Advanced Linear Continuous Control Systems
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Lecture - 25
State Transition Matrix (Part-I)

Now, we start with a State Transition Matrix. So, what is meant by State Transition Matrix?

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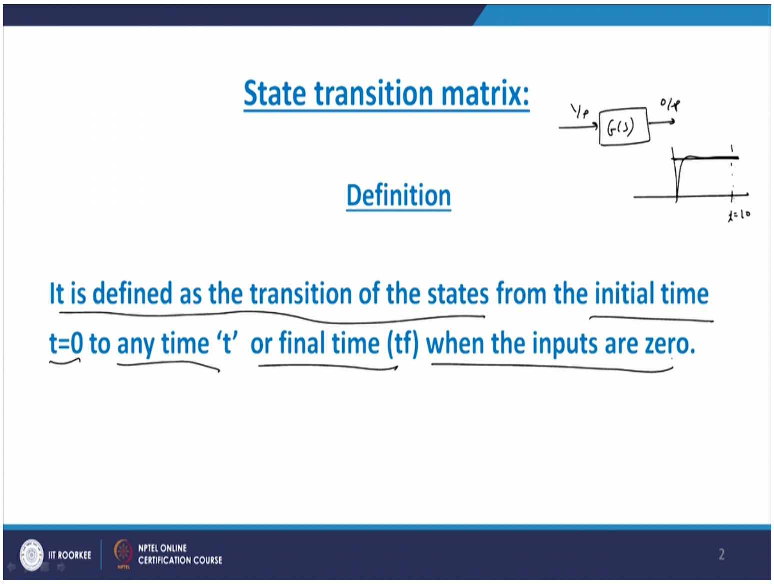

If you see a classical system the classical system here is a involvement of a plants say G of s we are we are applying input and we are getting a output. So, impede input is applied at t equals to 0, and we are expecting that the output as certain time interval say output say at say t equals to say 10 second. Now, this is the step input and now is a output. So, here output we have got at t equals to 10 second.

But, in case of a state space system along with this output there are states which are the important parameter in the system, that has been also traveled that is from t equals to 0 to t equals to f the state has been transferred. So, this transferring of the states we can we can say that state transition matrix.

Now, how will you define the state transition matrix? It is defined as the transition of states from initial time t naught to any time t or a final time tf when the inputs are zero, that is this trans state transition matrix has not concerned with the any type of input it is concerned with the some initial state that the t 0 to tf. Now, you see what are the significant, means say what are the importance of these state transition matrix.

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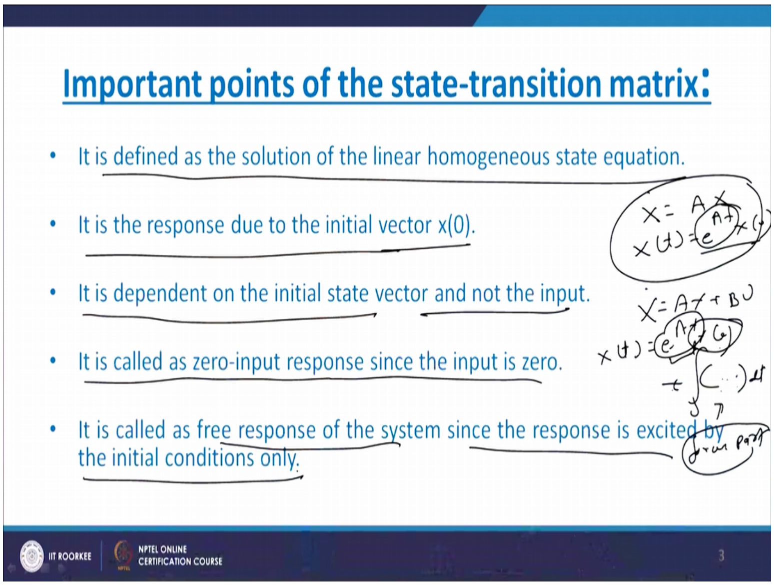

The first point it is defined as the solution of linear homogeneous state equations. We have seen last time, the X dot equal to A X X of t equal to e raised to A t X naught. So, this is e raised to A t is called state transition matrix and this is a homogeneous state equations.

