Advanced Linear Continuous Control Systems
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Lecture – 29
Controllability in State Space (Part-1)

Now, we start with Controllability in State Space. So, what is mean by controllability?

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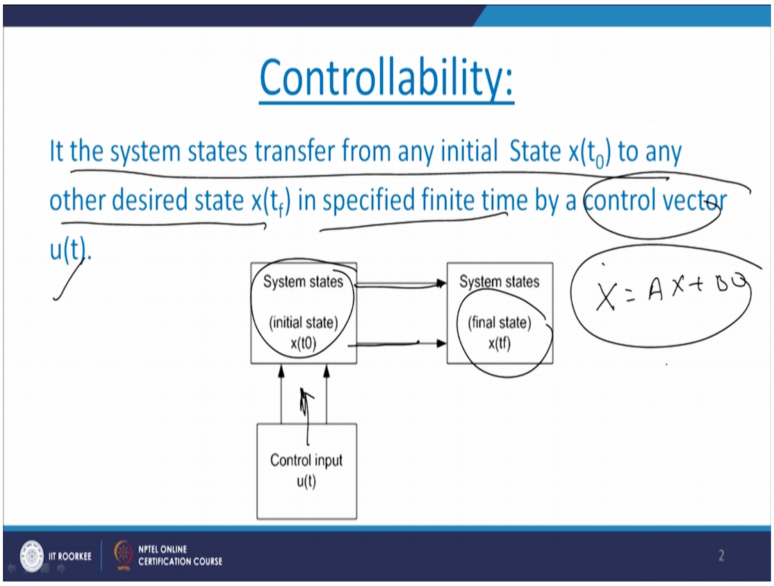

So, it is say that if the system states transfer from any initial state x of t naught to any other desired step x of t f, in a specified time by a unconstrained control vector u of t; that means, these are system states they are call initial state. Now, because of this control input they will transferred to a final state x of t f; that means, this we are talking about X dot equals to A x plus B u. The states at some initial time and it has been transferred to the final state, and the system is controllable. Then how will think about this in a general way for example, an industrial trip is going from say one place to another.

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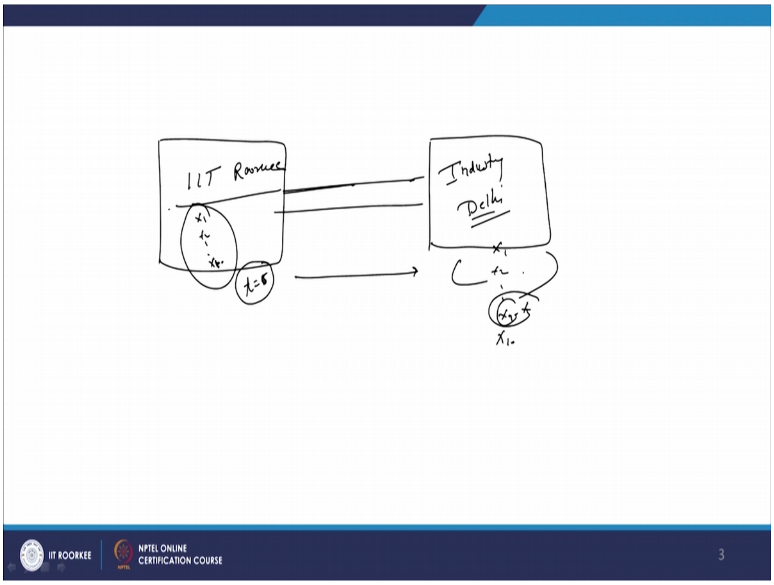

Let us say now, we are here in say in IIR Roorkee and now here I have to take out some students from IIT Roorkee to say some industry at Delhi. So, their IIT Rookee is there campus then this the industry is there, and I have to take let us say 10 students, now we will defined the 10 student as 10 states x 1 x 2 to x 10. So, the 10 states or 10 students are present in IIT Rookee before travelling; and now we have started travelling from IIT Rookee to some industry at say at Delhi which is in India.

So, what happen at t equals to say at 5 it has been started and it has to reach at a sometime. So, what happened all the states are available on initial time, sometimes you can say that it can be at a t t 0. So, these states are available at some initial time, but when this states or the student has been transferred to say Delhi say it should be x 1, x 2 up to x 10, but when the students trip has been transferred to Delhi out of these let us say this is x 9. So, let us say this is x 9 state or x 9 student is not present. So, as; that means, when this bus for which I am taking my students when these reach to Delhi and I would take the students to some industry, and when I started counting the students instead of 10 students I have observed 9 students.

