Advanced Linear Continuous Control Systems
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Lecture – 27
State Transition Matrix Using Cayley-Hamilton Theorem (Part-III)

Now, we start with the State Transition Matrix Using Cayley-Hamilton Theorem. In previous classes, we have seen the various method for calculations of state transition matrix. We have seen a power series method, then we have seen Laplace transform method and also we have seen that diagonalization method.

Today we have to see another method which can helps us for state transition matrix; that means, we can get a 10th state transition matrix by also or another method that method name is called Cayley-Hamilton theorem. This theorem is very important it can be useful in various application, particularly if you want to calculate inverse, but that also this particular method can be useful.

Now, we will see what is that theorem?

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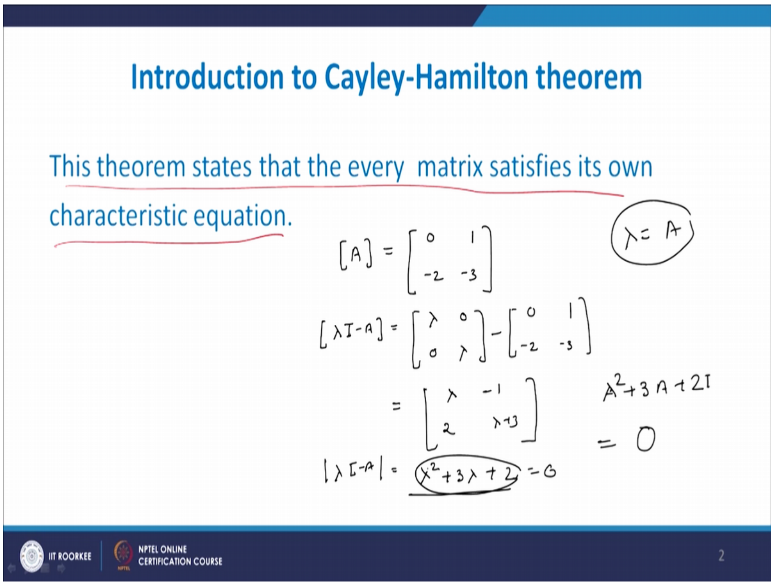

This theorem states that every square matrix satisfies its own characteristic equation. That is if you take any matrix if you determine its characteristic equation it will satisfied that is very important theorem. How it is possible? that is it says if you take A matrix; if you take this is a matrix A and let us say it is 0 1 minus 2 minus 3. And now we will take is characteristic equation that is lambda I minus A equal to lambda I minus A that is equal to; so here lambda 0 0 lambda minus A here is 0 1 minus 2 minus 3.

And if you solve it, you will get lambda minus 1 2 lambda plus 3 and if you take this determinant; lambda I minus A equal to lambda square plus 3 lambda plus 2 equal to 0.

Now, this is the characteristic equation in terms of lambda; normally we are calculating the Eigen values of this matrix and if you replace this Eigen value it become 0. But Cayley-Hamilton theorem says that here instead of this lambda; instead of this lambda you replace A, A is the original matrix if you replace A; in terms of lambda, you will get A square plus 3 A plus 2 I and that is also equal to 0, that is lambda and A; we can interchange that is the basic theory of the Cayley-Hamilton theorem.

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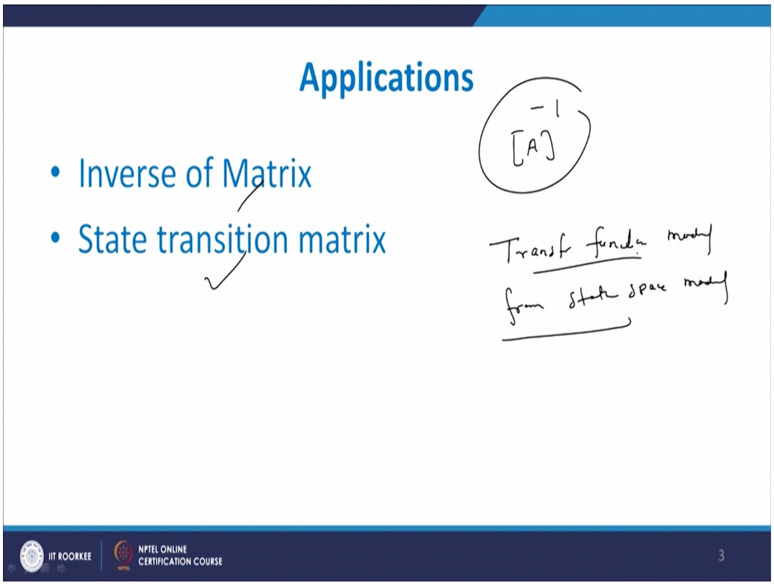

Now, its applications; the first application is inverse of matrix and second one is the state transition matrix or your topic is state transition matrix, but in here it will also determined inverse and we have seen that the applications of inverse means the determination of inverse is helpful in various approaches, it is useful in case of the Laplace transform approach.