It is response due to initial vector X naught it is response due to initial vector X naught here this is what, but if you consider a first system like this X dot equals to AX plus BU. So, here we will get X of t equal to e raised to A t X naught plus 0 to t some values will get here coefficient will get and we written d of t.

So, in this particular case this is u of t it has been taken care by X naught only whereas, if you take this part this portion that is a force portion force part in this particular part there is there is no involvement of X naught. So, therefore, what we are saying here it is dependent on the initial state vector not the input. The input is concern with the force part only, this part state transition matrix is concerned with only X naught. So, it is called as zero input response since input is 0, and whereas, if you take the force part this is called as zero state response. Here in this particular force part there is no initial state has been involved, but as far as now, we are discussing about the state transition matrix it is concern with the zero input response because this portion e raised to A t into X naught this portion, there is no input.

It is called as a free response of the since the response is excited by the initial condition only, that is due to initial conditions the state has been transferred and this is called state transition matrix.

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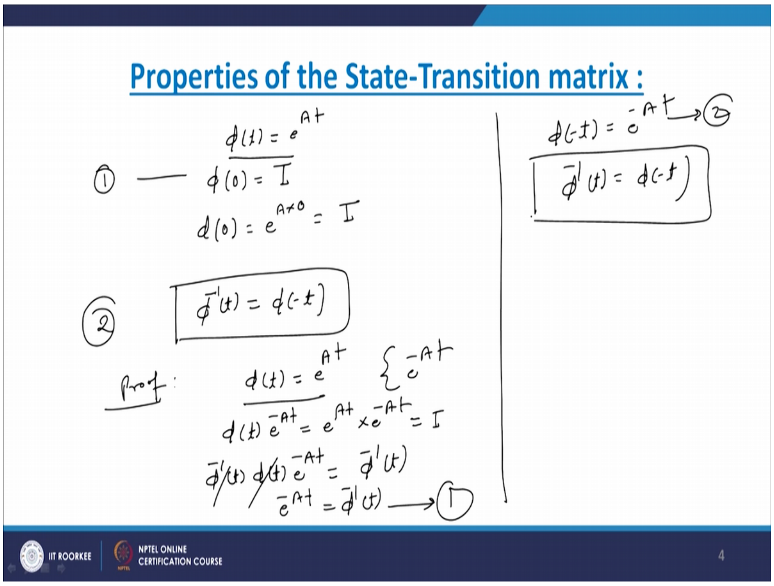

Now, we have to see a few properties of this state transition matrix. Say phi of t equal to e raised to A t that is a state transition matrix, and now we have to pre prove this phi of 0 equal to I this is a first part first property. So, how do you get I, identity matrix? So, in this particular equations, that is phi of 0 we replace we will get identity matrix.

Now, the second properties, we have to prove phi of minus t equal to phi of minus t that is phi inverse of minus t equal to phi of minus t. Now, how will you prove this one? So, now, we start with the same equation phi of t equal to e raised to A t. Now, through this equation we wow, post multiply by e raised to minus t, that is what we have to do to this equation we have to multi post multiply by e raised to minus A t.

So, after post multiplication what we will get? We will get phi of t to e raised to minus at equal to e raised to A t into e raised to minus at this is post multiplications, that is phi of t multiplied by e raised to minus at here e raised to A t multiplied by e raised to minus at we will get a anti matrix. So, now, our main purpose is to get phi inverse of t, equal to phi of minus t. Now, what we will do? We will pre multiply both side by phi inverse of t. So, what we will get? So, if you multiply this phi inverse of t into phi of t to e of e raised to minus at, so phi phi inverse of t, that is I. So, this is cancelled; what we will get e raised to minus at equals to phi inverse of t.

So, e raised to minus at equals to phi inverse of t now, this if the phi of minus t equal to e raised to minus at just like phi of t equals to e raised to plus at, but if you say phi of minus t equal to e raised to minus A t. Now, if you see this equation say 1, you can say this equation 2, if you compare these equations, so what we will get? We will get phi of phi inverse t equals 2 phi of minus t, this is a second property.