So, if the 9 students are observed one is to sent in that particular group in that particular states then what will happened? Can I take to student to industry because this is a responsibility for me to take care of the all students when I observed that the student is not there it means that; that it is a very tough work for me I have to identify where is the student and I will do this process and if I am not found that they it is better for me to take the student back; that means, our work cannot be completed.

It means that your system is not controllable, let me what we want as we apply input to the system whatever the states are there, that up to applying input the state must be transferred present, but it is not doing is work; that means, dot system is not controllable and when the system is not controllable. We cannot design any controller in that case therefore, this concept of a controllability is very much important in a state space application the state must not be ideal and was state is if the state is ideal or it is not all the it.

That means we cannot design any controller if you cannot if cannot test the controllability of the system. Location has come how to check the controllability of the system. Now, testing of the controllability so, if you see the literature there are two test, one is call Kalman test and other is called Gilbert test, both have been used to check the controllability of the system.

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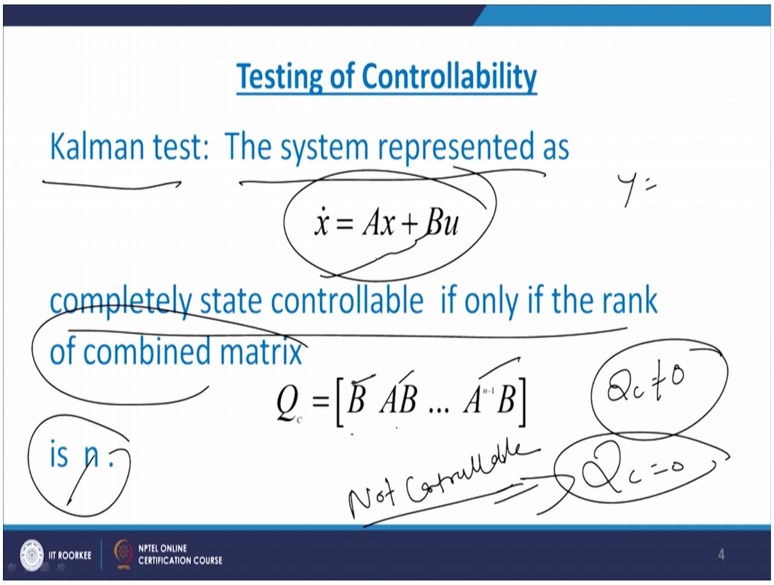

Now, first of all we see the first test Kalman test, the system represented by x dot equals to A x plus B u. Here I have not written equation of y because the state here, is concern with x and u there is no relationship with output therefore, when I am talking about Kalman test, I have written x dot equals to A x plus B u.

So, this system is completely state controllable if and only if the rank of this combined matrix that is B AB A raise to n minus 1 B is n the rank is n; that means, that determinant of Q c should be nonzero. If this Q c is 0 it means that system is not controllable; that means, determinant should be non-zero it means that rank of this system a is an and in that case we can say that system is controllable. Then the again here point is come why it is like this why we are saying that for controllability we required B A B A raise to n minus 1 B now will see the proof for this.

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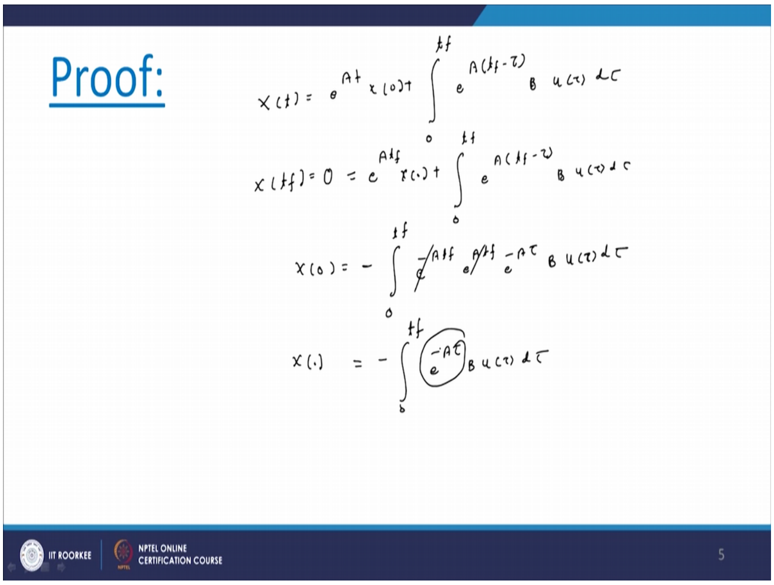