It is also useful in various say in case in case of getting a state space model from the given transfer functions. That is our that is this particular state, this particular inverse is helpful in getting the transfer function model from a given state space model; we have seen this part earlier.

So, it has wide applications and however, here basic purpose is to getting the state transition matrix. Now first of all we will see; how to calculate the inverse of the matrix using Cayley-Hamilton theorem. Now we start with the concepts, so our main input is A matrix.

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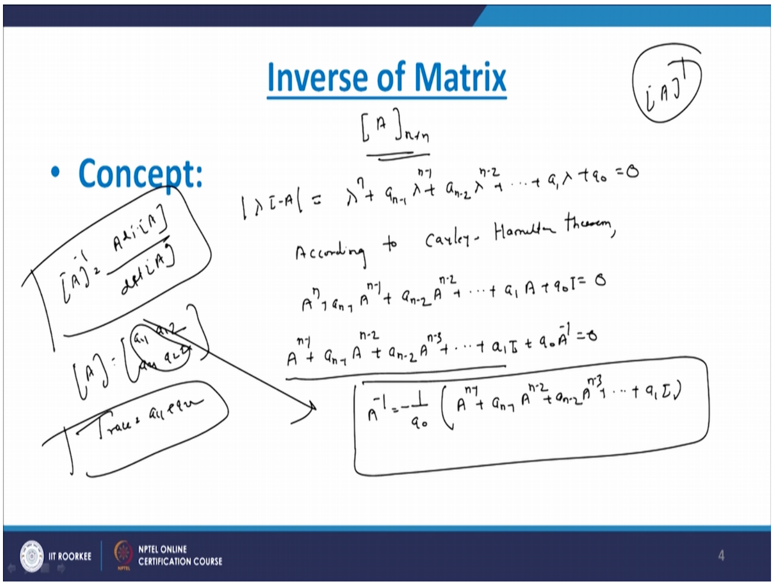

So input is A matrix and let us say it is n cross n matrix; now, we have to take lambda minus A that is a characteristic equations and we will take determinant of it; that is equal to lambda raised to n; a of n minus 1 lambda n minus 1 plus a of n minus 2, lambda n minus 2 plus a 1 lambda plus a naught equal to 0.

Now, this is the your characteristic polynomial in terms of lambda; when we are working on s domain, we can write down s instead of lambda that is s raised to n e of n minus 1 s raised to n n minus 1 up to a 0. Now our aim is to get the inverse; so now this is the characteristic equations. So, what we can do further? We will apply the concept of the Cayley-Hamilton theorem, as you seen the concept of Cayley-Hamilton theorem that is lambda can be inter change with A therefore, here we will convert this lambda into A.

So, according to Cayley-Hamilton theorem; we can write down the above equation as A raised to n a of n minus 1; A raised to n minus 1, a n minus 2, A raised to n minus 2 this process repeated. Then here a 1 into A equals to 0 is now this equation here written. Now our main purpose is to get the A inverse that is our base purpose is to determine A inverse.

Now, what we can do? To this equation, we pre multiple A inverse both side. So, if we multiple A inverse to this particular equations what we will get? So this n when we multiply by A inverse it becomes A n minus 1 plus a n minus 1 A, it become n minus 2 plus a n minus 2 A n minus 3 and this process repeated, it becomes a 1; we can say I plus a 0 into A inverse equal to 0.

Now, our purpose is to get A inverse; so A inverse we already got; now what we can do all these terms this A of n minus 1 to a 1 I; we move on right side. And this A 0 A inverse we keep it on one side. So, what will get? So, this A inverse equal to this and at this shift into right side.

So, what you will get? You will get A of n minus 1; a n minus 1, A n minus 2 plus a n minus 2, A n minus 3 and here is process repeating a 1 into I and this is basically your A inverse. We have seen various methods for getting A inverse; if you want to A inverse there is a various method one; A equals to adjoint of A it will divided by a determinant of A.