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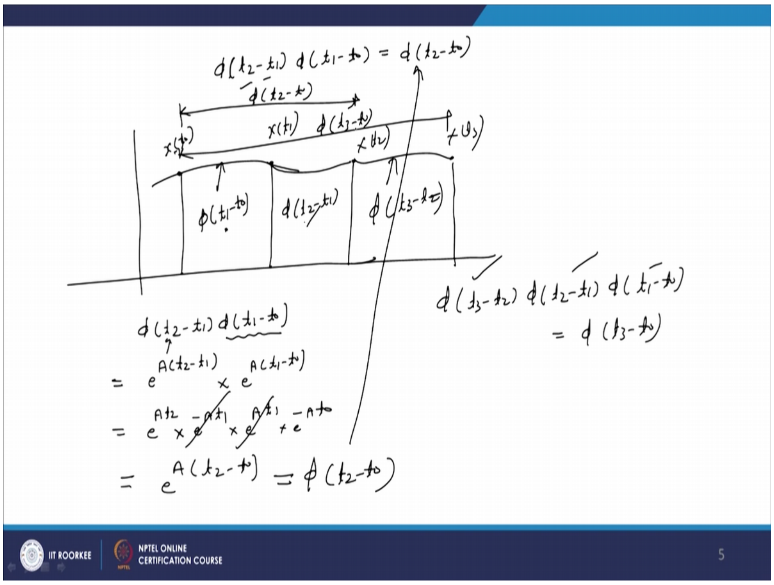

Now, we see the another property, then another property is phi of t 2 minus t 1, phi of t 1 minus t 0 that is equal to phi of t 2 minus t 0, phi of t 2 minus t 1, phi of t 1 minus 0 equal to phi of t 2 minus t 0.

Now, this we can express through a diagram. Now, here on point this one, is a another point here. Now, here X of t 0 and now here is X of t 1. Now, the state has been transferred from X of t 0 to t 1. So, that has been represented by phi of t 1 minus t 0 then same state. Now, you see take X of t 2, t 2. Now, here also the state has been transferred form X of t 1 to X of t 2 so it has been represented by phi of t 2 minus t 1.

Now, you just see this phi of t 2 minus t 1 phi of t 1 minus t 0 equals to phi t 2 minus t 0 that is this particular portion, phi of t 2 minus t 0. Now, we have to prove this part mathematically. How will you prove this? So, here phi of t 2 minus t 1 phi of t 1 minus t 0, now, you have to solve this. So, this is also a transition matrix state transition matrix. So, it has been represented by e raised to a t 2 minus t 1 you see this portion this part.

And about this phi of t 1 minus t 0 it has been written by A of t 1 minus t 0. Now, we solve this one. So, we can write down as e of A t 2 to A of minus A into t 1 that is manipulations e of A into t 1 multiplied by minus A into t naught. Now, you see here e of minus A t 1, minus A t 1 this will be cancelled. So, what we will get? E of a t 2 minus t 0 and that is nothing, but phi of t 2 minus t 0. Now, we have prove this part.

So, your transition for state from t 0 to t 1, t 1 to t 2, but it may possible that the state at X equal to t 2 may transferred to X equals to say t 3 that means, X equal to t 3 that is it can be represented by a phi of t 3 minus t 2. Now, how do you express this through state transition matrices? That is phi of t 3 minus t 2, phi of t 2 minus t 1, phi of t 1 minus t naught, that is equal to phi of t 3 minus t naught that is you see here phi of t 3 minus t 2 this one, phi of t 2 minus t 1, then phi of t 1 minus t 0 and this is a complete part is the involvement of phi of t 3 minus t naught.

So, similarly more state elements we can add it and we can express these in terms of state transition matrix.