Now, for this particular case we have seen the state response the state response, we can write down as x of t equal to e raise to A t x naught. So, 0 initial time 0. So, t f e raised to A t f minus tau B u tau d tau. Now, here is the state equation x dot t equals to e raise to a t x naught 0 to t f e raised a tau f tau of B tau d tau now what is the final state? Now we are conserving the final state is say your original. So, final state is a origin; that means, what I am saying x of t f this final the state origin that is equal to e raise to A t f x naught naught 0 to t f e raised to A tau f minus tau B u tau d tau.

Now, here a final state is origin. So, origin means we can write this as origin 0. So, 0 equals to e raise to A t f x naught 0 to t f e A e raised to A tau f minus tau B u tau d tau. Now this origin is there now we can write down equation of x naught as x naught equal to minus 0 to t f, now this is plus e raise A t f.

So, when we take one side we can write down as e minus A t f, now this e raise to A t f multiplied by e raise to minus A tau B u tau d tau. So, now, this cancelled. So, will can write this x naught equals to 0 to t f, e raise to A tau B u tau d tau. Now, about this part e raise to A tau. This a raise to t tau B u tau d tau. So, how to write down this e raise to A tau? So, we know the Cayley Hamilton theorem. Now here we will use the concept of Cayley Hamilton theorem that is every square matrix satisfies its own characteristic equation, we have already seen this in a state transition matrix.

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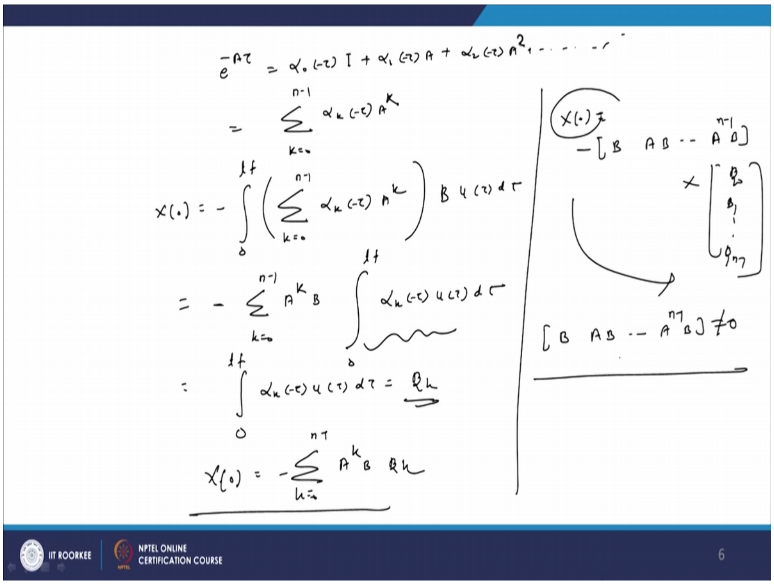

Now, this part we can write down in terms of Cayley Hamilton theorem e raise to minus A tau that is equal to plus alpha 1 plus alpha 2 and this process imitated. So, this equation can be written as k 0 to n minus 1 into A raise to k. Therefore, the equation can be written as x naught equal to minus 0 to t f k equal to 0 to n minus 1 you can do A k B u tau into d tau and this equation finally, can be written as k equal to 0 to n minus 1 A raise to k B 0 to t f alpha k tau u tau into d tau.

So, now this equation we are obtained, and now after word this equation can be written as 0 to t f particular this part, alpha k tau u tau d tau let us say equal to beta k. And therefore, finally this equation can be written as x naught equal to minus k k equals to 0 to n minus 1 a k into beta k. And this equation finally, we can simplified as x of x naught equal to minus B AB say A raise to n minus 1 B and this is multiplied by this beta k as beta naught beta 1 up to say beta n minus 1. Now, here x naught is there and this equation is there this is to be satisfied. So, this is satisfied that a initial condition h 0 equal to this is satisfied only when this particular part B AB A raise to n minus 1 B should be non-zero.