This is one and all this is alternative approach; we are we also seen a liberal algorithm to calculate the inverse, where this is based on the trace of the matrix; that means, we have seen the liberal algorithm means A suppose here a 11, a 12, a 21, a 22

So, the stress equal to a 1 1 plus a 2 2; so, based on the stress we have also determined the inverse; now this is an alternative approach to get inverse. Now we will see which method is count to be simple, we have seen that if you this particular approach liberal algorithm for determination of inverse as order in cases this is better, but for lower order we have seen that this adjoining of a determinant is helpful.

Now, will try to solve an example to calculate the A inverse of a given system matrix.

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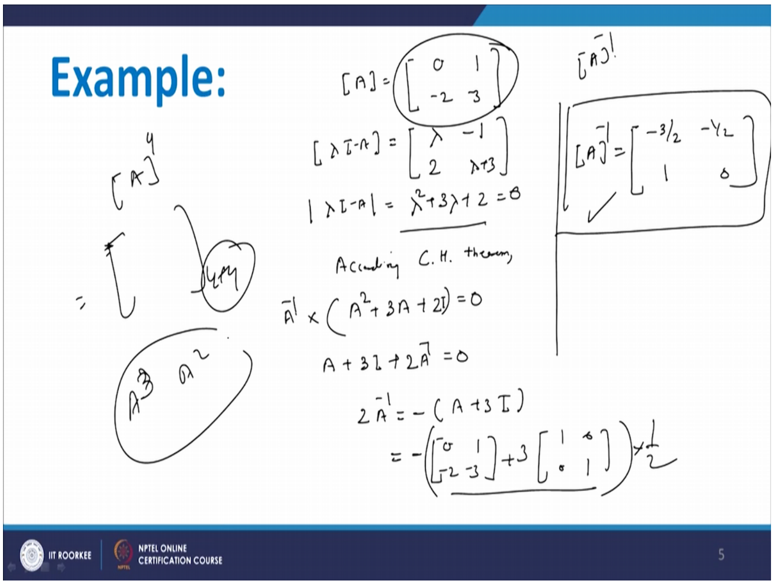

Suppose you have taken the matrix A equal to 0 1 minus 2 minus 3; this is A matrix. Now we have to take lambda minus A; if you take lambda minus A, it becomes lambda minus 1, 2 lambda plus 3.

Now we want characteristic equations; so, characteristic equations becomes determinant of lambda minus A. If you solve it you will get lambda square plus 3 lambda plus 2 equal to 0, but our aim is to 2 A inverse, but we have got a equation in terms of lambda.

But we have a Cayley-Hamilton theorem which states that every square matrix satisfies with the own characteristic equation. Now we can convert this lambda by A; so, if you convert this lambda by A. so according to Cayley-Hamilton theorem this above equation can be written as; there is according to Cayley-Hamilton theorem above equation can be written as A square plus 3 A equal to 0.

Now, A square 3 A plus to have got, but we want A inverse. So, best option is we apply or we multiply A inverse to both side; if you this is equations if you multiplied by A inverse, this becomes A, this become 3 I; 2 A inverse equal 0; that means, this A order of 2 change is to 1, this A and A inverse neutralize.

So, become identity matrix plus 2 into A inverse; now we have to solve this equations. So, we can write down as 2 A inverse minus A plus 3 into I. Now, what is A matrix? A matrix is 0 1 minus 2, minus 3, plus 3 identity matrix is 1, 0, 0, 1 and total this multiplied by 1 by 2. Now this particular matrix we have to solve then divide by 1 by 2 and after solving this; we will get the A inverse as minus 3 by 2 minus 1 by 2 1 0.

Here is a simple mathematics just you multiply 3, then additions then 1 by 2 we can come across this A inverse matrix. So, we will find that in a very simple manner; we have got the A inverse. I think particularly for n order system I think this method is better than this (Refer Time: 13:31) algorithm as well as the our conventional adjoining and determined approach.

But again is the problem is that issue increase the order of for example, here we have A raised to 4, A raised to 4 or we have this particular matrix, this matrix it is a 4 cross 4; that means, finally, this equation will get in terms of A cube, A cube, A square; that means, here we have to multiply this a matrices. So, it may increase your computations like this, but as far as we will find it for some lower order; this Cayley-Hamilton theorem is quite useful.

Now, we go for the second concept, state transition matrix ok; now we go with the concept of state transition matrix.

