Advanced Linear Continuous Control Systems
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Lecture – 34
Pole Placement By State Feedback (Part-II)

Now we start with a Pole Placement By State Feedback Part 2. In this we will study design of state feedback controller using pole placement approach that is state transformation and once we study this approach we will going to solve one numerical example.

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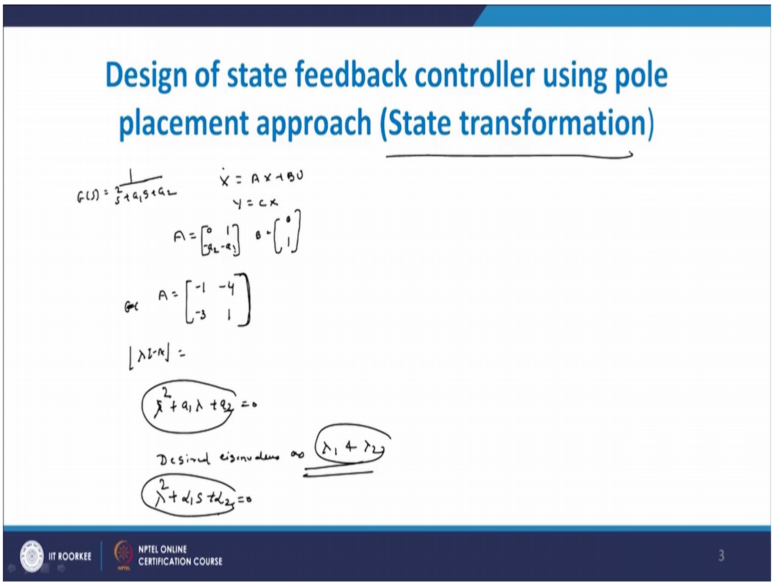

Now design of state feedback controller using a pole placement approach that is state transformation. Last time we have solved system or we have solved one example, in an example we have seen that the our given system that is x dot is equal to A X plus B U Y equal to c x.

In this case this A matrix it is in companion form that is if you write this g s g of s 1 upon s square plus a 1 s plus a 2 this can be written as 0 1 minus a 2 minus a 1 and B is say 0 1. So, we will find at if the if give us the system is a feedback transfer function or A matrix it is in companion form then we can directly determine we can directly design a controller.

So, controller design is possible if the if the given system it is in transfer function form, but now the there is a problem. Assume that this system it is not in transfer function form or A it is not in companion form because the advantage is that if it is system is in companion form we directly write down the any characteristic equations.

For example, if you write say if you write a system matrix A like this minus 1 minus 4 minus 3 say 1. So, if the matrix A like this if you want to determined this lambda I minus A then here we actually we have to solve it, but if the given system matrix it is in companion form we can directly write down the characteristic equations, that is for example, for this case 0 1 minus A 2 minus A 1 your characteristic equation is say lambda square a 1 lambda plus a 2 equal to 0.

So; that means, the given system matrix it is in companion form we can directly write down is characteristic polynomial, but here if the matrix A like this we cannot write down like this and other thing is that we have a desired characteristic equations. So, desired characteristic equation it is also in terms of lambda for example, if you take desired Eigen values as say lambda 1 and say lambda 2.

So, these are desired Eigen values; so, here this can be written in terms of characteristic equation as lambda square. So, here we will find at the desired Eigen value like this. So, we can write down instead of characteristic equation. So companion form characteristic equation, so there is a direct comparison and when this direct comparison is there we can we can design a controller.

So, this particular part we have seen a last time, but there is a problem when if the given system matrix it is not in companion form. So, in that case how to do it and in again one other thing is that the given system must follow the controllability test. So, in order to have a given system matrix into a companion form; that means, some transformation we have to use. So, along with the transformation we have to go through the controllability test. So, now, we will study the method by this method we can design the controller, controller in which the controllability test has also been checked.

Now, we will see the method how to design this controller by means of these state transformation.