So, when this is satisfied then and then we can get the controllability of the system or we can say that x 0 will be satisfied and therefore, this B A B A raise to n minus 1 B is the controllability test of the system; that means, the whatever the Kalman test, this Kalman test that is to check the controllability of the system we have to check the rank of B A B A raise to n minus B n.

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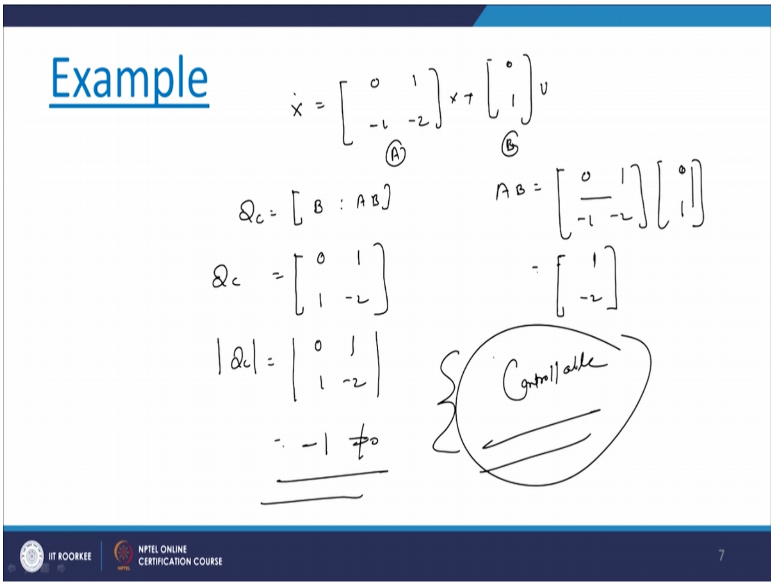

Now, we will see the example; suppose the system is represent by x dot equal to 0 1 minus 1 minus 2 x plus 0 1 into u I never question is we have to check the controllability of the system. Now, this is A matrix and this is a B matrix and now here we have to use our Kalman test that is x dot equals to A x plus B 1 B A B A raise to n minus 1 B. So, here we can write the controllability matrix Q c as B A B; that means, here we need the multiplication of A and B matrix. So, this A B we can write down as 0 1 minus 1 minus 2 and here B as 0 1 and if you solve it. So, here get 1 and here minus 2.

So, B is our 0 1 and here is 1 minus 2 this is Q c matrix and now what we want we want a determinant of this matrix. So, determinant of this matrix is Q c 0 1 1 minus 2, and if you solve it you will get here as minus 1 this is nonzero and therefore, we can say that the system is controllable. So, here simply determining the B matrix B as simply writing

B and A B or in terms of A raise to n minus B n when we can get the controllability of the system. So, here we will find that this controllability test form numerical point of view is quite easier, but you will find at we are not getting the inside of this controllability, why this means why this type of thing is happened here or let us say if it is not controllable, why did not controllable that information we cannot get.

Because whenever we are dealing with the system we must know that out of this different state which particular state is not controllable. So, that somebody we can make if required that time, but this type of informations cannot be achievable by means of this Gilbert this Kalman's test. So, in that case there is there is another test, that is called Gilbert test of controllability and based on this Gilbert test we can get in some information about the inside of the system; that means, which is the particular state, which is not controllable that information can be achieved by means of this Gilbert this Gilbert test.

So, we will see what is this; Gilbert test of controllability. So, the Gilbert test of controllability is base on the concept of diagonalization. In that time I had a explained you the concept of daigonalization that time I had told you that this diagonalization is helpful in checking the controllability of the system. And as I discussed earlier that before checking we will going for the controller, we have to check the controllability; that means, whatever we have study earlier that has been very much important in control in controllability as well as the control design. Now, we again see the concept which was in last time that is about diagonalization, and how this diagonalization is useful in the checking in the controllability we will see it.