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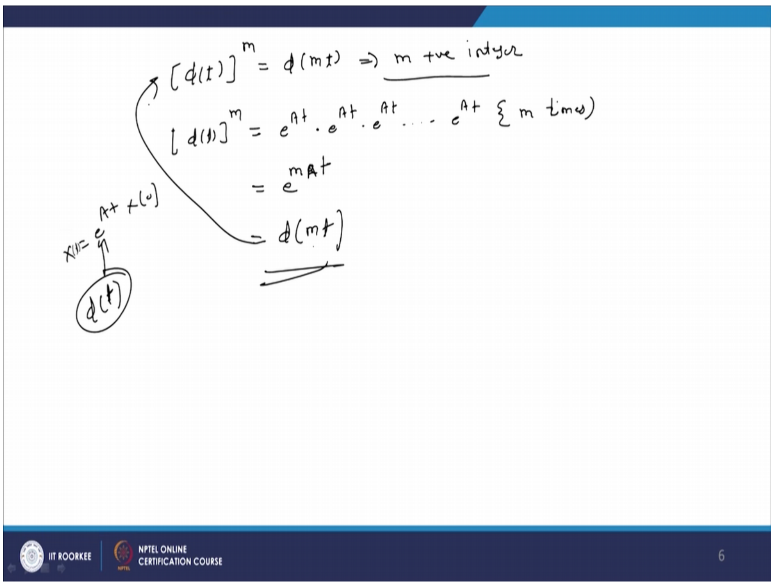

Now, the next properties say phi of t say raised to m equal to phi of m into t. So, in this case m is any positive integer. Now, this we have to prove. So, now, this phi of t raised to m. So, phi of t is nothing, but e raised to A t let us say m equals to 2 that means, e raised to A t multiplied by e raised to A t. Now, here m is there that means, e raised to A t has been written in terms of m times. Therefore, this equation we can write down as e raised to A t multiplied by e raised to A t e raised to A t up to e raised to A t.

And now up to how much terms we have to write? This we have to write m times, see here if phi of t raised to m e raised to A t e raised to A t m times. So, this we can write down as e how much time is a m into A into t, m into A into t and now we seen that this X t equals to e raised to t X naught. So, this is written as phi of t. So, here this can be written as phi of m into t here m term has been added therefore, this results has been proved.

So, these are the properties of the state transition matrix. Sometimes when we are doing the analysis of the system and many states are involved this type of properties can be useful. Now, the point has come this e raised to A t is there so that means, whenever required state transition matrix we have to calculate e raised to A t. Now, how to calculate it? So, if you see the literature there are various methods are available. So, these methods are useful to calculate the state transition matrix then we see the various methods for determination of state transition matrix.

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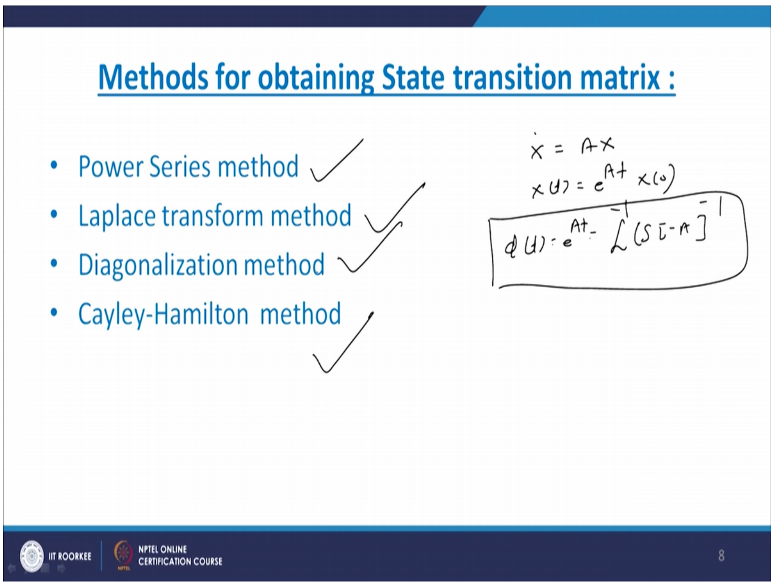

So, first one is Power Series method, Laplace transform method, Diagonalization method and Cayley-Hamilton method. So, these 4 methods can useful to get the state transition matrix. If you take a any example then apply this 4 method you will get the same results but now which method to be applied. So, this depends upon the type of systems. If you see the power series method we will see it, but you will find that as a order increases calculation is quite difficult and sometimes if we do not know whether we getting we will get result or not. This Laplace transform method is based on a s domain and Laplace is very important tool. Therefore, we will find that the Laplace transform method is quite useful.