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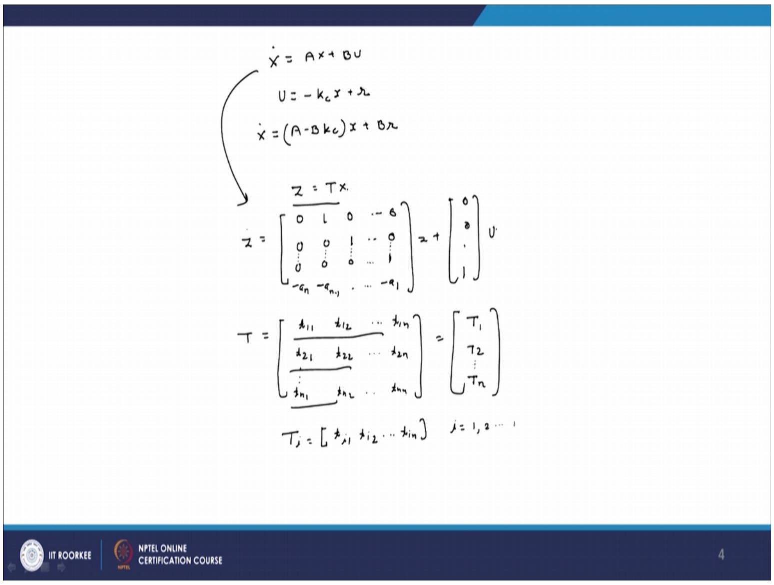

Now, our problem is X dot equal to A X plus B U, this is a plant. Now the control law which we have seen last time that is U equal to minus K c x plus r. Now, this is your controller and assume that A it is in general form, it is not in a companion form. Therefore, this X dot equal to A minus B K c x plus B into r and the what we want this given system matrix, it should be in a controllable phase variable form; that means, it has to go through some transformation. Let us say if use transformation Z equals to T X, this given system can be converted into controllable phase variable form.

So, there controllable phase variable form will like this Z dot equal to 0 1 0 0 0 0 1 0 and here 0 0 0 1 and here minus a n minus a n minus 1 minus a 1 this is Z plus 0 0 1 U. So, by means of this transformation it can be converted into a controllable phase variable form.

Now about this transformation T; that means, here for Z equal to T x this T can be represented by t 1 1 t 1 2 to t 1 n here t 2 1 t 2 2 t 2 n to t n 1 t n 2 t n n is now this particular portion particularly this particular row this row and this row can be represented by t 1 t 2 and here say t n

And here this T i that is t 1 t 2 t n that can be represented by t i 1 t i 2 t i n and where i is varying from 1 2 up to n. So, Z in all the state variables X is the state variable, but whereas, the X is our original state variable Z is the transforms state variable and this transforms state variable has been carried by t and if you do this transformations you will get the given system matrix in the companion form and we have seen that if the given system matrix is in the companion form we get directly determine the characteristic equations. So, we have desired characteristic equation we can easily compare it and we can get the results.

So, that is why here for this case we have done some transformation approach because our A is not in a controllable phase variable form. So, now here we have written equation like this.

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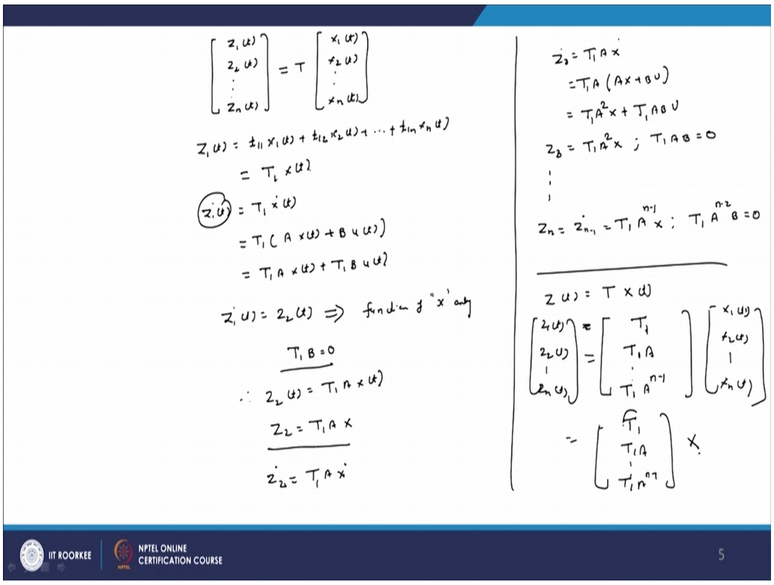

So, now here the states is there that is represented by Z 1 t Z 2 t Z n t; n states. Now here this is a transformation T and state say in terms of time T x 1 t x 2 t say x n t. So, now here there are n states that is z 1 z 2 z n and x state x 1 x 2 x n, these are the original states these are modify states.

Now what will do will take the first state z 1 of t. So, this z 1 t z 1 of t we can write down as this t as a transformation matrix that is t 1 1 t 1 2 t 1 n; so, it can be written as t 1 1 into x 1 of t plus t 1 2 x 2 of t plus t 1 n x n of t equal to t 1 into this x of t. Now here we have written first state that is z 1 of t equal to t 1 into x of t.