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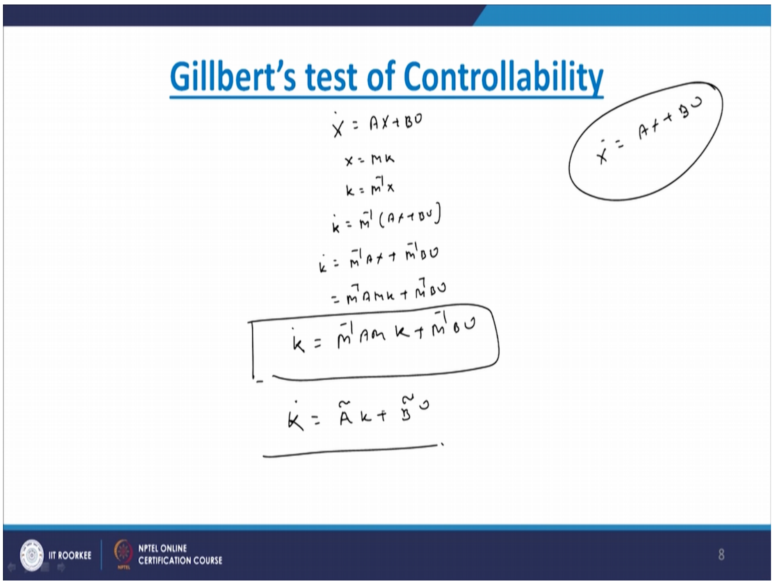

So, we start with the our original model as X dot equals to A x plus B u as I told earlier that we have not taken y, because y is not required here it is concerned with x and u only. Now in case of diagonalization we are defining a new state that can be any variable just now x we can define a k variable. So, now here within state transformation say x equal to M into k, and this k equal to M inverse x and now here we can write down k dot equal to M inverse x dot that is equal to A x plus B u. So, afterward we can write down as k dot equal to M inverse A x plus M inverse B u.

And now here M inverse A what is x x equals to M k plus M inverse B into u that is k dot equal to M inverse A M into k plus M inverse B into u and this is nothing, but k dot equal to A bar k plus B bar into u. So, now this we you equation we have got normally we equation just like x dot equal to A x plus B u and where we have converted the (Refer Time: 18:19) equation in this form this a bar it is in a diagonal form and therefore, this equation.

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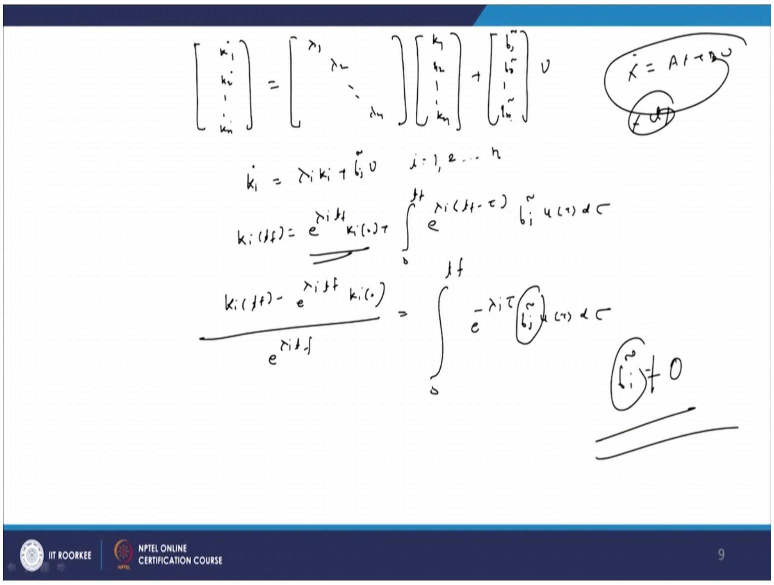

We can write down as k 1 dot k 2 dot up to k n dot that is equal to lambda 1 lambda 2 up to say lambda n here k 1 k 2 k n and here B 1 bar B 2 bar B n bar into u. So, now this equation that is k i dot equal to lambda i into k i B i B i bar into u that is i equal to 1 2 up to say n and if you solve this equation we can write down as k i of t f e raised to lambda i into t k i 0 plus 0 to t f e raised to lambda i t f minus tau B i bar u tau into d tau.

So, now here just like x dot equals to A x plus B u, we determine x of t here we have define k f k i of this is a final time equals to lambda a t f x k i 0 this is our unpost part and this is the post part now which simplify this. So, this equation can be written as k i t f minus e i lambda i t f, k i 0 divide by e lambda i into t f equal to 0 to t f e raise to minus lambda i tau B i bar u tau into d tau.

So, now this equation has been got now, but here till that we have not understand that where is the controllability part has come; So, now, if we see if these particular elements, this is concern with u and k 1. So, k 1 is one of the states if any of the elements is 0 here. So, similarly here if any of the elements 0 here, there is no connection between the states and you therefore, these particular part this is B i bar this is nothing, but concern with a B i B 11 B 2 bar B n bar so, that should be non-zero.

