Advanced Linear Continuous Control Systems
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Lecture - 31
Observability in State Space (Part-I)

Now, we start with Observability of the System in a State Space. Now what is mean by observability? Last time we have seen the controllability of the system, the controllability means concerned with the that there are some states which are at initial time and it has been transferred at time t equals to t f and we are all states, we can we can found all the states that mean that the system is controllable.

But the all the state which has to reach really there observed or not, that is also very important. Because sometimes it may possible that that all the states which are present that has been reach or there has been transferred, but they cannot be observed and the if the states cannot be observed is means that system is not observable and that is also a create a problem particularly in controller design. Now first of all we see what is mean by observability of the system.

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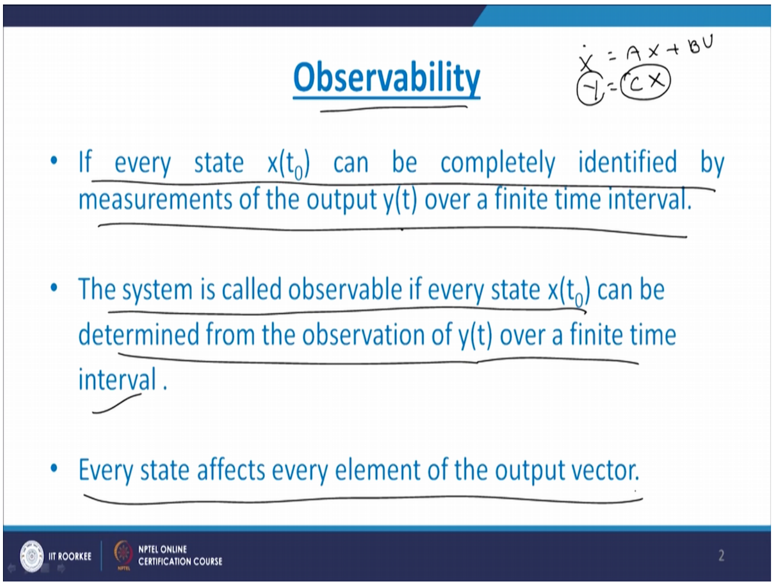

Observability so, every state x of t 0 can be completely identified by measurement of output y t over a finite time interval. Now here this part is concerned with the output because if you see the state space model X dot equal to A X plus B U Y equal to C X. So, now, the output is also related to the state this output y is also related to state x.

Therefore, every state can be completely identified by measurement of output y t over a finite time interval. System is observable, if every state can be determined from the observation of y of t over a finite time interval. Every state affect the every element of the output vector; that means, so all this points indicate that there exist a relationship between the state and output. Now we will see through this concept through general example.

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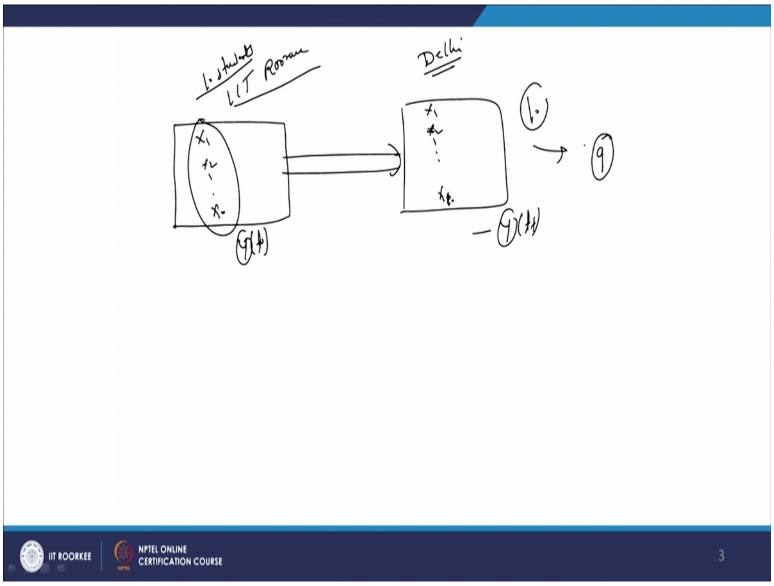

So, in last two parts means particularly in a state controllability observability i have explained you the one example. Now here as I told earlier as a industrial trip is going from Roorkee to Delhi. So, there all say extent these are ten students, there are ten students, they are at IIT Roorkee and they are going to Delhi and now X 1, X 2 X 10. Now here there are 10 state and whatever output is required here is Y. Let us say this is also Y, Y at t naught and say Y of t four, this is final time.