Now, we have seen that there are different states. So, if you want Z 2 what we have to do we have take that the derivative of the that states So, what happened if you take Z 1 dot of t that gives you Z 2 t. So, here we take the derivative on the both sides. So, you will get here Z 1 dot t equal to t 1 into x dot t now here t 1 into x dot t and we know x dot T that is equal to A X plus B U.

So, we write this x dot T in terms of AX plus B U. So, we can we write this as T 1 A x of t plus B into U of t. Now we solve this equation. So, you will get T 1 A into x of t plus T 1 B to U of t and now here Z 1 dot T it is function of Z 2 of t and this is basically the function of only x which is function of x only.

Because as we see that is Z equals to T x, so Z is only function of x. So, here we have got Z 1 dot T equals to Z 2 T it is function of x only. Therefore, this T 1 B we can consider as 0. So, here we write this T 1 B equal to 0. Therefore, this Z 2 of T equal to T 1 A into X of t. For simplicity we can write this Z 2 equal to T 1 into X.

So, now, we have got Z 2, but we want Z 2 Z 3 Z 4 up to Z n. So, here we again differentiate this equations. So, after differentiate in this equations will get Z 2 dot equal to T A X dot. So, here Z 2 dot equal to T a x dot So, here T a x dot we replace as A X plus B U and now you solve it. So, TA square X plus this is T 1 T 1 T 1 T 1 A B into U.

So, now this Z 2 dot equal to T 1 A square X plus T 1 A B into U. Now similar to here as Z Z elements are the function of X, here also this T 1 a B we can take as 0. So, here write this as Z 2 dot equal to Z 3 T 1 A square X is T 1 A B equal to 0. So, now, you have got Z 3 similarly we can get Z 4 Z 5. So, finally, we will get Z n that is Z n minus 1 dot equal to T 1 A raised to n minus 1 X and here we will write T 1 A raised to n minus 1 2 B equal to 0.

So, by means of this transformations taking first as a variable will find that will get the equations in terms of Z and X and other variables we have consider as 0.

Therefore we write this equation as Z of T equal to T of X that is X of T. So, here we can write this as this is your x 1 of t x 2 of T up to say x n of T and here also we have Z 1 of T Z 2 of T Z n of T. So, will find that this Z 1 of T equal to T 1, now here will get T 1 into a and here will get this T 1 A raised to n minus 1 into X. So that means, this part we can write as directly T 1 T 1 A T 1 A raised to n minus 1 x and this T 1 this must means must satisfied the conditions that is this T 1 should satisfy the condition T 1 B equal to T 1 into a B; so, T 1 a raised to n minus 2 into B.

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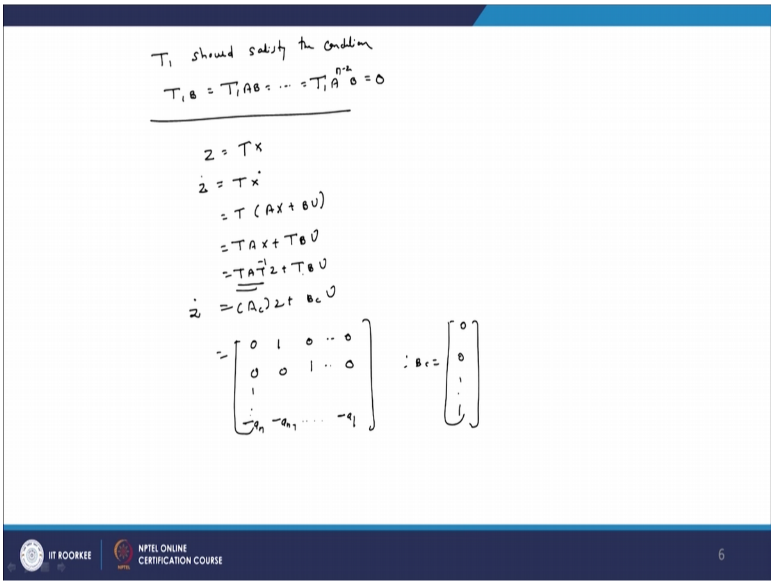

So, now by means of some transformations we have got different conditions for we have got the conditions for T 1 and here so, we write down this as Z equals to T X. So, Z dot equal to T into X dot. So, here T X dot is here AX plus B U, now here we can write down as T AX plus T B into U. So, this x is a function of Z.