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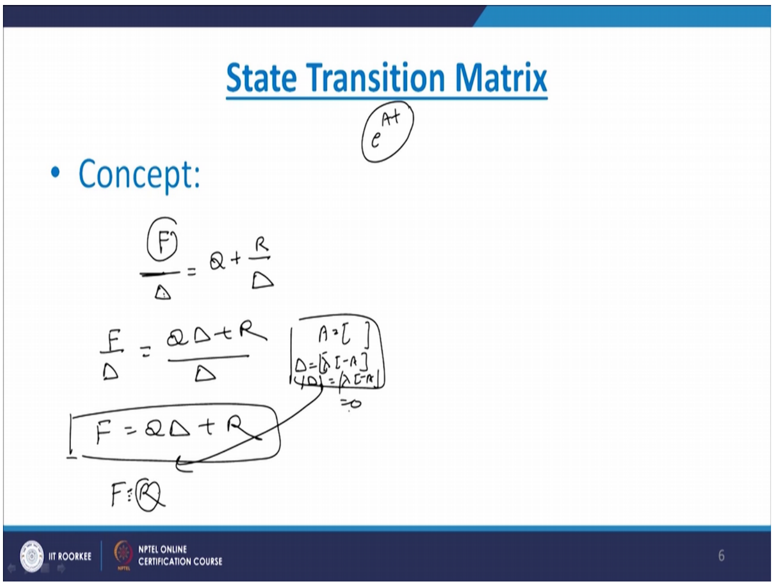

State transition matrix means e raised to A t; that is e raised to A t and I as discussed earlier we have seen the various method to get the e raised to A t. I will find that this Cayley-Hamilton theorem is useful to get in the state transition matrix.

Just looking after the problem just it satisfies means any square matrix satisfies characteristic equation, it is hardly believable that how can this particular concept is useful for state transition matrix. Now, we will see some logic behind it; the logic says that suppose if we had taken one particular A functions say we have taken a function F and let us say it is a very higher order, it is very higher order.

To this F; if you divide by delta definitely the degree of this del is lesser than F; F is very higher degree. So, F is divided by delta; so what we will get? We will get one quotient and also we will get the remainder. So here in this F divided by delta will be Q plus remainder R divided by del and; that means, this above equations F by del equal to deled Q deled plus R and if you required this what we get equation F Q delta plus R.

Assume that delta is the characteristic polynomial of any matrix, this delta is the characteristic polynomial F of matrix. That is if you are having we take say this A matrix A is matrix and this delta equal to lambda minus A. So, definitely if you take determinant of this it become lambda minus A. So, this determinant is 0 this determinant lambda minus A equal 0. So, this means this it is delta 0, if you replace here what we will get, F equals to R.

Now, this F equals to R and R is the remainder and definitely degree of this R is also lesser than this delta. So, we get F equals to R now this is the basically the concept that is F equals to R. Now this concept we have to use in getting the state transition matrix; now how we will do it?

Now, we take a polynomial which is represented in terms of lambda and let us say it is f of lambda.

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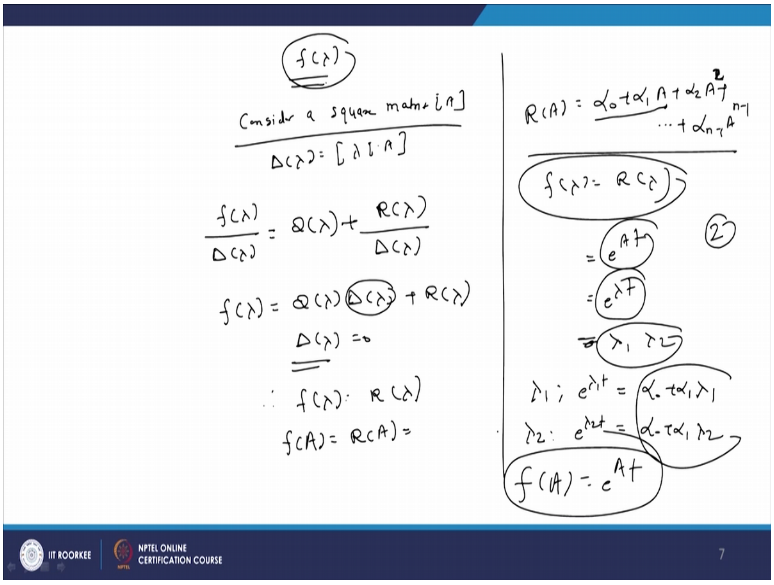

F of lambda is the polynomial, then consider a square matrix A, consider a square matrix A and now it can be written as del of lambda equal to lambda I minus A; this is f of lambda and as I told that a f of lambda is higher degree than del lambda now we divide this.