Assume that we are concerning a teacher is particular output; the teacher is concerned with this states. So, Y output is a teacher where there is a involvement of 10 states and here at Y equals to t f is also these involvement of 10 states. Now what happened all 10 students are there? Now the all 10 students has been reached here.

Now, here we found at that that all students has be reached that mean the system is your controllable, but concerned with the output Y, Y output here I am talking about the teacher. When teacher start counting the 10; student 1, 2, 10 that particular student is set a say it is inside the particular bus, I mean I am I counting that student, but it is found that even the student present the bus I cannot identify him; that means, in the output instead of 10 students which has been there in the bus at Delhi, I am identifying only 9, but 10 student is reach there, but I cannot identified that means that this system is not observable.

And there also when I am not observing that particular student, then definitely it is tough for me to take decision whether to go to industry or to go back to Roorkee. It means that whatever states are there that a initial time they have to reach an final time along with there that means, that controllable when they are reach here; that means, that controllable and when we contacted it that much be observable therefore, controllability and observability both are the same thing.

Now, we will see how they are useful in a controller design. This control design part will come later on, but now as these issues is related to our controller design; I would like to discuss few points. In a state space analysis, the controller is designed base on the state information whereas, in classical control in classical control the controller has been design base on output and input, but in a state space the working of controller is base on the states as well as the input output both. But you see that in the output, output Y is concerned with the state if any of the state is not observed or it means sometimes it is not possible to measure that state, then what is the option for available for me? I have to identified the state.

Sometime you have seen that we are going for some doctor for some treatment and sometimes doctors say that you do the test, but because of time limitations over non availability we are not doing the test, then doctor just putting some hands here somewhere here near to heart, he will identified that particular states; that means, that so, if that particular parameter if it is not identified, then he cannot prescribe the medicine; that means, that state which cannot be measured that need to be identified.

So, if identified this particular state, then we can say that controller design is possible, but if the state is there, but we cannot identified, then we cancelled. We can say that system is not observable and therefore, there is no need to it is not possible to design a controller.

Similarly, here as I told you that all 10 states output I am talking about teacher all students have been reach, output is teacher when students are reach here all though they are reached, but we cannot identified or teacher cannot identified that particular single student that is outer of the 10 he identified only 9 students. Therefore, this system cannot be observable.

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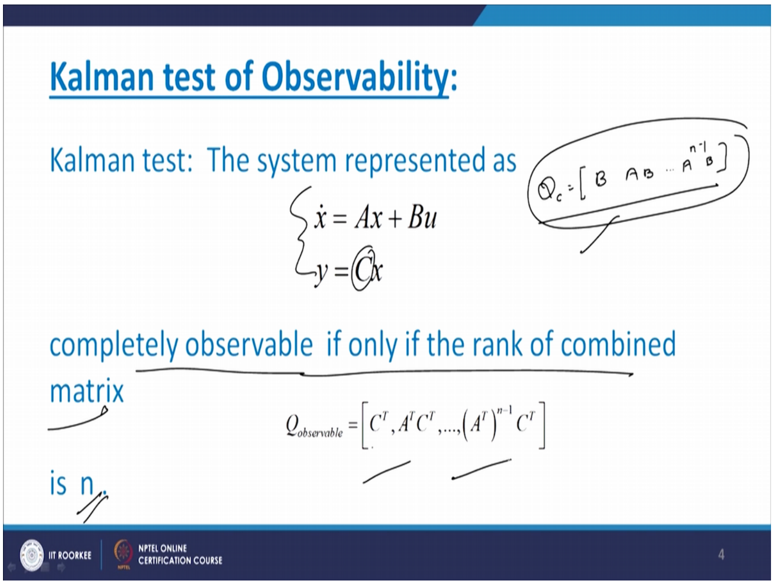

Now the next thing mathematically how to check the observability of the system? And observably is there an observability means this concern with the output. Therefore the important role of C is here. We have seen that in case of state controllability, the control state controllability Q C, there is a involvement of only B A B A raised to n minus 1 B.