If any of the elements in this particular column is 0 then we can say that system is not controllable; that is particularly if you see this part this B r bar if any of this B 0 there we no connections between the input and the as well as the states therefore, to check the controllability of the system what we can do that that none of the 0 in this particular column should be; that means, none of the elements in this column should be 0. Now, this is the case which you have taken when there are eigenvalues are distinct, but the case may be different when you are eigenvalues are repeated.

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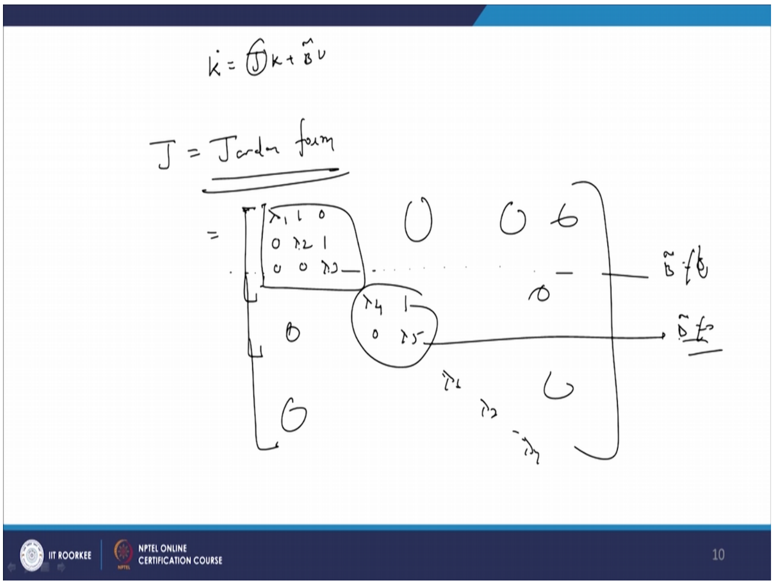

That means, let us say here k dot equal to j k plus B bar into u and d j bar is particularly in the Jordan form this is called Jordan form. Jordan form it means that you are having the repeated eigenvalues. So, when the repeated eigenvalues are there how to check the controllability of the system. So, the Jordan form is coming it means that, in the Jordan form let us say here lambda 1 say 1 0 0 lambda 2 one 0 0 say lambda 3, then there can be another this is one block is one block and their another block let us say lambda 4 1 0 lambda 5 and here let say lambda 6, lambda 7 up to say lambda n if and rest of the elements are 0's all are 0's.

Here these are diagonal elements lambda 1 lambda 2 lambda 3 lambda 4 lambda 5, 6 and these are the Jordan blocks and the main thing is that how to check the controllability in this case. So, in that case we have to see the they end of the Jordan blocks let us say this is end of the Jordan block the here whatever elements of B bar that should be non 0; that means, for this case this one and for this case that B bar should be non zero.

If any anyone here or the element concerned with the end of the Jordan block is 0 then we can say that system is not controllable. So, here form this we can get the inside of the system; that means, which particular state is not controllable; that means, here also we have seen that if any of the state let us say this is B 1 bar is equal to 0 it means that this k 1 dot is not exist in our systems therefore, that that particular state we can assuited and finally, if you take this checked into of account the controller design cannot be possible.

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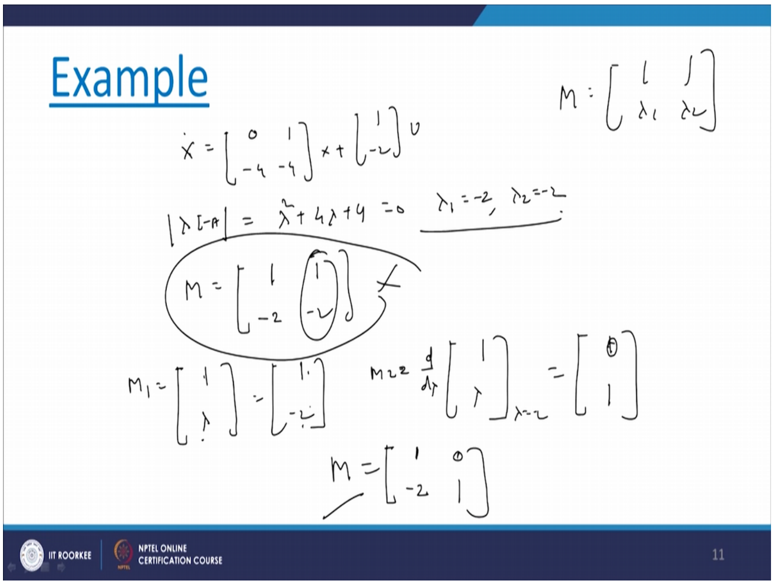

