this k c equal to 0 1 B AB inverse into phi of A. So, this is the controller which is determined, and particularly the phi of A here we are considered as the desired characteristic equations. So, what are the desired characteristic equation is there; we write in terms of lambda and replace that lambda by A.

So, this equation we have determined for when n equals to 2. Now, how will you get, write how will write the control design equation when for anywhere is of n.

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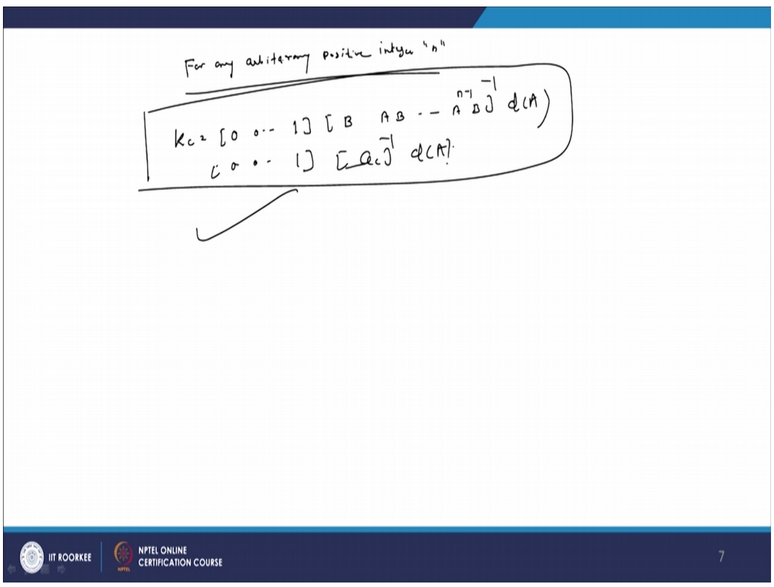

So, for any arbitrary for any arbitrary positive integer n we can write down as this k c is controller as 0 0 0 1 B AB A raise to n minus 1 B inverse into phi of A. This is the the controller design, so you will find that here k c equals to 0 1 B AB inverse phi of A that means same thing here 0 1 this is Q C inverse into phi of A.

Now, just see in when, so we have a state space model X dot equals to AX plus BU we have control law. Now, here the controllability matrix: so here what we are done here. Here this A minus B k c we can write down in terms of a bar. So, desired eigenvalues that is lambda 1 lambda 2 lambda 1 that can be written in terms of this one. And this phi of A bar we have written in terms of A bar square alpha 1 A bar alpha 2 as there is for n equals 2. And now here we have a simplified. And here this alpha 2 I alpha 1 A plus A square we are written in terms of phi of A. And finally, this equations we have written and we come across the finally all doing all the steps the controller k c equals to 0 0 0 1 Q C inverse into phi of A, and this is a final controller for the n-th order system.

Now, this controller designed and this basically called Ackermann formula, this controller which we divide Ackermann formula and this depends upon the Cayley-Hamilton theorem.

Now, we start solving an example. So, we will take the same example, which were taken earlier that is. We have solve earlier example which is based on state transformation. So, same example we will we will try by means of Cayley-Hamilton theorem or that means we will solve by Ackermann’s formula.

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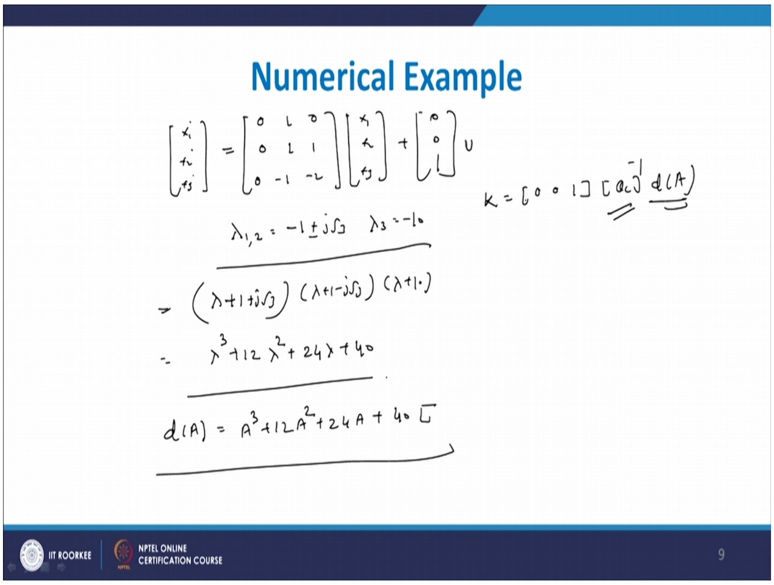

So, here original system has X 1 dot X 2 dot X 3 dot 0 1 0 0 1 1. 0 minus 1 minus 2. here X 1 X 2 X 3 0 0 1. So, this is the same plant. Here this is no need for any output equations. Now the desired poles, the desired port which we are considered earlier lambda 1 2 as minus 1 plus minus 0 3 and here lambda 3 equals to minus 10 and based on this we can get the equation has lambda plus 1 plus j root 3 lambda plus 1 minus j root 3 plus lambda plus 10.