There is no C is involved in the concept of controllability observed controllability, but now in talking about the observability we see that system is represent by this one is completely observable if and only if the rank of the combined matrix C transpose A transpose C transpose A transpose raised to n minus 1 considered was this is n. And here we will find at there is no involvement of B matrix. There all related to e A A and C matrix. A is A system matrix, C is the output matrix and there all concerned with this states.

Now, how this equation has come? We have already seen the origination of this equation B A B A raised to n minus 1 B and this we have determined or this equation we have got from the state equations. Similar here also from the state equation itself, we can get this particular equation. Now we start deriving the results C transpose A transpose C transpose A transpose n minus 1 C transpose definitely when we start any system our model is same X dot equal to A X plus B u Y equal to C X plus D.

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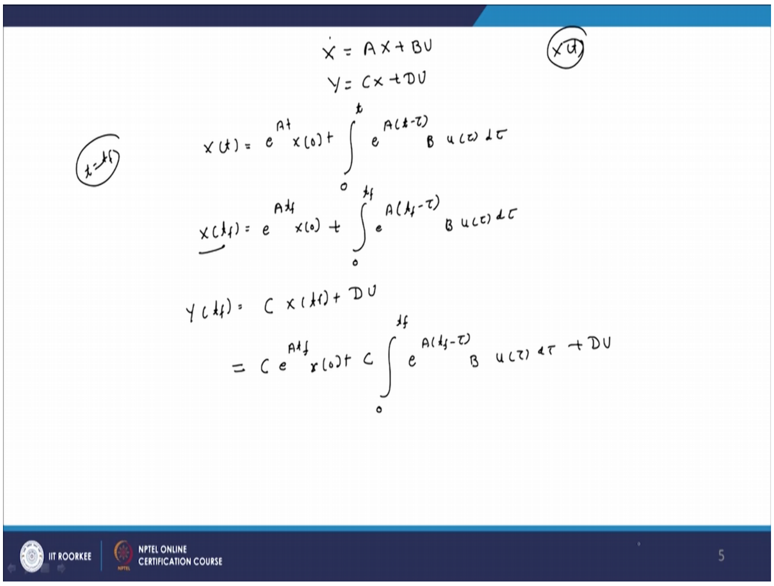

This is our state space model and now here we are concern with the observability; that means, when we derive any equation the output must come in to existence. In case of a controllability of the system, we are talk about the X of t, but here we have to we are concerning with the output Y of t also. So, Y is depending upon X therefore, we write first of all the equation of for X of t, now this X of t equal to e raised to A t X 0 0 to t e raised to A t minus tau B u tau t tau. Now this is state equation we are written. This X of t, but now our final time is say t equal to t f.

Now this equation we have to write in terms of final time t f. Now we can write this as X t f e raised to A t f X of 0 plus 0 to t f e A raised to tau f minus tau B u tau d tau. Now see every equation we have written; that means, t has been replaced by t f everywhere. Now this is equation for X t f, but we are concerned with the output. Therefore, output Y this is to be time t f is in as C matrix X of t f plus D into U. Now you replace this X of t f in this particular equation. So, we will get C e A t f X 0 plus C 0 to t f e A t f minus tau B u tau d tau plus D into U. For in this particular equation, we will find at all these parameter C B D U all are known and here also the same thing.

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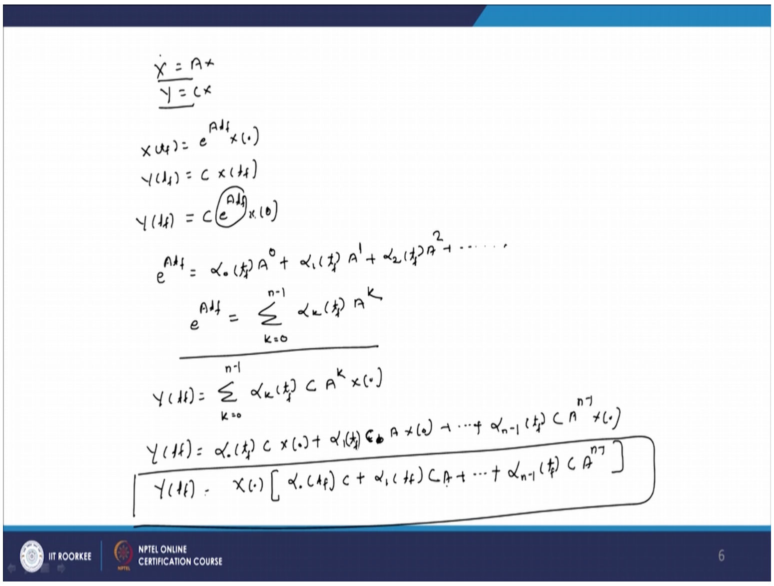