So, we can write down that x equal to T inverse Z. So, will get T A T inverse Z plus T B into U and now this T A T inverse that we can write down as A c into Z and here B c into U. So, this is Z dot and this should be of the form 0 1 0 0 0 0 1 0 and this process repeated we can write down as minus a n minus a n minus 1 minus A 1 and B c as 0 0 say 1.

So, now here we will find at this is B c this is 0 0 1 and we have T into B. So, here this T into B the T involved this involvement of T 1 T 1 A T 1 A raise to n minus 1 A raise to n minus 1. So, we if we compare the equation of B c with respect to T B and see what condition will get.

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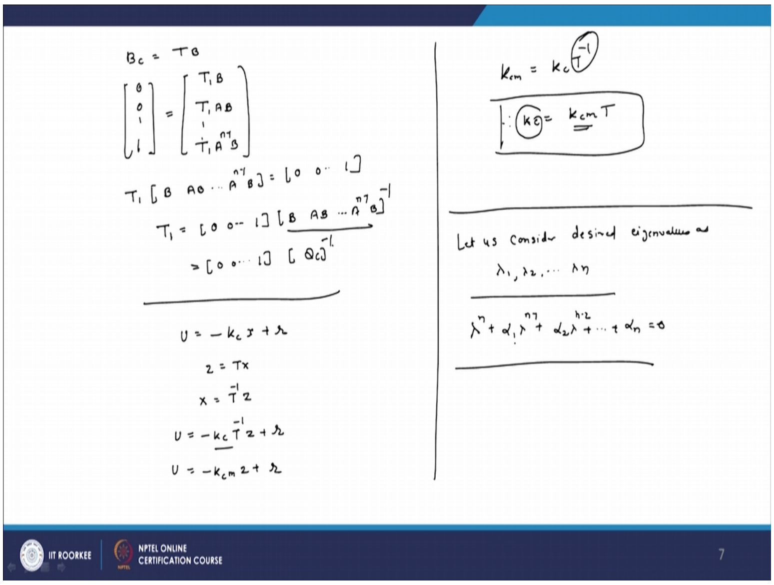

So, here we can write down as this B c equal to T into B. So, this B c is 0 0 say 1 and conditions of T is say T 1 B T 1 into a B and here T 1 A raise to n minus 1 into B. Now if you solve it so, you will get T 1 B A B A raise to n minus 1 B equal to 0 0 0 1 and hence this T 1 equal to 0 0 1 as B A B.

Here our main purpose is to design a controller and as we discussed earlier that controllability's very much important because in controllability we are transferring the state from initial time to final time. So, we will find at here we are design in the controller in the controller design here; we will find that this B A B A raise to n minus 1 B equation will come; that means, if the system is not controllable. So, this part will not exist and our controller design is not possible. So, here these particular part is concerned with the controllability matrix that is.

So, here we will find that this T 1 depend upon the inverse of the controllability matrix if the system is not controllable. So, the T 1 cannot exist and this Q c is like this. Now we come into the final equations that what is the final formula of the controller design. So, now, in order to get final formula now we go to the basic control law. So, my basic control law says U equal to minus K c x plus r this is my control law till is the process has been done in order to get the your in order to get the controllable phase variable form and in a controllable phase will form will find that here this 0 0 1 this 0 0 1 is compared with the T into B and we are determine the condition about T. So, so this conditions you are determined controllability condition also we have got and now here our purpose is to design the controller. So, here you are written U equals to minus K c x plus r and here the transformation is Z into T X.

Now, here X equal to T inverse Z. Now, the U equal to minus K c T inverse Z plus r; now this U. So, this portion we can write down as minus K c m into Z plus r and therefore, the final controllable conditions that is K c m equal to K c into T inverse and a final controller K c equal to K c m into T. So, here this K c m which I was shown here this K c m we are obtained for the state Z but our original system is it in terms of X dot equal to A X plus B U. So, when AX plus B U is there your controller is K c.

So, K c m is the upper transformation of the state and this because of this K c m or the K c which we obtain it is the (Refer Time: 23:07) direct comparison with the companion form, but as original system matrix it is not in companion form. So, that part is taken by this T inverse there were final controller is K c equal to K c m into T. So, K c m is nothing, but the comparison between the companion form and T the transformation matrix which gives you the final controller K c.

Now, we know that the Eigen values are invariant after doing transformations we were seen that the given system matrix we are convent to into diagonal form then we are seen about the Jordan form. So, different forms are there, but Eigen values cannot change. So, here also after doing a transformation Eigen values cannot change. So, when the Eigen values cannot change it means that property of the system cannot change. So, therefore, what we are assuming that we have a desired Eigen value. Let us say let us consider considered desired Eigen values as A lambda 1 lambda 2 say lambda n.