So, if you solve this equations you will get: lambda cube plus 12 lambda square plus 24 lambda plus 40. So now, this is the desired characteristic equation. And now here according to Cayley-Hamilton theorem we take phi of A as A cube plus 12 A square. plus 24 A plus 40 into I. This is phi of A.

As the controller k involved 0 0 1 controllability inverse Q C inverse into phi of A, so phi of A or D we can determine means phi of A formula is given. So, replacing the values of a in this equation you will get phi of A. Now, there is need to determined controllability inverse and 0 0 is 1 is exist and after multi function we can get the controller k c.

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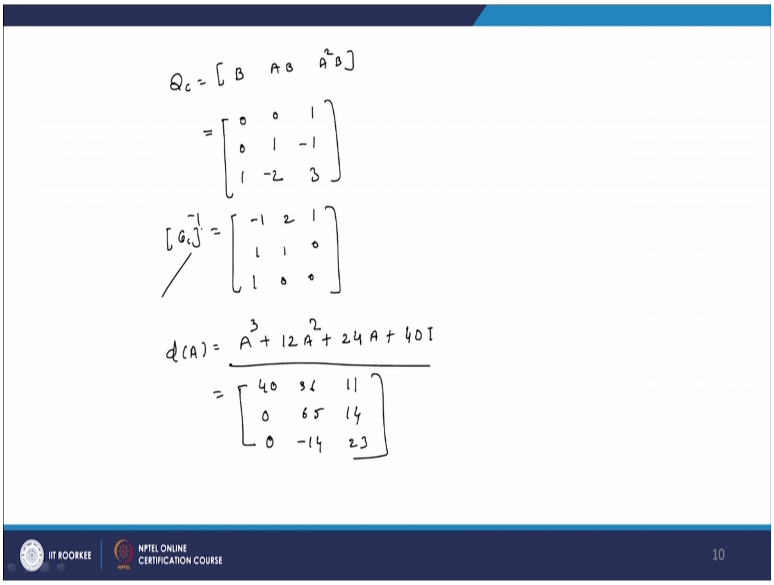

So now, first of all we determined the controllability matrix Q C as B AB. A square B. So, the values of B here. 0 0 1, AB as 0 1 minus 2 1 minus 1 3 this is Q C.

Now, this Q C is there, but in actual formula we required Q C inverse; so the inverse of this we will get Q C inverse as minus 1 2 1 1 1 0 1 0 0. Now, Q C inverse we have got. Now, we required a phi of A, this phi of A equal to A cube plus 12 A square plus 24 A plus 40. So, what we have done here; you will find that here we have got the equations that is a desired equation has lambda Q plus 2 lambda square 24 lambda plus 40 that is by desired. And now what is say that the Cayley-Hamilton that this. This equation has to satisfy the overall characteristic equations.

So, characteristic equation means we are trying with A matrix; so A A square A and this one. So, here you will find that this A cube plus 12 A square plus 24 A plus 40, so here if you solve it or replace in the values of a in this equation and after solving we come across the result as 40 36 11 0 65 for a 14. 0 minus 14 plus 23.

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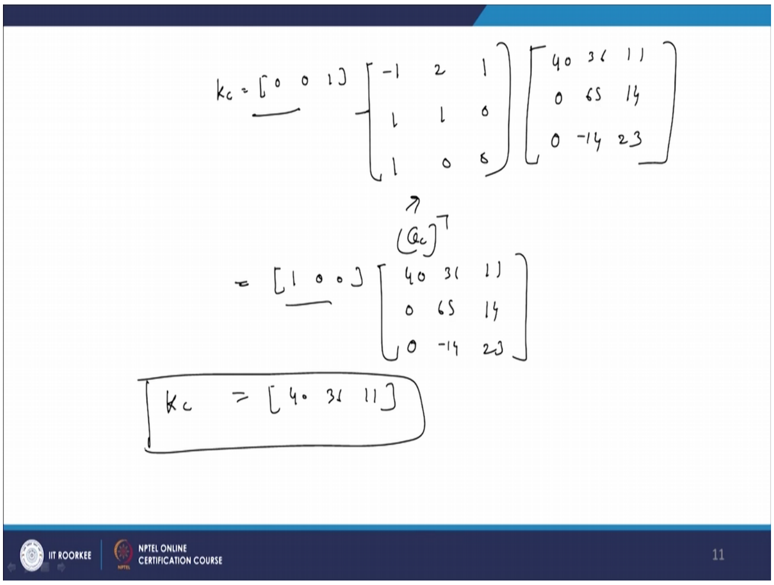

So, this phi of A we determined, Q C inverse we determined, and now the we can write down the formula as k c equal to 0 0 1. and we will get minus 1 2 1 1 1 0 say 1 0 0. This is basically your Q C inverse. And this phi of A, we can write down as 40 36 11 0 14 then 0 minus 14 this 23.