Now this complete equation we can write down because this are known in terms of directly X dot equal to A X and Y equals to C X. What we what we written? Because this C B D U if you move on the left hand side, we get output and that output will get in terms of C that is C X. Therefore, here final equation we have writ 10 X dot equals to ax Y equals to C X. Now here from this equations, we can write down X of t equal to e A t f X of 0 and here Y of t f C X of t f we can write down this as C e A t f X of 0 that is Y of t f equal to C e raised to A into t f into X naught.

Now up till now we have reached now about this part e raised to A t 4. In case of controllability of the system that is when we have derived the controllability equations, we have written this equation that is equation means we have written e raised to A tau t f e raised A t f in terms of Cayley Hamilton theorem.

So, now this equation that is equation is this e raised to A t f can be written as alpha 0 t into A raised to 0 plus alpha 1 t A raised to 1 plus alpha 2 t A square. Therefore, e A raised to t f can be written as k equal to 0 to n minus 1 alpha k t into A raised to K. Now we replace this equation in this particular equations. So, we can get Y of t f, we will get Y of t f that is k equal to 0 to n minus 1 alpha K t to C A raised to K X naught. Therefore, Y of t f now you now you simplify this. So, you can get alpha 0 t C X naught plus alpha 1 t C A X naught and this up till alpha n minus 1 of t C A raised to n minus 1 X of 0.

Now here alpha 0 t C X 0. Now here alpha 1 of t to C, C into X naught alpha n minus 1 t C A raised to n minus 1 X 0 and here Y of t f, you can take X 0 common. So, here X 0 X of 0 alpha 0 into final time t f. now this t because we are talking final time. So, now, t we are replace here is t f C plus alpha 1 t f to C into A this means you see C into A plus alpha n minus 1 C A raised to n minus 1.

Now this is the equations this is the final equations; that means, Y of t f we have got in terms of X 0 alpha 0 t f C alpha 1 t f C A alpha n minus 1 t f C A, but in this case we will find at C C A and C A raised to n minus 1. This plays a very important role. That means particularly if you calculate the rank of this matrix particularly C C A C A raised to n minus 1 that should be full, that should not be 0 or they should not be falls because false this Y of t f cannot exist.

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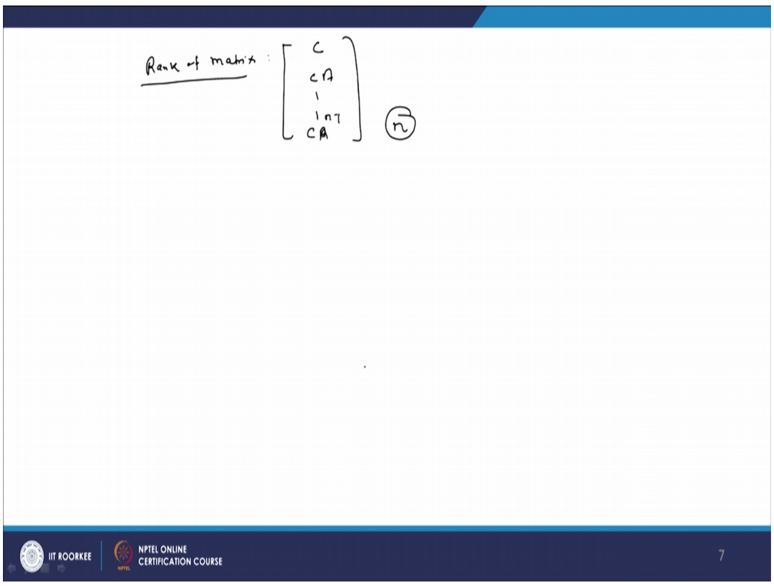

Therefore, we have to determine the rank of matrix of C C A C A raised to n minus 1 and this must be n the rank of this matrix C C A C A raised to n minus 1 must be A; that means, whatever we have our statement Q observe matrix C transpose A transpose C transpose A transpose raised to n minus 1 C transpose has been proved and this we have proved from this equations that is C C A C A raised to n minus 1 because if this is part if it is not observable or if the rank fall in this particular case; that means, any of the state has been shield at from this observations and therefore, it is difficult in terms of a controller design. Because all the states are very important because there sometimes they are somewhat inter related. So, if you control one state, if one state cannot be control, it also affects our results. Therefore, all states must be observed that is a monitory for the controller design.