We are already seen this approach earlier to get the state equation, that is X dot equals to AX X of t equal to e raised to A t into X naught and now these phi of t equal to e raised to A t equals to Laplace inverse of SI minus A inverse. By means of this we can calculate this state transition matrix using Laplace transform method that is you have take sI minus A inverse.

Now, the diagonalization method, this method is based on the concept of converting the given system matrix into the diagonal form and in this case there is need to go for the model matrix. So, if you see the calculation wise this method is quite cumbersome, but Laplace transfer is quite better.

Now, the next is Cayley-Hamilton theorem this is a very important theorem and it has wide applications. So, we will find that this method is also useful in getting the state transition matrix. Now, we will see through, but this is the method we already seen only, so method which we have to see power and diagonalization and Cayley-Hamilton theorem. Now, we start with power series method.

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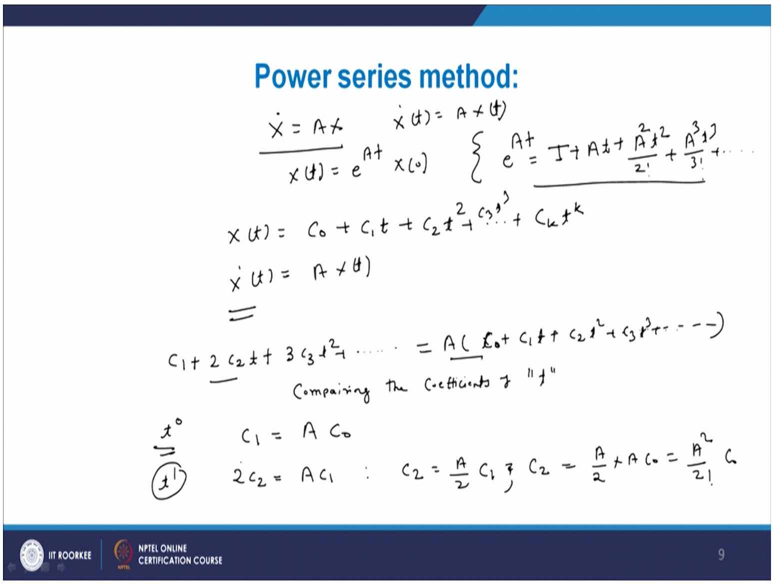

Now, we start with power series method. So, here we start with X dot equal to AX or this we can write down as X dot t equal to A into X of t. And here X of t state equation equal to e raised to A t into X naught and here this e raised to A t that is equal to I, A into t A square into t square factorial 2 plus A cube t cube factorial 3 and this processor. Now, we have to prove this. Why? The state transition matrix e raised to A t equals to this way.

Now, we start doing it. So, first of all what we will do we will assume that say X of t equal to C 0 plus C 1 t plus C 2 t square plus say C k into t raised to k X of t. Now, we are concerned with the X dot t. So, here X dot t equal to A into X of t, now if you see X dot t so now, differentiate these equations. So, if you differentiate X dot t this term of C 0 will not present. So, terms of C 1 C 2 up to C k will be present therefore, we write the equation as this is C 1 plus this depression of these 2 into C 2, 2 say 2 into C 2 2 into t plus 3 C 3 into t square and this process repeated that is s C through C 3 is there.

So, here C 3 t cube means 3, 3 C 3 into t square equal to A into X of t. So, we can write down as A into X 0 C 0 C 1 t C 2 t square like this or we can write C 3 t cube. So, these terms we have got and now our main purpose is to get this result. So, what we can do? Means in what way we imposes. So, we can see that we can compare the various terms of t and some somewhere we can achieve the result.