So, if you divide is f of lambda divide by del of lambda equal to this is equal to Q of lambda plus R of lambda divided by del of lambda. And after solving it become f of lambda equal to Q of lambda equal to del of lambda plus R of lambda. Now we assume that this del of lambda equal to 0 and therefore, what we can say this f of lambda equal to R of lambda. But now here is coming the concept of Cayley-Hamilton theorem, this lambda can be interchanged with A matrix.

So, therefore this f of A is equal to R of A and this R of A can be written as; this R of A we can write down as alpha 0 plus alpha 1, A plus alpha 2 A square plus this process repeated till alpha n minus A raised to n minus 1; that means, degree of this R of A is lesser than this both del lambda, del of lambda f of lambda or del of a and f of A and now this is a R of lambda. Now here this is f of A equals to R of A or we can say this f of lambda equal to R of lambda.

Now, we want to determine the state transition matrix; now state transition matrix it nothing, but e equal to A t or this is e equal to say lambda t. So, here depend upon the Eigen values; suppose if you are insisting we have 2 Eigen values lambda 1 and lambda

2. So, this for 1 lambda 1 this e of lambda 1 t equal to alpha 0 plus alpha 1 into a alpha 1 into say let us say in terms of lambda.

And for another lambda 2 e raised to lambda 2; 2 t equals to alpha 0 plus alpha 1 into lambda 2. That means, what are be the Eigen values for given system matrix; suppose there are 2 Eigen values then here this n equals 2 minus 1 that is alpha 1 into A; that means, our equation equals to we have take alpha 0 into A.

So, as the as a number of Eigen values we are increasing or the order of the matrix increases, this term will increases therefore, we have to take R of lambda, R of A or R of lambda equals to alpha n minus 1 A raised to n minus 1. So, here e raised to lambda 1 t e raised to lambda 2 t and now you have to solve these equations to get the alpha 0, alpha 1 and if you calculate it finally, we will get the equation of f of A as a equals to A t e raised to A t.

So, this is the concept which I have told at basic concept as I told that is it is f of A equals to R of A f of lambda equals R of lambda. And then we have to calculate Eigen values and for each Eigen values, we have do as a functions and that is written in terms of alpha 0, alpha 1 for first, second like this and we have to solve these equations to get the values of alpha 0 and alpha 1. Now, we solve a example and again see it in detail how to get the state transition matrix.

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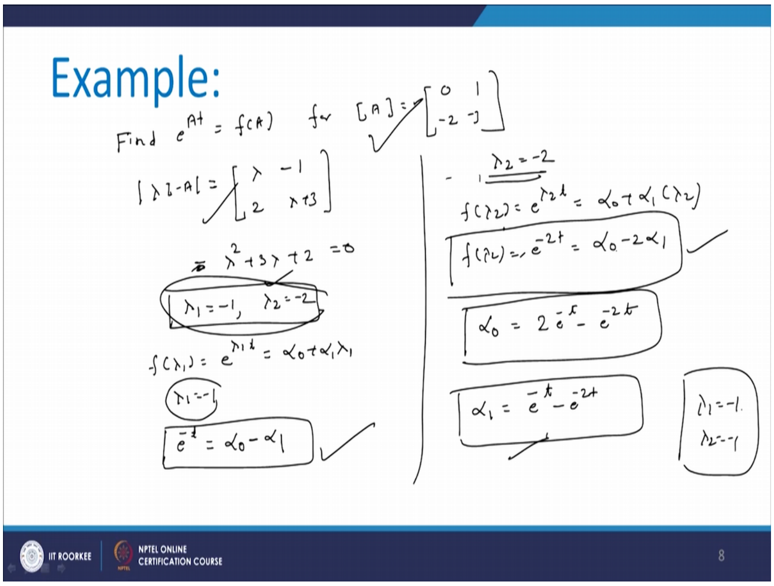

So, our question is find e raised to A t that is equal to f of A for a given matrix A equal to 0 1 minus 2 minus 3; this is the problem, now we have to use in Cayley-Hamilton theorem.

Now, the first step is we have to determine its characteristic equations. So, characteristic equation for this matrix A that is say lambda I minus A that is equal to lambda minus 1 2 lambda plus 3 and this is equal to lambda square plus 3 lambda plus 2 equals to 0; so, lambda square plus 3 lambda plus 2 equal to 0.

Now, we calculate the Eigen values of this; so, Eigen values for this will get as lambda 1 equal to minus 1, lambda 2 equal to minus 2; these are 2 Eigen values. Now this 2 Eigen values you have determined and our purpose is to get e raised to A t. Now, we will take first function that is f of lambda 1 equal to e raised to lambda 1 t; that is equal to alpha 0 plus alpha 1 into lambda.