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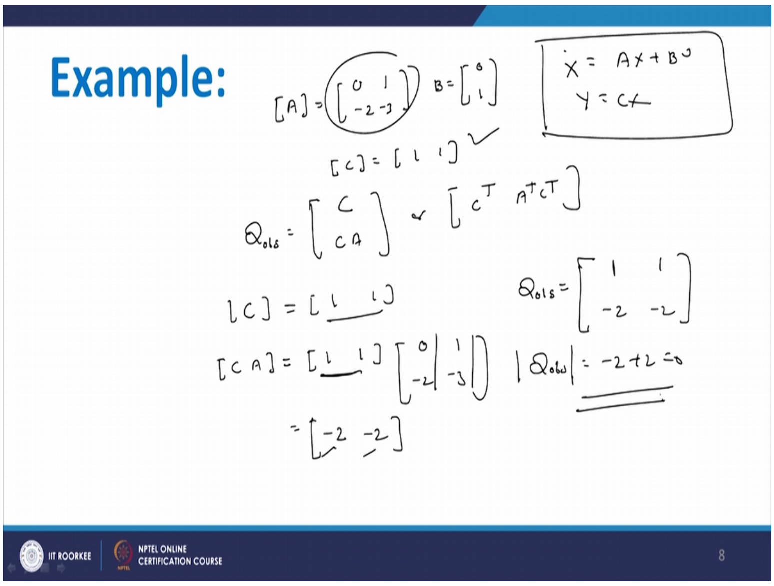

Now we see example. So, we have taken simple system. So, let us say A matrix as 0 1 minus 2 minus 3 and say B matrix as 0 1 and our C matrix is like this C equal to 1 1 and this plant is nothing, but X dot equal to A X plus B U and Y equal to C X. Now our main purpose is to check the observability of the system. We are check controllability as well as up to controllability of such type of example.

Now, here point is that we have to check the observability of the system and now the observability we are checking by Kalman test. And we are already seen the test if you want to check the observability we have to carry out the or we have to determined the determinant of these particular matrix. A determined is 0 it means that rank falls that is C transpose A transpose C transpose.

Now here now we talking about this particular example and now for simplicity we can write the observability matrix as Q observable as C C A or we can write this as C transpose A transpose C transpose that is here C transpose A transpose C transpose. So, here we can write down the matrix C C is here 1 1 this one.

Now we want to determined here C into A. So, C is 1 1 and A matrix is 0 1 minus 2 minus 3. See here this is A matrix we are written here. Now we have to solve it. So, this multiplied by this and this particular row multiplied by this columns. So, after multiplication of this we will get this as minus 2 and this as also a minus 2. So, for checking the observability we have got C we have also got C A and therefore, we can write observability matrix as 1 1 this 1 1 and here this minus 2 minus 2 we can write down like this and now we determined the determinant of this matrix write minus 2 minus 2 this is 0.

So, from this example we find that the determinant of this observe matrix is 0 ; that means, rank fall that in other word we can say that these particular system is not observable. So, we will seen that in this particular test, the problem which we have found that that we are not getting the insight of the concept of observability; what we want that we have to see that out of these we state is responsible for making the system an observable because that also helps us to go for some we conversion because sometimes in a practical sense nothing is perfect. Therefore, some approximation some adjustment we are always making in there system therefore, this type of things cannot observed in such cases.

Therefore here we have to go for the Gilbert test of the observability, the how to solve the Gilbert test or how to perform the Gilbert test. We already seen the Gilbert test that is base on the concept of diagonalizations and we have seen that diagonalizations, all states are separate whatever thing is going on is individual state we can easily observed. In other words the states have been decoupled. Now we see the, what type of results will get if you go for though Gilbert test of observability.