So, please see it. So, now, what we will do comparing the coefficients of t. Now, you compare the coefficient of t. So, now, see here, for if see that the term t raised to 0. So, you will find that t raised to 0 means 1. So, here for this it is C 1 and here is A into C naught. Similarly we compare that terms t raised to 1. So, here that is t. So, we can write down as see here t raised to 1, 2 C 2, so 2 C 2. Now, here we find A into C 1 into t's. Just see here A 1 C 1 into t, so we can write down as A into C 1. So, from this we can write down as C 2 equal to a by 2 into C 1 C 2 equals to a by 2 into C 1. But we know C 1; what is C 1? The C 1 is A into C naught. So, we replace the value of C 1 in this equation. So, we will get C 2 equal to A by 2 into A into C naught equal to A square by 2. So, we can write down as a factorial 2 into C naught.

So, why are your factorial, we will see later on. Now, here we have compare the coefficient of t 1. Now, similarly we have to compare the coefficient of t 2.

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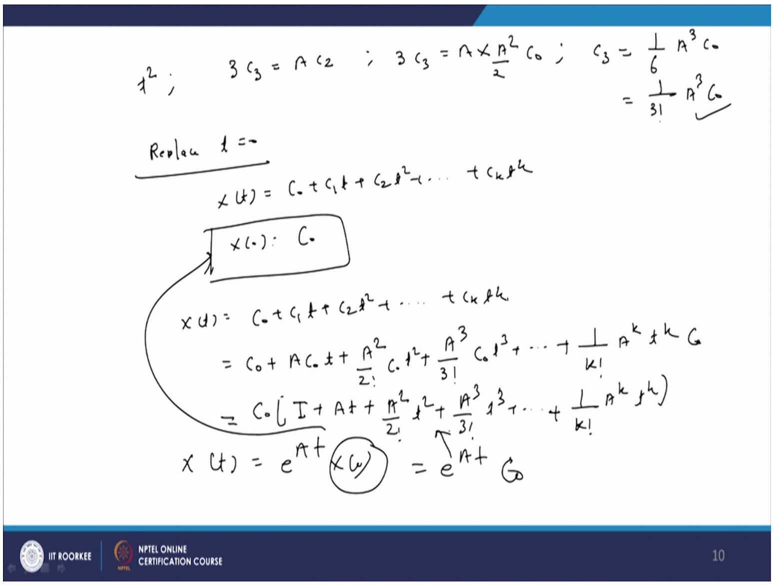

So, if you compare the coefficient of t 2 we will get 3 C 3 into A into C 2, so here 3 C 3 into A. Now, 3 C 3 into a what is C 2? C 2 we are already got A square factorial 2 into C naught. So, we replace these values here. So, you will can write A square by 2 into C naught.

Now, we have got A square by 2 C naught, but you know if we know also C naught. So, C naught is, so you will find that that C naught is no value. So, we can keep like this only. So, we can write down as this C 3 equal to 3 we will take this side and we will get a A cube into C naught, so here we can write down this as 6 into A 3 into C naught and if you write this as this as a factorial 3 that is 3 into 2, 6 A cube into C naught.

Now, similarly we have to push it further. Suppose if you replace, if you replace t equals to 0 in equation of X of t that is X of t equals to C 0, C 1 t C 2 t square plus C k into t k. So, you will get X of 0 equal to C naught. So, now, we have got the value of X naught C 0, C 1, C 2, C 3 assume that we got up to C k. Now, we replace all this element in the given equations and now we will see what results we will get that is X of t equal to C 0, C 1 t plus C 2 t square up to C k into t k.

So, we have got all values of C 0, C 1, C 2, C into C k into t k. So, you can also find this process A equals C 0, C 1, C 2 into C 3 into t cube this portion and we have got all values and now we try to replace. So, if you replace it C 0, what is the value of C 1? C 1 you will find you get C 1 equals to A into C naught, C 2 equals to A square factorial into C naught and C 3 equal to 1 by factorial 3 A cube into C naught.