That is f of lambda 1 equal to e raised to lambda 1 t equal to alpha 0 plus alpha 1 into lambda 1. Because the e and A lambda can be replaced, can be inter changed that is lambda 1 and A can be inter changed.

Therefore e equals to lambda 1 t, alpha 0 plus alpha 1 into lambda 1 and you know why we have we have vertical why we have vertical alpha 2 because here this is because here A raised to n minus 1 that is n is to therefore, we have taken this equation till alpha 0 plus alpha 1 into lambda 1.

Now the first is replace say lambda 1 equal to minus 1 in above equation. So, what we will get e raised to minus t equal to alpha 0 minus; this alpha 1 is minus 1. So, it become alpha 0 plus 1 this is a first equation you got; e raised to minus t equals to alpha 0 minus alpha 1.

Now, coming to the another equations that is lambda 2 equals minus 2. Now we replace lambda 2 equals to minus 2 here; that means, here function of lambda 2 e equal to lambda 2 into t plus alpha 0 plus alpha 1 into lambda 2. Now here we solve it that is replacing lambda 2 by minus 2 gets minus 2 t, alpha 0 minus 2 alpha 1.

Now, this is equation number 2 this is f of lambda 2. Now this equation this and this and now we can solve this 2 equations and we can get the unknown values of alpha 0, alpha

1. That is there are 2 equation, 2 unknowns; so, we can easily get the values of alpha 0 and alpha 1.

So, after solving these 2 equations; we will get the values of alpha 0 alpha 1. So, this alpha 0 after solving; we will get as 2 into e raised to minus t minus e raised to minus 2 t is alpha 0 and this alpha 1 equal to minus t minus e raised to minus 2 t.

So, now we have got alpha 0 1; our basic purpose is to get the f of A and that is equals to e raised to A t.

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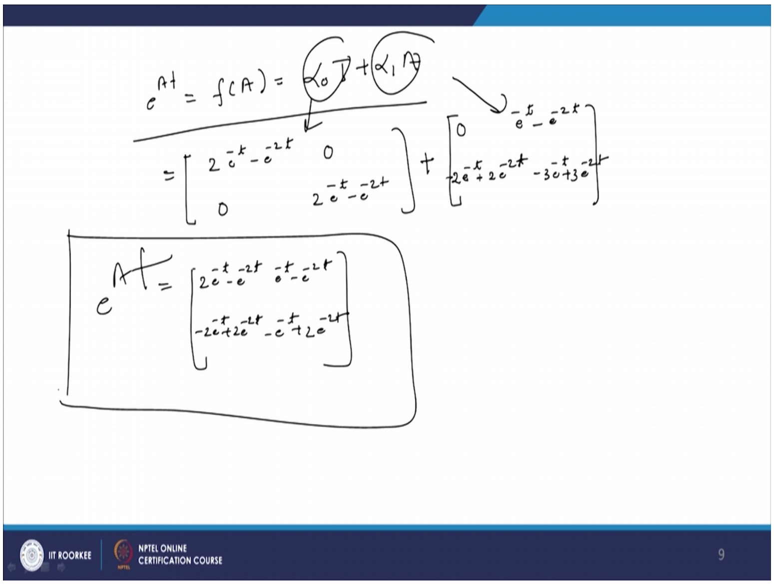

Now this e raised to A t; equal to f of A that is equal to alpha 0 I into alpha 1 into A; this basic equations. Now why this is to obtained? Because of the Cayley-Hamilton theorem because A and lambda can be inter changed because you see all this results all this calculation; we have done in terms of lambda; we have got alpha 0 and alpha 1.

But to get e raised A t, we have changed this alpha by A that is e raised to A t equals equal to f of A; alpha 0 I plus alpha 1 into A, now replace the value alpha 0 alpha 1 and A; A is the symmetric matrix this matrix a this matrix A. Here in this equation you just see what will get; so here this alpha 0 into I; I is a identity matrix.

And we have already got alpha 0 as 2 e raised to minus A t; e raised to minus 2 t. So, here we can write down this as 2; e raised to minus t minus e raised to minus 2 t, this is 0 0. So, then here 2 e raised to minus t minus e raised to minus 2 t like this plus alpha into I

Now, here alpha 1; alpha 1 we have already determined this one and this is multiplied by A. So, when this alpha 1 is multiplied by A so we will get equation as or result as 0 e raised to minus t minus e raised to minus 2 t minus 2 e raised to minus t plus 2 e raised to minus 2 t.