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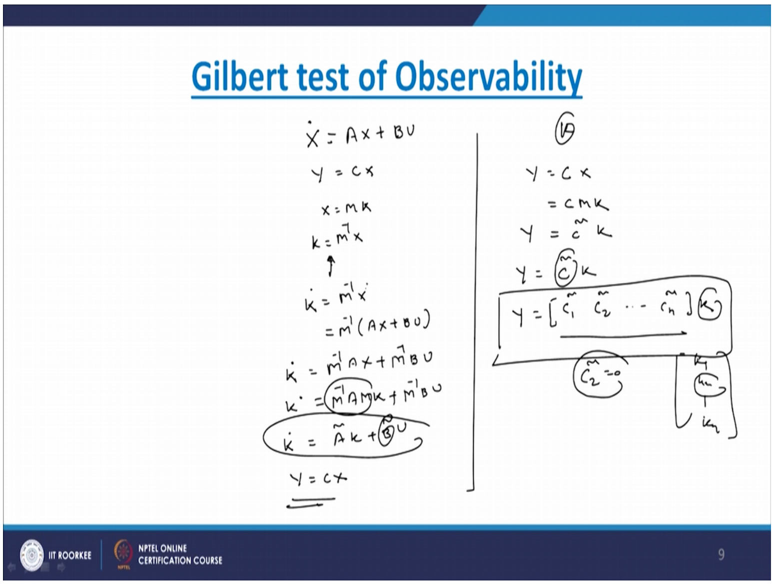

So now, this concept is base on diagonalizations. So, I am taking a model as X dot equal to A X plus B U and Y equal to C X. So, X dot equals to ax plus B U Y equals to C X and now the concept of diagonalization is based on state transformation. You can take any other state, you can assume any variable. So, for as we are assuming the variable K every times earlier also here also we are assumed with assuming a new state in terms of k. So, here you can take X equals to M K.

Now K equal to M inverse X and now this equation can be simplified as K dot M inverse X dot. Now here M inverse X dot is equal to A X plus B U. Now this K dot equal to M inverse A X plus M inverse B into U. Now this is M inverse A this X is M K and here is M inverse B into U, now K dot is like this. Now this K dot equal to this M inverse A M. We can write down as A bar K and this is B bar into U.

So, this is about the new state transformation in terms of K, but here is our problem is about the observability. It concerned with the output. Therefore, all though we derived, but this cannot solve our purpose. Therefore, we are concerned here is the output equations. So, output is our Y equal to C X and now here if you write Y equal to C X, now here X is M into K now here C bar into K. Now we got Y C bar into K; that means, there is a involvement of say C 1 bar C 2 bar say C n bar into K.

So, when we are seen the Gilbert test for controllability, we have talk about this particular equation A A bar B bar and we have seen that in a controllability, if any element of B B bar is 0, then system is not controllable that is concern with the B bar. Now here the observability test is concerned with the C bar and C bar involve C 1 bar C 2 bar C n bar. Now output Y it depend upon K, now K is our new states there very important now and if assuming that if the any of the elements in C 1 bar to C n bar is 0, let us say if the C 2 is bar 0; that means, the K this involvement of various states K 1 K 2 up to K n.

So, this is C 2 by 0; that means, the K 2 state which is available for measured in earlier because of this 0 this is not available for measurement and therefore we can say that the system is not observable in this particular case. That mean directly we can we have got the information about the particular state which is not observable where as if you see the earlier examples earlier methodology, C C A A raised to n minus 1. Here we have not got any information about the v state is observable or v state is unobservable or whether r state are unobservable no information, but whereas, here we get can get all the information.

Therefore, from practical point of you for analysis point of view, I think we will defiantly go for this Gilbert test. But as far as the numerical point of view if our solving the examples or from academic point of view, we will go for the Kalman test. But both tests are important and both tests are are helpful in check in the controllability and observability that is another word we can say that that is the first step of designing of the controller because whatever we are learning is final aim is to controller design.

If this test are filled it means that controller design cannot be possible. So, that in other words we can say that this Gilbert test as well as Kalman test is an necessary condition for the check in the controllability and observability and now we are talking about the observability; that means, it is also a necessary conditions for designing of the controller.