So, base on the desired Eigen values we can write down the characteristic equation. So, the characteristic equation we can write down as lambda n say alpha 1 lambda raise to n minus 1 plus alpha 2 lambda raise to n minus 2 plus alpha n equal to 0, this is your desired characteristic equations; this basically we want and this lambda 1 lambda 2 we knows; that means, we have fix it and from this characteristic equation we have determined.

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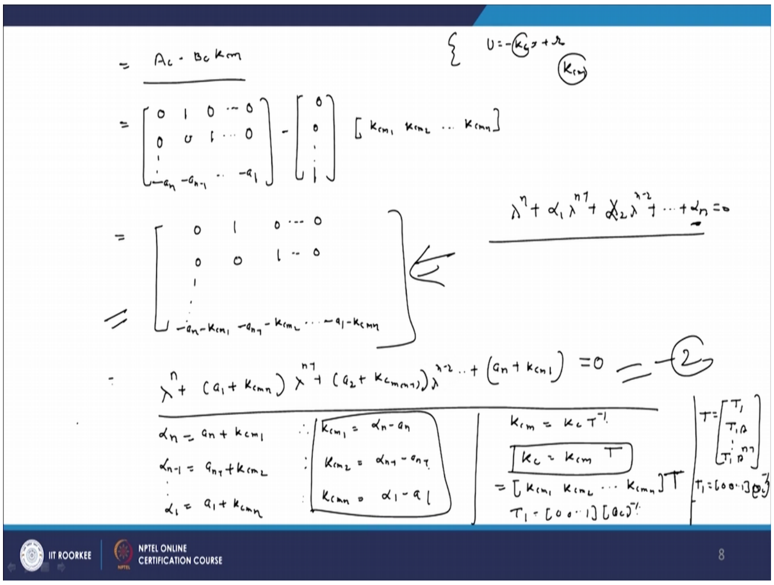

Now, we have a original equations the transformation it is in terms of this A c minus B c into K c m.

Why this K c m because when we were write down a state feedback equation as U equals to minus K c x plus r that is with respect to state x. But when we were deals with the we are state transformation in terms of Z instead of K c. We have got K c m and therefore, A c minus B c into K c m we have written. Now if you write this equation as this ac as 0 1 0 0 0 0 0 1 0 and here minus a n minus A of n minus 1 say minus a 1 and what is bc; bc is 0 0 1 this is multiplied by K c m as K c m 1 K c m 2 and here K c m n and after solving this you will get equation as 0 1 0 0 0 0 1 0 and last row will get as minus a n minus K c m 1 minus a of n minus 1 minus K c m 2 and this process repeated will get minus a 1 minus K c m n.

So, this product or this result we have got after the multiplication of this with this and subtracting this from the original ac matrix we have got this particular equations. So, here we will find at this is our desired characteristic equations and this particular equation we have got that is of a transformation we have got this. So, this particular matrix we can also write in terms of characteristic equations. So, here this matrix can be write down in terms of characteristic equation as lambda raise to n a 1 plus K c m n lambda raise to n minus 1 plus a 2 plus K c m n minus 1 plus a n plus K c m 1 equal to 0.

So, we will find that after doing transformation we got ac bc K c m and this equation we have solved; this we have got in terms of a companion form and we have written the characteristic equations. Now this particular characteristic equation we can easily compared with this equations. So, let us say if write this equation has 1 and this equation we have obtained as 2.

Now we compare these 2 equations you just see the result. So, this is equations and now here I am writing down the equation as alpha 1 lambda n minus 1 plus alpha 2 lambda n minus 2 and this process repeated plus alpha n equal to 0. So, this is a desired and now this is through transformation.

Now, if you compare it so, what will get alpha n? This is compared with this a n plus K c m; that means, this equivalent to this there is a n plus K c m 1 and after solving this we will get K c m 1 equal to alpha n minus a n. Now here alpha n minus 1 that gives you a of n minus 1 K c m 2.

So, we can write down as K c m 2 equal to alpha n minus 1 minus a of n minus 1 and this process repeated. Finally, we write alpha 1 as a 1 plus K c m n. Therefore, K c m n we can write down as alpha 1 minus n.