Now we see one example say x dot equal to 0 1 minus 4 minus 4 x plus 1 minus 2 into u. Now here if you calculate the eigenvalues of the system. So, we will find that the caustic equation for the system is lambda i minus a equal to lambda square, plus 4 lambda plus 4 equal to 0 and here lambda 1 equals to minus 2 and another lambda 2 equals to minus 2 and here as this is a in companion form. So, the model matrix M you can get at 11, here we will get it as minus 2 and about this particular minus 2, but this particular part is so on.

Because when the eigenvalues are repeated, so, in this case the first particular part the eigenvector corresponding to this eigenvector corresponding to this we can write down as 1 this is lambda equal to 1 minus 2; and corresponding to this M 2 what we have to take we have to take differentiation of 1 by lambda that is at lambda equals to minus 2. So, here will get 1 say 0 and here 1 and therefore, the model matrix in this case we will get it as 1 minus 2 0 1 and here M inverse A M will get it as minus 2 1 0 minus 2 this and this B bar this B bar B bar that is equal to M inverse B.

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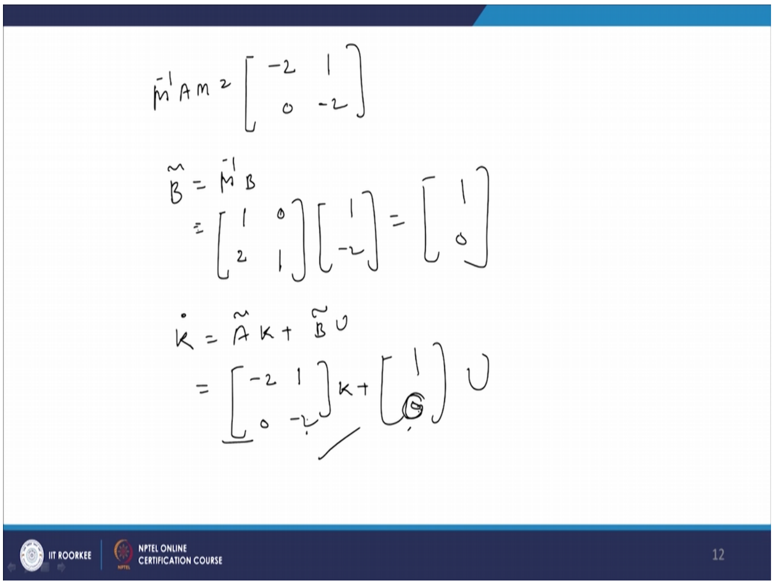

So, here ever M inverse is 1 2 0 1 if you calculate the inverse of this M will get 1 0 2 1, and B as 1 minus 2 and if you solve it will get o1 0 therefore, this k dot equal to A bar k plus B bar into u. So, we can write down this as minus 2 1 0 minus 2 k plus B bar as 1 0 into u. Now, you just see here; now here comes 0 and here it is minus 2 and as discussed earlier the particularly end of the Jordan block. So, this is Jordan block n the corresponding element should be non-zero, but here comes 0 that mean that this system is not controllable.

So, in this way we can check the controllability of the system. In one this particular example the most important point we observe that here about the writing the elements in the as I model matrix, here we will find that we are started with the writing the M and because this is a companion form or controllable canonical form.

We have to write down a Vandermonde matrix as 11 lambda 1 lambda 2, but in this particular case eigenvalues are repeated that is why initially we have in eigenvalue as 1 lambda that is 1 minus 2, but another one is minus 2. So, same eigenvalue cannot be written therefore, what we have done periodical differentiation of this one and then we have got 0 1 and in this way M we are determined and then M inverse A, M and B bar equals to M inverse B and finally, we come this result and, but this particular inverse is 0.

0 is corresponding to the Jordan block therefore system is not controllable. So, here one example I have solved you try to solve many example as possible and check the controllability of the system. You can also check the controllability of the system for multi input multi output system, and see the what way you are getting the result.

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Now, these are some references.

Thank you.