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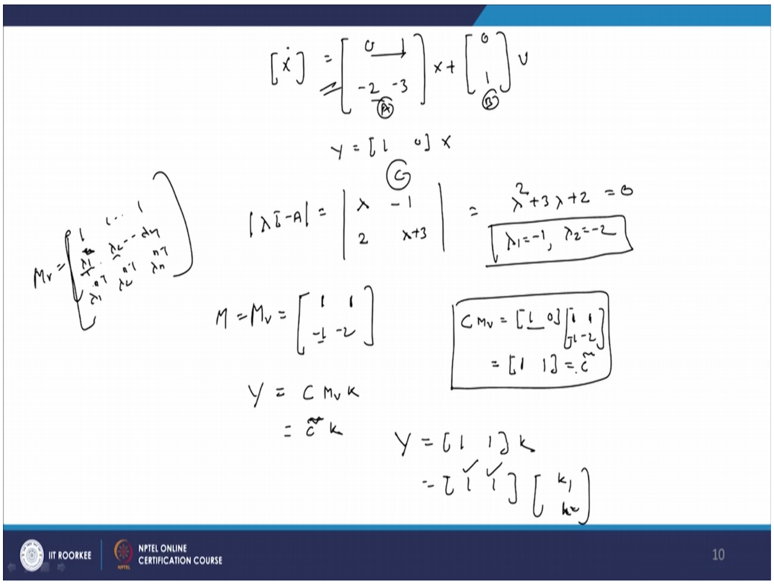

Now we see an example now will take the same system which we have taken earlier say X dot as 0 1 minus 2 minus 3. This X say 0 1 U and Y as 1 0 into X. Now this is A matrix, this is B matrix and this is C matrix; A B and C. Now our aim is to check the observability of the system using Gilbert test and Gilbert test is base on the model matrix; that means, we need to determined the eigenvectors.

So, the first step in this case we have to calculate the eigenvalues of this matrix. So, now, lambda I minus A ; so, it find at here this is lambda this is minus 1 2 lambda plus 3. So, if you solve it will get lambda square plus 3 lambda plus 2 equal to 0 so, if you solve, it will get lambda 1 equals to minus 1 lambda 2 equals to minus 2. So, there are 2 Eigenvalues lambda 1 equals to minus 1 lambda 2 equals to minus 2 and this system is definitely a stable system and now where to check the observability of the system.

So, now the concept of diagonalization we are using; that means, for each eigenvectors for each eigenvalues we have to calculate the eigenvectors and eigenvectors there are 2 different processes we already seen and, but we have seen that if your given system matrix, it is an companion form that is 0 1 minus 2 minus 3. So, if this is the arrangement, we will directly write the eigenvectors of a given eigenvalues. So, for this case we can write down the model matrix M V. This M which your take earlier, this M and what I am talking about M V their same, but V is is the Vandermonde matrix because Vander monde has developed this one that is why for this case, we are writing down the model matrix as M V this also as called as M matrix.

So, here we do 1 1 this is the arrangement and this minus 1 and minus 2. We already seen how to write in this that is the general form of this ; so 1 1 like this lambda 1 square lambda 1 lambda 2 lambda n and here lambda 1 raised to n minus 1 lambda 2 raised to n minus 1 say lambda n raised to n minus 1 1 1 1 lambda 1 lambda 2 lambda n lambda 1 n raised to n minus 1 lambda 2 raised to n minus 1 lambda n raised to n minus 1 say here, 1 1 and for this lambda 1 lambda 2 has been written as minus 1 and minus 2.So, M we have got directly.

Now we are talking about Y Y equal to if you see here C M into K. Now here we write down this has C M V into K that is here C bar into K. Now we have to determine this, we have to determined here C into M v. So, C is here 1 0 and here our M V M V is 1 1 minus 1 minus 2 and here we can write down this as 1 multiplied by this. So, here we will get 1 and here we multiplied will get 1 X, this is K. Now here we have got solving this 1. We have got 1 1. So, C M V after solving this 1 0 1 one minus 1 minus 2 we got 1 1 and now this Y equal to 1 1 this is K and this is nothing, but your C bar into K and this is nothing, but 1 1 at 2 state, we can write down K 1 K 2.

So, there are 2 states K 1 and K 2 that are transfer state for each K 1 K 2, we have got this 1 and this one ; that means, this states are available for observations and therefore, we can say that by means of this concept the our system is we can say observable system.