So, now all the values of K c m 1 K c m 2 K c m 3 K c m n we have determined. Now what is the final your problem is the K c m final controller that is equal to K c into T inverse, but our controller is not K c m our controller is K c. Why not K c m because this K c m we were determined after doing a transformation, but our original system in a general form and for the general form we required this K c equal to K c m into T and this K c m if we can write down as K c m 1 K c m 2 K c m n into T and we have seen in law previously this T that is equal to T 1 T 1 a T 1 a raise to n minus 1 and this T 1 is equal to 0 0 1 into qc inverse that is controllability inverse.

That is again write down here this T 1 as 0 0 1 into qc inverse. So, the final controller is this K c into K c m into T. So, K c m equals to K c m 1 K c m into K c m n and and this particular T is equal to T 1 T 1 a T 1 raise to T 1 a raise to n minus 1 and whereas, single T 1 equal to 0 0 1 qc inverse. So, this is the basic methodology to design their controller now all this mathematics we have seen. So, this we can easily understand if you solve some numerical example.

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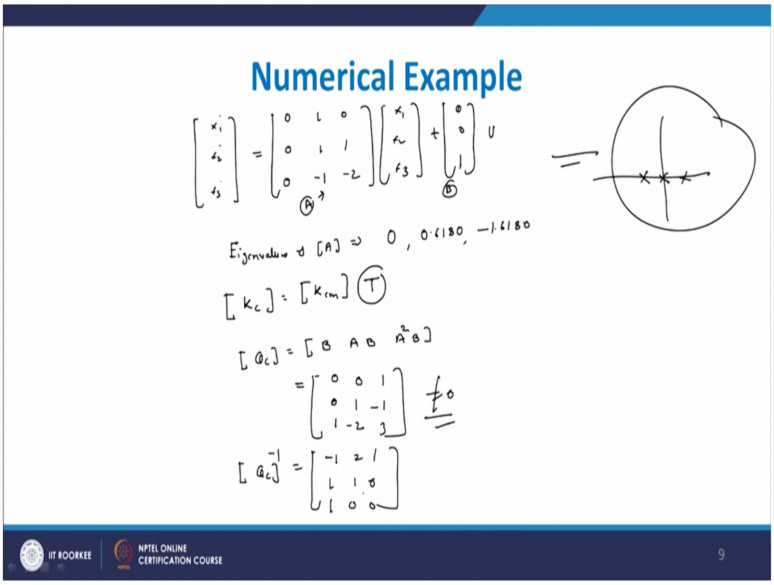

Now, we start solving some numerical example base on this technique. So, now, our example is x 1 dot x 2 dot x 3 dot equal to 0 1 0 0 1 1 0 minus 1 minus 2 x 1 x 2 x 3 plus 0 0 1 into U. Now the Eigen values of this matrix A, this is matrix A, this is matrix B. The Eigen values of this matrix say r Eigen values of a belong to 0 0 point 6 1 8 0 minus 1 point 6 1 8 0. We know that Eigen value govern the performance of the system.

As this given system matrix a has 1 Eigen value at origin 1 Eigen value of the right hand side other on the left hand side. Now this is our condition, this is the origin that is here, second is here and third is here. So, this system is unstable system, but if a Eigen values. Let us say they are coming on the left hand side of they explain and if they are satisfactory there is no need to design a controller. Controller we are design only when there is a requirement just like as if some if the patient is not ill and still is going to doctor and he is taking the medicine, so, that create a problem.

So, we take a medicine only when we have some problems here. Also control design we can do only when system is unstable or performers is not satisfactory for this particular problem. We observe that your system is unstable performers is not satisfactory. Therefore, there is need to design a controller now in order to design a controller. So, our formula is this your formula for designing the controller that is K c is a final controller equal to K c m K c m into T and we have seen that this T equal to T 1 T 1 a T 1 a raise to n minus 1 and T 1 equals to 0 0 1 qc inverse that is controllability inverse. Now first of all which is the controllability all though this controllability fit will come in t; that means, if the we can say if the system in not controllable the T 1 will itself come 0.

But anyway we require controllability equations afterward we will now check the controllability of the system. So, the controllability of the system is given as Q c equal to B A B A square B. Now here we can write down as this B as 0 0 1. If you multiply this A B matrix is A and B we will get 0 1 minus 2 1 minus 1 sorry, now if you determined the determinant of this matrix will found that this is non zero and therefore, we can easily get this qc inverse as controllability inverse as minus 1 2 1 1 1 0 1 0 0.

So, system is controllable therefore, we can design a controller, but if when we design a controller definitely we required where we want place where we want to place the poles and what is our desired performance therefore, here for this problem now we decide the our desired poles.