So, we replace these values and we will find C 0 into A into C 0 into t plus C 2 A square factorial 2, C 0 into t square plus A cube factorial 3 C 0 into t cube and this process repeated and we will find that this is 3 our terms are up to k. So, we have 1 by a factorial k into A raised to 3. So, k, so we a raised to k into t raised to k.

Now, we will find that in all these cases in all these cases the term C 0 is common. So, we will take C 0. So, we can write down I plus A into t A square factorial 2 into t square, A cube factorial 3 into t cube plus 1 by factorial k a raised to k into t k. So now, we have got this one. Now, what is the X of t? So, X of t is equal to e raised to A t into X naught. And what is a X naught? This X naught we are already calculated as C naught and therefore, this is e raised to A t into C naught and therefore, if you compare this equation this is the same equation.

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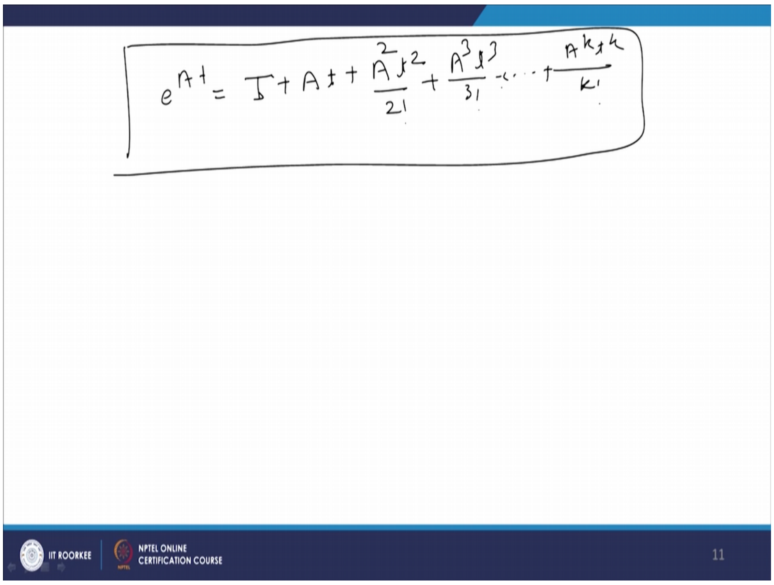

Now, here e raised to A t is nothing but this part that means, we can write down e raised to A t equal to I plus A into t plus A square t square factorial 2 plus A cube t cube factorial 3 and like this A k t k factorial k. Now, this is the direct formula. So, by means of this formula we can easily get the state transition matrix.

So, in that case what is the input is available to you input is available to is A, A matrix. So, you replace the A matrix and solve it we can get the result.

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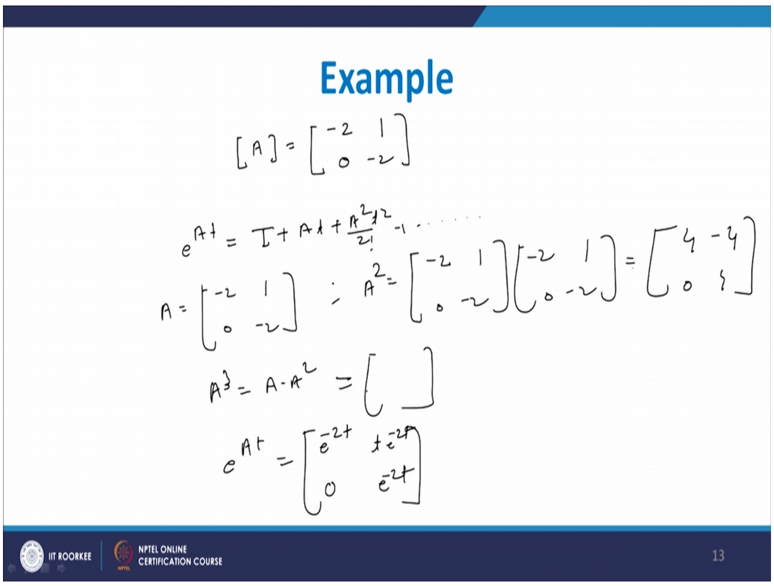

Now, we try to solve one example. Suppose if you are taking a matrix equal to minus 2, 1, 0, minus 2. And we want to determined we have to determine the state transition matrix e raised to A t by means of power series approach. So, here e raise to A t equal to same formula we write is I, A into t plus A square t square factorial 2 and like this.