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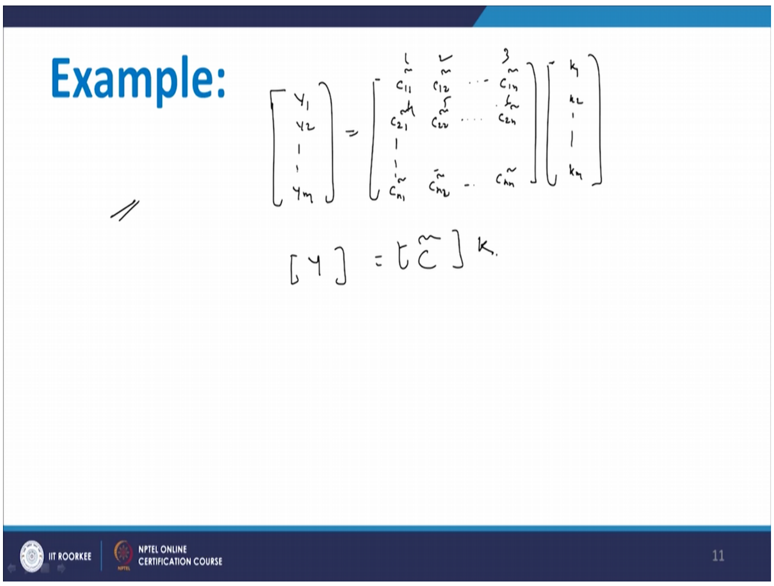

Now the another problem is that if there is a involvement of state space model and state space model involved the multi input, multi output system in; that means, if we have output Y 1 Y 2 Y M and their involvement of say C 1 1 bar C 1 2 bar say C 1 n bar, then here C 2 1 bar C 2 2 bar C 2 n bar C n 1 bar C n 2 bar C n n bar and here is say K 1 K 2 say up to K M, K 1 K 2 this is K M this is K M. Now this is complete Y equal to C bar into K and now C bar involved all these elements it can be related 1, 2, 3 say 4, 5, 6 and any of the elements. So, all the elements are involved.

Now in that case how to check the observability of the system and this output Y we have got from the concept of diagonalizations, the C bar we have got it is. So, how to check the observability of the system? So, in that case, if any of the column elements should be 0; if any of the column elements are are 0, then we can say that the system is not observable. So, that is the case when we we are talking about the multi input multi output system; that means, if any of the column elements are 0's that is the any columns any of this, then we can say that system is not observable. And most of the time, we have a multi input multi outputs system.

But portable robotic arm systems, then we have process control system, a power system everywhere multi input multi output system. Therefore, if you talking about the practical point of view, we have to go through this multi input multi output plant, but as far as the sometimes academic point of view, we have we are talking about the single input single output system. And there the most important concept we understood that if any of the element in the C matrix is 0 ; that means, that that state is not available for the observation and hence we can say that that particular system is not observable. Now, these are some references.

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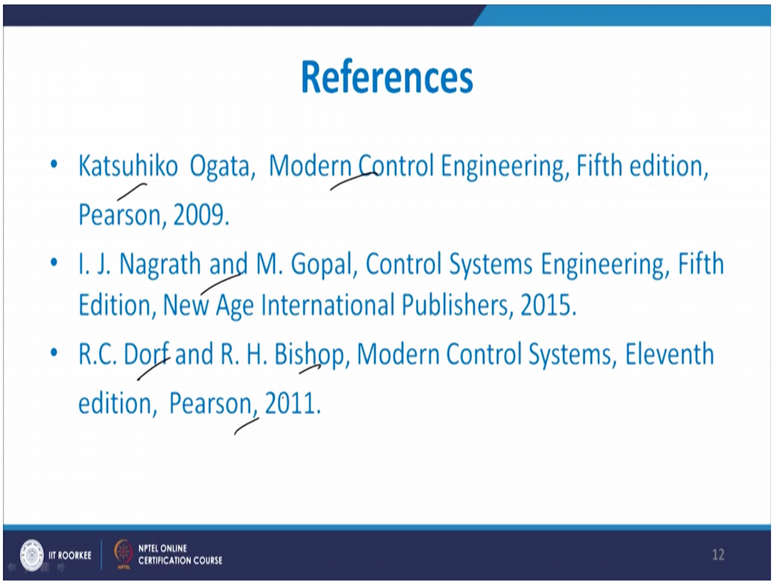

Thank you.