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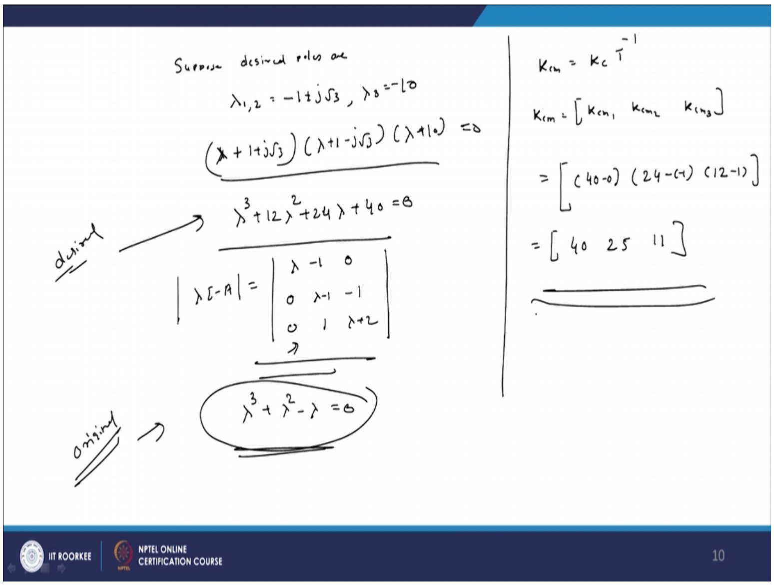

So, the desired pole suppose desired poles are say lambda 1 comma 2 minus 1 plus minus j root 3 and lambda 3 equal to minus 10. Now here the question has come why we were taken a 3 poles.

Because lambda 1 lambda 2 and lambda 3 we can take lambda 4 and here also, but here why we have taken 3 poles because you find that your original system is of third order that is why we have take consider only the 3 poles. Now base on this 3 poles we can determine the characteristic equations. So, the characteristic equation for this for this poles poles or sometimes you can also called as a Eigen values. So, here lambda plus j root 3 lambda plus 1 minus j root 3 plus lambda plus 10 equal to 0. So, all this 3 Eigen values we have considered and know after multiplications of this will get lambda cube plus 12 lambda square plus 24 lambda plus 40 equal to 0.

Now, this is our desired characteristic equations, but our aim is to get the controller, but problem is that this matrix it is not in companion form, but to get the values of K c m. We have to determine the characteristic equation for the same and then compare it. So, here we determine the characteristic equation for this lambda I minus a equal to lambda minus 1 0 0 lambda minus 1 minus 1 0 1 lambda plus 2 and here after solving this we will get lambda cube plus lambda square minus lambda equal to 0.

When the system it is not in companion form we cannot directly compare to get the controller. So, therefore, here as I had told earlier the step to be followed about the K c m. Now K c m in nothing, but the characteristic equation of this and characteristic equation of the desired if you compare it. We can get the values of K c m. So, here both this equations if you compare. So, what will get; that means, we can write down this as this K c m equal to K c into T inverse and now here this K c m equal to K c m 1 K c m 2 K c m 3.

Now if you see here now here 40 40 value is there, but here there is no terms; that means, the value is 0. So, here you will get K c m 1 as 40 minus 0. Now come into K c m 2. So, we will find that this is a 24 and here lambda is minus 1. So, we will get 24 minus of minus 1 and here lastly this is 12 minus 1.

So, this K c m basically is you are determined from this equation alpha n minus a n alpha n minus a a minus 1 alpha 1 minus a n, where alpha n alpha n are the desired and these are the the coefficient of the original characteristic equations. So, here that is why we are written lambda cube plus lambda square minus lambda this is the characteristic equation of the original plant original system.

This original and this is the your desired now we are compared it we will get K c m as K c m 1 K c m 2 K c m 3 40 24 plus minus 1 12 minus 1. So, if you solve it we will get 40 25 into 11. So, K c m K c means we have got, but finally, we have not got controller, controller is your K c and for K c we required T.