Now, here you find that we will get A, A square, A cube it means that we have to calculate a many multiplications, multiple multiplication has to be solved and we will find out do this calc to do these calculations or to carry out this calculation sometimes it is quite cumbersomes.

Just we see here suppose if you take A is like this same thing minus 2, 1, 0, minus 2. Now, we have to determine A square, so A square is minus 2 1 0 minus 2 here minus 2 1 0 minus 2 and if you solve this one you will get 4 minus 4 0 into 4 and we have got A square, similarly you can get A cube A cube equals to A into A square.

So, we have to solve you get some values and finally, you have to replace in these particular equations. So, you will find that there are some terms. So, everywhere these terms are added and finally, if you get e raised to A t as e raised to minus 2 t t into e raised to minus 2 t 0 e raised to minus 2 t. That means what you do? You have to determine A square, A cube, A 4 like this they replace these values.

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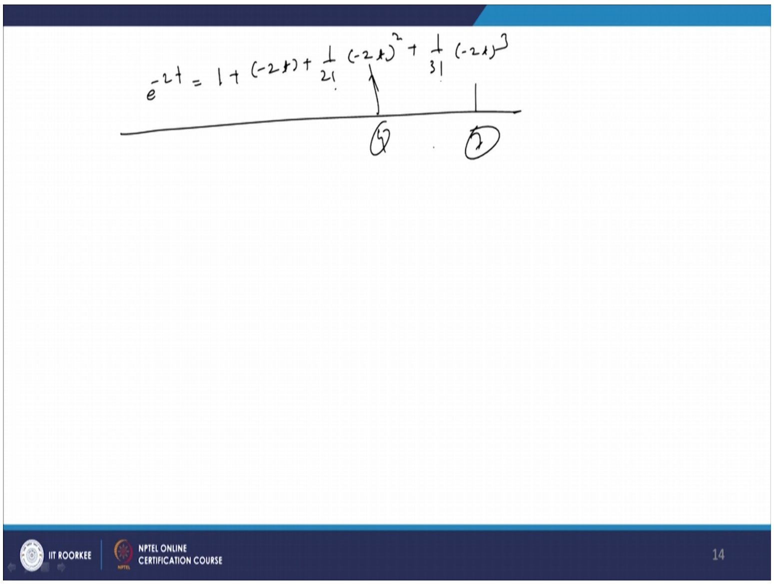

And from this same expression taking different elements we will get e raised to minus 2 t 2 t, that is e raised to minus 2 t portion is nothing but 1 plus minus 2 t plus 1 plus factorial 2 minus 2 t square plus factorial 3 minus 2 t whole cube. This is e raised to minus 2 t. Similarly we will get t e raised to minus 2 two t 0 e raised to minus 2 t. So, this is very simple calculations you can easily do it.

But here the main intention of this example is that sometimes it may not possible that you can get a similar type of results that is instead of minus 2 sometimes we get 4, it may get 7. In that case to make these elements or make adjust this element by means of power series exam power series expansion is quite difficult. Therefore, as we increase the order we will not recommend this method to get the state transition matrix because calculation is quite difficult. That means, in that is case it is better to go for the Laplace transform approach or there are other approaches that is called Cayley-Hamilton theorem or diagonalization that also helpful, but this approach sometimes is quite difficult.

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Now, you see the references Gopal, D Roy Choudhury, Kuo, Varmah and so these references can you can use or if try to solve many example as possible, using this power series expansions and as I told you that you identifying the various cases where this method fails. And the same problem you can solve by a Laplace transform approach and you prove that a Laplace transform is better than the power series method.

Thank you.