Now, after solving these equations particularly we solve this with this. So, we will get 1 0 0,. and here this we written as it is 40 36 11 0 14 and here 0 minus 14 23. And again cross multiplying we get 40 36 and 11. So now, this is the controller k c.

So, we will find that by means of Ackermann formula we have a design a controller. So, this controller we will stabilize our system and whatever the performers we have specified that can be achieved by mean the Ackermann’s formula. So, based on Ackermann formula we have got the controller design and if you are seen the various. steps here. So what we have done here, here we have taken the plant as A plant then we have taken a controller U equals to minus k c x plus r, and then this is a characteristic equations then lambda i minus A bar. Now, here we have a desired eigenvalues in terms of lambda 1 lambda 2 lambda n, so this is the equations.

Then as our the system has changed to A minus B k is k c into A bar, so we have written equation in terms of A bar. But, as for simplicity purpose we have taken n equals to 2. And then we have solve this equation and somewhere we have got lambda alpha 2 I alpha 1 A plus A. A square. then this has be written in terms of phi of A.

So, alpha 1 B k c AB k c. A bar and again after simplify everything we have got this particular equation 0 1 Q C inverse into phi of A and similarly we have written for the n-th order plant that is here 0 0 0 1 this is Q C inverse into phi of A. And based on this, one example we have solved. Following in this example what we have done we have desire decided the desired poles.

So, the desired poles we have written, then we have determined the characteristic equations. And then once this characteristic equations we have determined there has been convert in means of ah A matrix that is according to Cayley-Hamilton theorem then controller k c equal to 0 0 1 Q C inverse into phi of A. And then we use the different states. And finally, we have got controller k c as 40 36 and 11.

So, here we have seen the two methods for the controller design, but here this an one important issue which we you might have identified.

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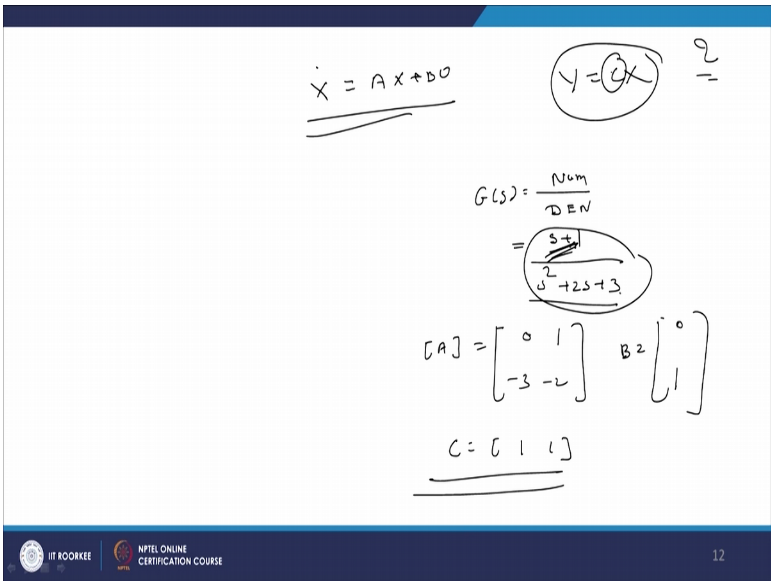

Here, the all the controller has been design based on AX plus BU. There is a no role of Y equal to C X after. State has been controller controlled by means of state equations, but the Y equals to C x. So, that part cannot taken into existence. So, by doing all this things can we how; mean did we got means having achieved completely desired performance. Definitely, there is something problem is involved, it is because if you write any transfer function G s in terms of numerator and denominator. So, if you write down this as let us say s plus 1 s square plus 2 s plus 3 and if you convert this in terms of AB matrix so A is: 0 1 minus 3 minus 2, and B matrix as 0 1 and C matrix as 1 1.

So, the system of like this and we are mainly confident on s square plus 2 s plus 3 and we have designed a controller, but this s plus 1. So, s plus 1 is called the zeros of the system. So, when the zeros are not place are not we have not be utilized in a controller design. So, definitely when we determine the actual performance this may create some problem to our system performance.

Therefore, there is need to tackle this issues. So, in a next part we will see how to handle the effect of zeros or in other words how to handle the effect of C matrix in a controller design.

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Now, these are some references: Katsuhiko Ogata, and D. Roy Choudhury, then Nagrath Gopal and Dorf. So, you can use these references. For more detail study try to solve as many examples as possible.

Thank you.