Here we will get minus 3 e raised to minus t plus 3 e raised to minus 2 t. So, here this alpha 0 I is nothing, but this part and this alpha 1 A equals this and now we add this all this elements individually, so we will get the result as 2 e raised to minus A t minus e raised to minus 2 t.

Here we will get e raised to minus t minus e raised to minus 2 t and here will get as minus 2 e raised to minus t plus 2 e raised to minus 2 t. Here get e raised to minus t plus 2 e raised to minus 2 t and now this is e raised to A t. So, we will find at in this way we have determined the state transition matrix; transition means transition forms of state from initial time to final time this results we have got by means of Cayley-Hamilton theorem.

But now here we have taken case, where their Eigen values are distinct that is lambda 1 equals to minus 1, lambda 2 equal to minus 2 and if you have easily we have got the result that is replacing lambda 1 equals to minus 1, then we are replace lambda 2 equals to minus 2 and somehow we got results.

But now here is a problem; suppose if say lambda 1 and lambda 2 both are same, assume that the case the case is that lambda 1 equal to say minus 1 and again say lambda 2 is also equals to minus 1 or say repeated Eigen values then how to manage this equation?

Because if you take lambda 2 minus 2; we come across the similar type equations and there are they are difficult to solve therefore, we have to use certain logic in order to solve these type of example. Now, we are going to solve example assuming that there are repeated Eigen values of a given matrix.

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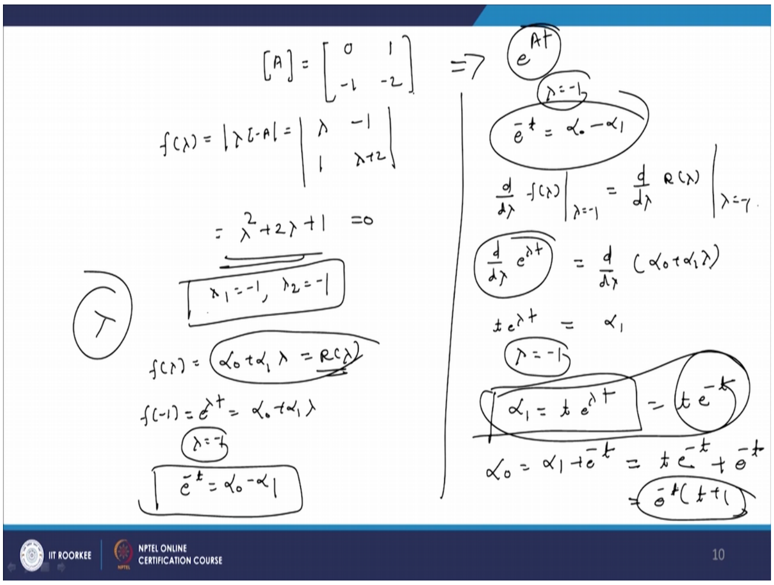

Now we will take a example; say A matrix is equal to 0 1 minus 1 minus 2; this is given matrix A and purpose is we want to determine e raised to A t this the A.

Now the what is the first step we have to determine lambda I minus A. So, the function f of lambda equal to lambda I minus A that is equal to lambda minus 1 1 lambda plus 2 like this. Now solve these equations; so, after solving we will get lambda square plus 2 lambda plus 1 equal to 0 and if solve this equation or we calculate the rules this equation.

So, what we will get? We will get lambda 1 equals to minus 1 and again same lambda 1, lambda 2 equal to minus 1; same Eigen values; lambda 1 minus 1 lambda 2 equals to minus 1; they are repeated Eigen values and our is to get this sate transition matrix. Now how we will proceed further?

So, first of all we proceed with initial step that is f of lambda equal to alpha 0 plus alpha 1 into lambda that is equal to R of lambda. Now here function f of lambda that is minus 1 that is equal to e raised to lambda t, again alpha 0 plus alpha 1 into lambda because they are repeated we have taken a generalized term lambda only.

So, here write replace lambda equal to minus 1, so we can write down as minus t e raised to minus t equal to alpha 0 minus alpha 1; now this is a first equation we have got now. So, e raised to minus t we have got now the problem is if you replace here another lambda by minus 1. So, if you replace another lambda, lambda equals to minus 1 as because this is another lambda we came across same equation this.

But problem is that if both equation was same how to determine the alpha 0 and alpha 1; then which procedure we applied? So, in that case we have to use the concept of differentiations. So, what we can do here? With differentiate this equation did if differentiation with respect to lambda of function of f of lambda say at lambda equals to minus 1, equal to differentiation of lambda equals to R of lambda at lambda equals to minus 1.