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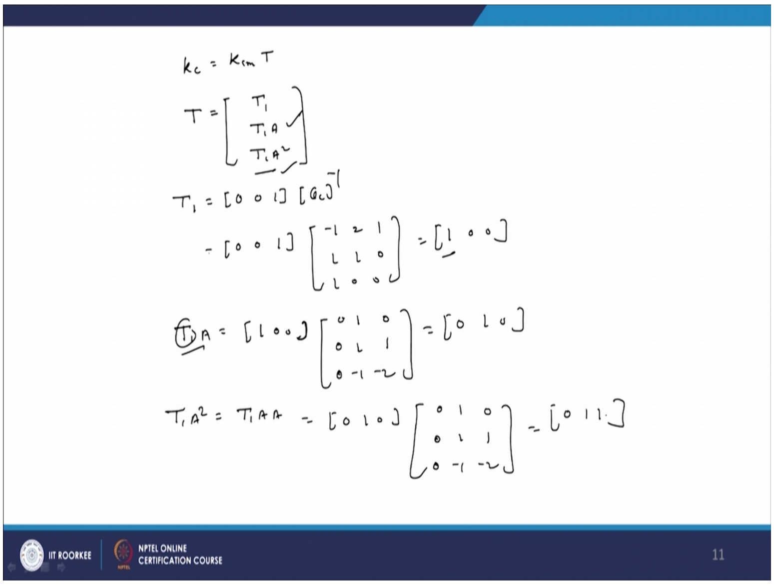

So, now they need to determine the transformation matrix T. So, here K c equal to K c m into T we can write down as this T equal to T 1 T 1 a T 1 a square. So, this T 1 equal to 0 0 1 Qc inverse controllability inverse and we have already determined the controllability inverse like this and 0 0 1 is there. So, if you multiply this 0 0 1 and Qc inverse which we already determined we can write down as minus 1 2 1 1 1 0 1 0 0 if you solve it will get 1 0 0.

So, we have got T 1, but the T involved T 1 A as well as T 1 A square. So, you have to determine T 1 A as well as T 1 A square. So, if you multiply T 1 into A. So T 1 is T 1 is 1 0 0 as we are determined T 1 like this and again we write the original a matrix as 0 1 0 0 1 1 0 minus 1 minus 2.

Now after solving will get 0 1 0 then lastly we required T 1 into A square T 1 into A square that is T 1 A into A so, T 1 A we are determined. So, you will get 0 1 0, this as 0 1 0 0 1 1 0 minus 1 minus 2 and we can get as 0 1 1.

So, we have got T 1 A T 1 T 1 A as well as T 1 A square, now we have also got K c m.

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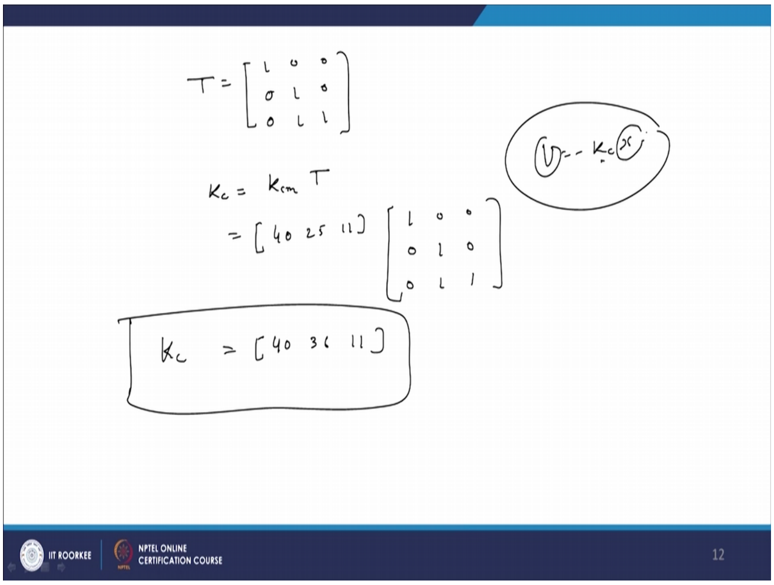

Now, we multiply the K c m with T that is finally, we have got T as 1 0 0 0 1 0 0 1 1 and your K c is K c m into T and the K c m which we are determined earlier. So, K c m is 40 25 and 11. So, we write this as 45 25 11 and the T as 1 0 0 0 1 0 0 1 1 and after solving this we come across 40 36 and 11 this is controller K. So, this is the basically a controller which can be help to stabilize the system and to find that this particular controller take care of all the states maybe x 1 x 2 x 3 as here there are 3 states. It can control x 1 x 2 x 3 if you are 10 states it also controller 10 states.

But if you see the classical part it is consider the input and output it. Consider the all state. So, if 40 36 11 is nothing, but the controlling up all states. So, control input is going through this K c that is your U equals to minus K c x that means, this control input will supplied through this K c into original state x that these are here. We are controlling basically the states of the system.

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Now, you see the references the mainly the some portion of the the proof you will find out here. Some more example also you can find here as well as here.

Thank you.