Now, here R of lambda is this one alpha 0, alpha 1 lambda, f of lambda we have written like this. Now this d of d lambda what is f of lambda, f of lambda equal to e to lambda t and here differentiation of lambda that is equals to alpha 0 plus alpha 1 into lambda.

Now, if we differentiate this equations; so, we will get this equals to t e raised to lambda t and here we will get if you differentiate this alpha 0 become 0 and here in terms of lambda; we get alpha 1. And now if you replace here this lambda equals to minus 1 so we will get alpha 1 equal to t e raised to lambda t and now about alpha 0 this alpha 0 equals to alpha 1 plus e raised to minus t.

And now here alpha 1 is t and now lambda is that is this t e raised to minus t. So, here t e raised to minus t plus e raised to minus t. So, with take e raised to minus e t common; we will get t plus 1. Now here alpha 0 1 and what is a alpha 1? Alpha 1 is this this part this complete part alpha 1 and this complete part equals to alpha 0; that means, here we can finally, write down this is alpha 0 equal to e raised to minus t plus 1.

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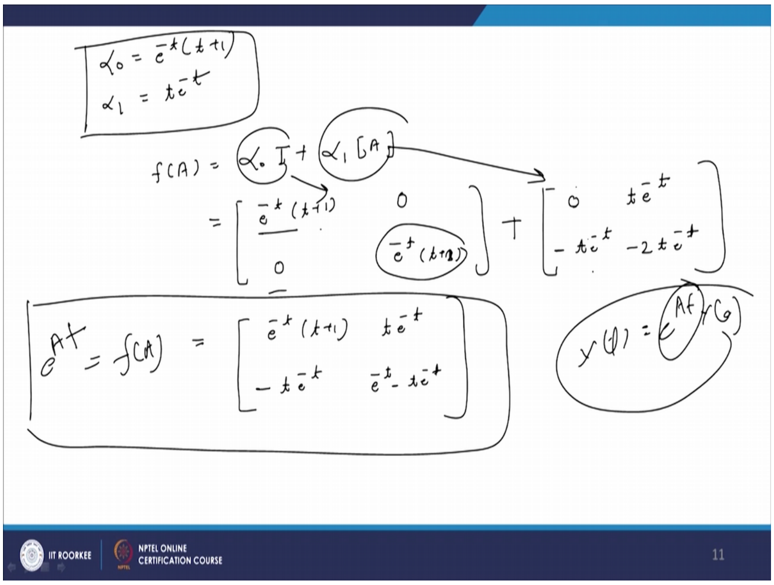

And here this alpha 1 equal to t e raised to minus t; these are 2 alpha 0 and alpha 1 we have determined. So, we will find that even know though they are repeated by differentiate this equation; we have come across this particular result alpha 0 and alpha 1. And now the rest of the procedure is very simple; that means, here f of A equal to alpha 0 into I plus alpha 1 into a.

Now, if you replace the values of alpha 0 as e raised to minus t; t plus 1, 0, 0 e raised to minus t t plus 1 this is this particular portion this part and now alpha 1 into A, so if you multiply this alpha 1 this one t e raised to A t to this particular matrix A; what we will get? So, we will get result as 0 t e raised to minus t minus t e raised to minus t minus 2 t e raised to minus t.

Now, this is nothing, but the result of this portion and now if you add this. So, after adding this we will get the state transition matrix as e raised to minus e t, t plus 1 see here this part plus this; we will get as t e raised to minus t and 0 plus minus t e raised to minus t; we can write down as minus t e raised to minus t and now this portion; t plus 1 minus 2 t e raised to minus t.

So, after adding this we will get e raised to minus t; minus t e raised to minus t. And now this is f of A or we can see e raised to A t as the result. Now this is the state transition matrix of a given system, when there are say repeated Eigen values.

So, in this way we have complete the part concerned with the state transition matrix. We have seen 4 methods first of all we have seen a power series method, then we have seen Laplace transform method, then we have seen the concept of diagonalization is to getting the state transition matrix and this last one is Cayley-Hamilton theorem method.

So, based on the problem we can use any of the method, but you will find that using all this method, we will come across the same result. And this concept is also used in the case of the getting the state equation; we have saying the state equation also that is x of t equal to e raised to A t into x naught for unforced system; that is the state equation is also depends on this state transition matrix.

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Now, there are some of the references.

Thank you.